Computational Methods for Kinetic Processes in Plasma Physics

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Over views:
Simulation methods Macroscopic and Microscopic Processes

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Context

1. Brief history of PIC simulations
2. Introduction
3. Basic structures of 3D RPIC code
4. Basic structures of 3D GRMHD code
5. Summary and Future Improvements

Brief history of PIC simulations

In late 1950s John Dawson began 1D electrostatic “charge-sheet” experiments at Princeton, later at UCLA.

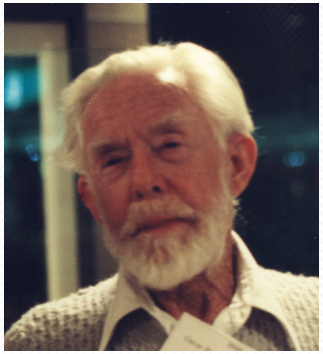
1965 Hockney, Buneman -- introduced grids and direct Poisson solver

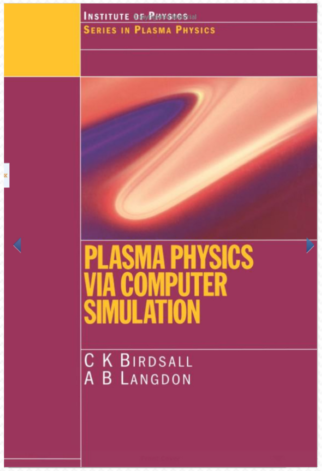
1970-s theory of electrostatic PIC developed (Langdon)

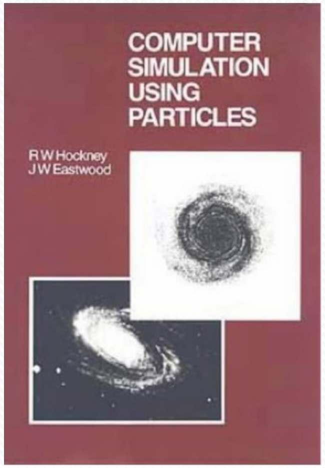
First electromagnetic codes
1980s-90s 3D EM PIC takes off
(Dawson, Rev. Modern Physics, 1983)

“PIC text books” come out in 1988 and 1990

Integrated circuit doubles ~ every two years (Moore’s law)







Experiments

Simulations

Plasma Physics

Theory

Nonlinear phenomena
PIC
MHD
Hybrid
Vlasov

Linear
Quasi-linear

Applications

Particle acceleration
Instabilities
Radiation
Anomalous resistivity
Reconnection
Relativistic jets

Characteristic time and length scales

$$\omega_P = \left(\frac{4\pi\ ne^2}{m}\right)^{1/2}$$

Plasma frequency

$$\lambda_D = \frac{V_{thermal}}{\omega_p} \propto \left(\frac{T}{n}\right)^{1/2}$$

