

Calculus I: Limits and Integrals

Limits

Limit Properties

- $\lim_{x \rightarrow a} [c \cdot f(x)] = c \cdot \lim_{x \rightarrow a} f(x)$
- $\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x)$
- $\lim_{x \rightarrow a} \left[\frac{f(x)}{g(x)} \right] = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}$ if $\lim_{x \rightarrow a} g(x) \neq 0$
- $\lim_{x \rightarrow a} [f(x)]^n = \left[\lim_{x \rightarrow a} f(x) \right]^n$

Continuity Test

Used to determine if a function is continuous at a point

1. $f(c)$ exists
2. $\lim_{x \rightarrow c} f(x)$ exists, so $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$
3. $\lim_{x \rightarrow c} f(x) = f(c)$

Example: Determine if $f(x) = x^2 + 2x - 6$ is continuous at $x = 1$

- $f(1) = (1)^2 + 2(1) - 6 = -3$
- $x^2 + 2x - 6 = -3$
- The limit exists at $x = 1$ because $\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x)$
- The function is continuous at $x = 1$ because $\lim_{x \rightarrow 1} f(x) = f(1)$

Integrals

Definition: The sum of infinitely many, infinitesimally small areas

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x \text{ where } \Delta x = \frac{b-a}{n} \text{ and } x_k = a + k \cdot \Delta x$$

Integral Properties

- $\int_a^a f(x) dx = 0$
- $\int_b^a f(x) dx = - \int_a^b f(x) dx$

- $\int_a^b c \cdot f(x) dx = c \cdot \int_a^b f(x) dx$
- $\int_a^b [f(x) \pm g(x)] dx = \int_a^b f(x) dx \pm \int_a^b g(x) dx$
- $\int_a^b f(x) dx + \int_b^c f(x) dx = \int_a^c f(x) dx$

Fundamental Theorem of Calculus

If $f(x)$ is continuous on $[a, b]$ and $F'(x) = f(x)$:

- $F'(x) = \frac{d}{dx} \int_a^x f(t) dt = f(x)$
- $\int_a^b f(x) dx = F(b) - F(a)$

Integration by Substitution

$\int_a^b [f(g(x)) \cdot g'(x)] dx = \int_{g(a)}^{g(b)} f(u) du$ where $u = g(x)$ and $du = g'(x) dx$

Example: Evaluate $\int_1^3 3(3x - 1)^2 dx$

- Let $u = 3x - 1$, so $du = 3dx$
- $\int_{3(1)-1}^{3(3)-1} (u)^2 du = \int_2^8 u^2 du = \left[\frac{1}{3} u^3 \right]_2^8$
- $= \frac{1}{3} (8)^3 - \frac{1}{3} (2)^3 = 168$
- So, $\int_1^3 3(3x - 1)^2 dx = 168$

Integration by Parts

$\int u dv = uv - \int v du$ where $v = \int dv$ OR $\int f(x)g'(x) dx = f(x)g(x) - \int f'(x)g(x) dx$

Example: Evaluate $\int 3x^2 \cdot \ln \ln x dx$

- Let $u = \ln \ln x$ and $dv = 3x^2 dx$
- So, $du = \frac{1}{x} dx$ and $v = \int 3x^2 dx = x^3$
- $\int 3x^3 \cdot \ln \ln x dx = \ln \ln x \cdot x^3 - \int x^3 \cdot \frac{1}{x} dx$
- $= x^3 \ln \ln x - \int x^2 dx = x^3 \ln \ln x - \frac{1}{3} x^3 + C$

Common Integrals

- $\int k dx = k \cdot x + C$
- $\int e^x dx = e^x + C$
- $\int \sin \sin(x) dx = -\cos(x) + C$
- $\int \sec^2(x) dx = \tan \tan(x) + C$
- $\int \sec \sec(x) \cdot \tan \tan(x) dx = \sec \sec(x) + C$

- $\int x^n dx = \frac{1}{n+1} x^{n+1} + C$
- $\int \frac{1}{x} dx = \ln|x| + C$
- $\int \cos \cos(x) dx = \sin \sin(x) + C$
- $\int \csc^2(x) dx = -\cot \cot(x) + C$
- $\int \csc(x) \cdot \cot(x) dx = -\csc(x) + C$

For more information on limits and integrals, make an appointment with a [content tutor](#). Appointments are available in person at the [Student Success Center](#) or online.

Adapted from Hass, J., Weir, M.D., & Thomas, G.B. (2012). *University Calculus: Early Transcendentals* (2nd ed.). Boston: Pearson Education.

This work is licensed under a [Creative Commons Attribution-NoDerivatives 4.0 International License](#).