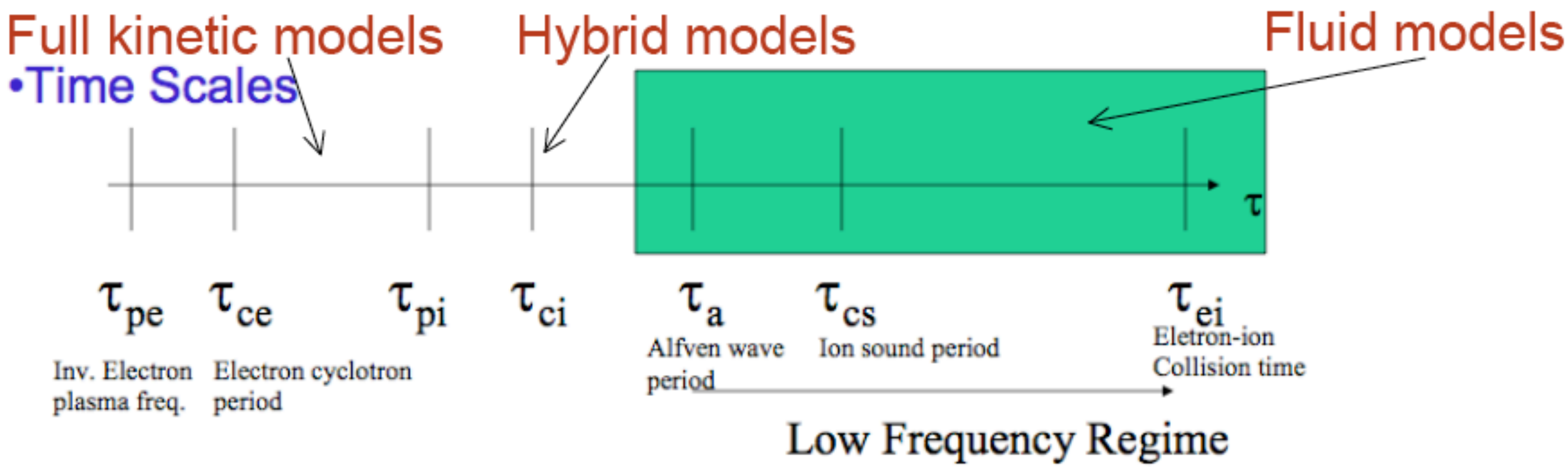
Debye length

$$\lambda_{skin} = c/\omega_p$$

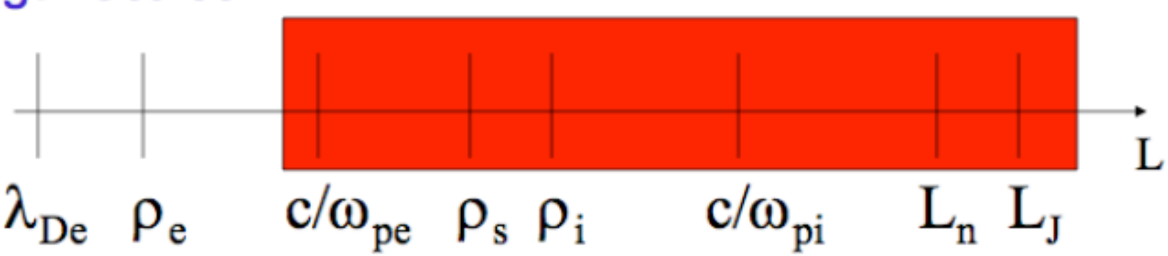
skin depth

$$\omega_c = \frac{eB}{mc}$$

Larmor



•Length Scales



$\rho_e = v_{te}/\omega_{ce}$ $\rho_s = \sqrt{T_i/T}\,\rho_i$ $L_n = \nabla n/n$

Magnetic field in the Universe

- Magnetic and gravitational fields play a important role in determining the evolution of the matter in many astrophysical objects
- Magnetic field can be amplified by the gas contraction or shear motion.
- Even when the magnetic field is weak initially, the magnetic field glows in the short time scale and influences gas dynamics of the system

Plasmas in the Universe

- The major constituents of the universe are made of plasmas.
- When the temperature of gas is more than 10^4K, the gas becomes fully ionized plasmas (4^{th} phase of matter).
- Plasmas are applied to many astrophysical phenomena.
- Plasmas are treated in several ways
  - particle-in-cell (PIC) (microscopic)
  - magnetohydrodynamics, MHD (macroscopic) (not covered)
  - hybrid (fluid electron and kinetic ions) (not covered)
  - MHD with test particles (fluid mixed with particles) (not covered)
  - particles with photons (not covered)

3D Relativistic particle-in-cell code

Kinetic processes are included in this code

- Particle acceleration can be investigated
- Calculation of radiation can be calculated
- Simulation system size is limited due to necessity of resolving Debye length (Skin depth)
- Global dynamics of plasma such as large jets cannot be simulated

This simulation method is complimentary to MHD method which will be described briefly later

Collisionless plasma can be described by Vlasov-Maxwell equations with distribution function f(**x**, **v**, t) (6 dimensions):

$$\frac{\partial f}{\partial t} + \vec{v} \bullet \frac{\partial f}{\partial \vec{x}} + \frac{q}{m}(\vec{E} + \frac{\vec{v} \times \vec{B}}{c}) \bullet \frac{\partial f}{\partial \vec{v}} = 0,$$

$$\nabla \bullet \vec{E} = 4\pi \int q f d^3\vec{v}$$

$$\nabla \times \vec{B} = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi}{c}\int q\vec{v} f d^3\vec{v},$$

$$\nabla \bullet \vec{B} = 0, \quad \nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}.$$

Direct calculation of this set of equations – 6D
Improvements have been made, but difficult to calculate using this method

$$\frac{d\vec{v}_j}{dt} = \frac{q_j}{m_j}(\vec{E} + \frac{\vec{v}_j \times \vec{B}}{c})$$

$$\frac{d\vec{x}}{dt} = \vec{v}$$

$$\nabla \times \vec{E} = -\frac{1}{c}\frac{\partial \vec{B}}{\partial t}$$

$$\nabla \times B = \frac{1}{c}\frac{\partial \vec{E}}{\partial t} + \frac{4\pi \vec{J}}{c}$$

$$\nabla \bullet B = 0, \quad \nabla \bullet E = 4\pi\rho$$

$$\rho(\vec{x}) = \sum_j q_j \delta(\vec{x} - \vec{x}_j)$$

$$\vec{j}(\vec{x}) = \sum_j q_j \vec{v}_j \delta(\vec{x} - \vec{x}_j)$$

Basic controlling equations

Newton-Lorentz equation

$$m_{i,e}\frac{d\mathbf{v}_{i,e}}{dt} = q_{i,e}(\mathbf{E} + \mathbf{v}_{i,e}\times\mathbf{B})$$

Maxwell equation

$$\frac{\partial\mathbf{B}}{\partial t} = -\nabla\times\mathbf{E}$$

$$\frac{\partial\mathbf{E}}{\partial t} = c^2\nabla\times\mathbf{B} - \frac{1}{\varepsilon_0}\mathbf{J}$$

$$\mathbf{J} = \sum(n_i q_i \mathbf{v}_i - n_e q_e \mathbf{v}_e)$$

Fields

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \qquad \frac{\partial \mathbf{D}}{\partial t} = \nabla \times \mathbf{H} - \mathbf{J}$$

$$\mathbf{B} = \mu_0 \mathbf{H} \qquad \mathbf{D} = \varepsilon_0 \mathbf{E}$$

$$\varepsilon_0 = 1 \qquad \mu_0 = 1/c^2$$

In Tristan code

$$\frac{\partial \mathbf{B}}{\partial t} = -c\nabla \times \mathbf{E} \qquad \frac{\partial \mathbf{E}}{\partial t} = c\nabla \times \mathbf{B} - \mathbf{J}$$

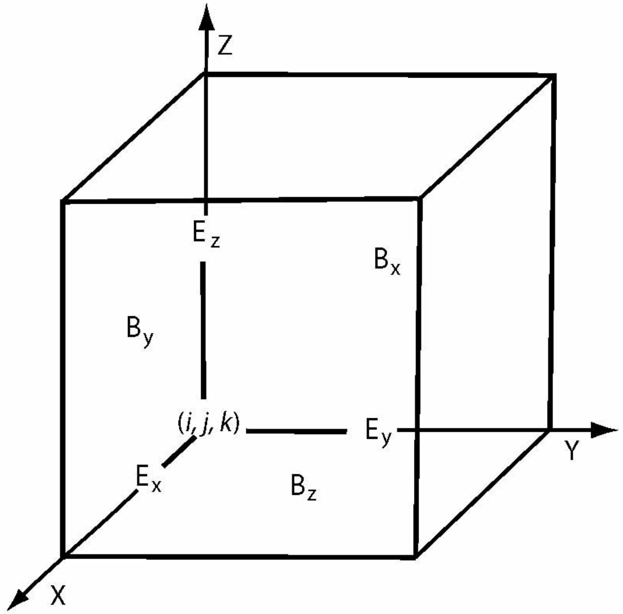
Two major methods of calculating current

1. Spectral method (UPIC code) (note by Decyk)
   We will review this method in details later after we do handout exercises

2. Charge-conserving current deposit (Villasenor & Buneman 1992)
   We will review this method with Umeda's method later

Ampere equation

$$\frac{\partial \mathbf{B}}{\partial t} = -c \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e_x & e_y & e_z \end{vmatrix} = c[\ \mathbf{i}(\frac{\partial e_y}{\partial z} - \frac{\partial e_z}{\partial y}) + \mathbf{j}(\frac{\partial e_z}{\partial x} - \frac{\partial e_x}{\partial z}) + \mathbf{k}(\frac{\partial e_x}{\partial y} - \frac{\partial e_y}{\partial x})]$$



Yee lattice

In Yee Lattice e_x, e_y, e_z, b_x, b_y, b_z are, respectively staggered and shifted on 0.5 from (*I, j, k*) and located at the position

$$\begin{aligned} e_x(i,j,k) &\rightarrow e_x(i+.5,j,k), \\ e_y(i,j,k) &\rightarrow e_y(i,j+.5,k), \\ e_z(i,j,k) &\rightarrow e_z(i,j,k+.5), \end{aligned}$$

and

$$\begin{aligned} b_x(i,j,k) &\rightarrow b_x(i,j+.5,k+.5), \\ b_y(i,j,k) &\rightarrow b_y(i+.5,j,k+.5), \\ b_z(i,j,k) &\rightarrow b_z(i+.5,j+.5,k). \end{aligned}$$

Field update

$$\frac{\partial}{\partial t} b_x = (b_x^{new}(i, j+.5, k+.5) - b_x^{old}(i, j+.5, k+.5))/\delta t$$
$$= c[(e_y(i, j+.5, k+1) - e_y(i, j+.5, k))/\delta z$$
$$-(e_z(i, j+1, k+.5) - e_z(i, j, k+.5))/\delta y].$$

Here $\partial t = \partial x = \partial y = \partial z = 1$

$$b_x^{new}(i,j,k) = b_x^{old}(i,j,k)$$
$$+c[\ e_y(i,j,k+1) - e_y(i,j,k) - e_z(i,j+1,k) + e_z(i,j,k)\ ].$$

$$b_y^{new}(i,j,k) = b_y^{old}(i,j,k)$$
$$+c[\ e_z(i+1,j,k) - e_z(i,j,k) - e_x(i,j,k+1) + e_x(i,j,k)\ ],$$

$$b_z^{new}(i,j,k) = b_z^{old}(i,j,k)$$
$$+c[\ e_x(i,j+1,k) - e_x(i,j,k) - e_y(i+1,j,k) + e_y(i,j,k)].$$

Electric field update

$$\frac{\partial \mathbf{E}}{\partial t} = c \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ b_x & b_y & b_z \end{vmatrix} = c[\,\mathbf{i}(\frac{\partial b_z}{\partial y} - \frac{\partial b_y}{\partial z}) + \mathbf{j}(\frac{\partial b_x}{\partial z} - \frac{\partial b_z}{\partial x}) + \mathbf{k}(\frac{\partial b_y}{\partial x} - \frac{\partial b_x}{\partial y})]$$

$$\begin{aligned} \frac{\partial}{\partial t} e_x &= (e_x^{new}(i+.5,j,k) - e_x^{old}(i+.5,j,k))/\delta t \\ &= c[(b_z(i+.5,j+.5,k) - b_z(i+.5,j-.5,k))/\delta y \\ &- (b_y(i+.5,j,k+.5) - b_y(i+.5,j,k-.5))/\delta z], \end{aligned}$$

$$\begin{aligned} e_x^{new}(i,j,k) = e_x^{old}(i,j,k) \\ +c[\, b_y(i,j,k-1) - b_y(i,j,k) - b_z(i,j-1,k) + b_z(i,j,k)\,], \end{aligned}$$

Particle update

Newton-Lorentz equation

$$\mathbf{v}^{new} - \mathbf{v}^{old} = \frac{q\delta t}{m} < \mathbf{E} + \frac{1}{2}(\mathbf{v}^{new} + \mathbf{v}^{old}) \times \mathbf{B} >$$

$$\mathbf{r}^{next} - \mathbf{r}^{present} = \delta t \mathbf{v}^{new}$$

Buneman-Boris method

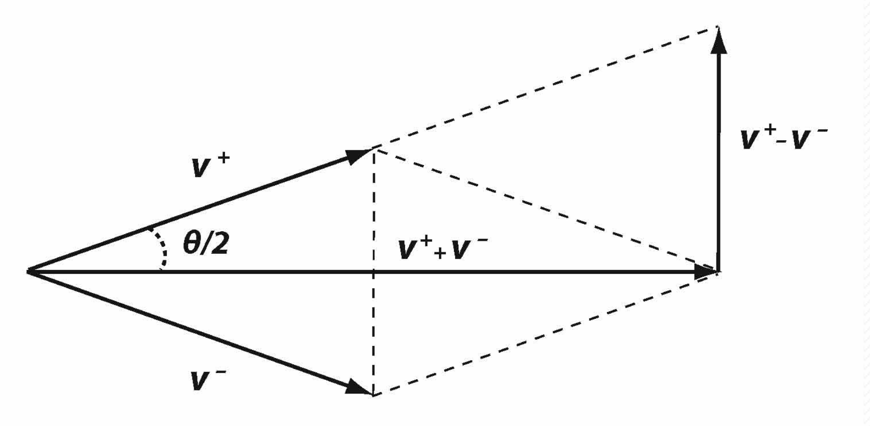
Half an electric acceleration

Pure magnetic rotation

Another half electric acceleration

Buneman-Boris method

$$\frac{\mathbf{v}^{n+1/2}-\mathbf{v}^{n-1/2}}{\Delta t}=\frac{q}{m}\left(\mathbf{E}^n+\frac{\mathbf{v}^{n+1/2}+\mathbf{v}^{n-1/2}}{2}\times\mathbf{B}^n\right)$$



$$\mathbf{v}^{-}=\mathbf{v}^{n-1/2}+\frac{q}{m}\mathbf{E}^n\frac{\Delta t}{2}$$

$$\mathbf{v}^{+}=\mathbf{v}^{n+1/2}-\frac{q}{m}\mathbf{E}^n\frac{\Delta t}{2}$$

rotation $$\frac{\mathbf{v}^{+}-\mathbf{v}^{-}}{\Delta t}=\frac{1}{2}\frac{q}{m}(\mathbf{v}^{+}+\mathbf{v}^{-})\times\mathbf{B}^n$$

$$\mathbf{v}^{+}=\mathbf{v}^{-}+\frac{2}{1+\mathbf{T}^2}(\mathbf{v}^{-}+\mathbf{v}^{-}\times\mathbf{T})\times\mathbf{T}$$

$$\mathbf{T}=\frac{q}{2m}\Delta t\mathbf{B}^n$$

Buneman-Boris method (cont)

4 steps

$$\mathbf{v}^{-} = \mathbf{v}^{n-1/2} + \frac{q}{m}\mathbf{E}^{n}\frac{\Delta t}{2}$$

$$\mathbf{v}^{0} = \mathbf{v}^{-} + \mathbf{v}^{-} \times \mathbf{T}$$

$$\mathbf{v}^{+} = \mathbf{v}^{-} + \mathbf{v}^{0} \times \mathbf{S}$$

$$\mathbf{S} = 2\mathbf{T} / (\mathbf{1} + \mathbf{T}^{2})$$

$$\mathbf{v}^{n+1/2} = \mathbf{v}^{+} + \frac{q}{m}\mathbf{E}^{n}\frac{\Delta t}{2}$$

$$\mathbf{r}^{n+1} = \mathbf{r}^{n} + \mathbf{v}^{n+1/2}\Delta t$$

Relativistic generalization

$$\mathbf{u} = \gamma_v \mathbf{v}, \quad \gamma_v^2 = 1 - \frac{v^2}{c^2} \qquad\qquad \gamma^2 = \left(1 + \frac{u^2}{c^2}\right)$$

$$\frac{\mathbf{u}^{n+1/2} - \mathbf{u}^{n-1/2}}{\Delta t} = \frac{q}{m}\left(\mathbf{E}^n + \frac{\mathbf{u}^{n+1/2} + \mathbf{u}^{n-1/2}}{2\gamma^n} \times \mathbf{B}^n\right)$$

$$\mathbf{r}^{n+1} = \mathbf{r}^n + \mathbf{v}^{n+1/2}\Delta t = \mathbf{r}^n + \frac{\mathbf{u}^{n+1/2}}{\gamma^{n+1/2}}\Delta t$$

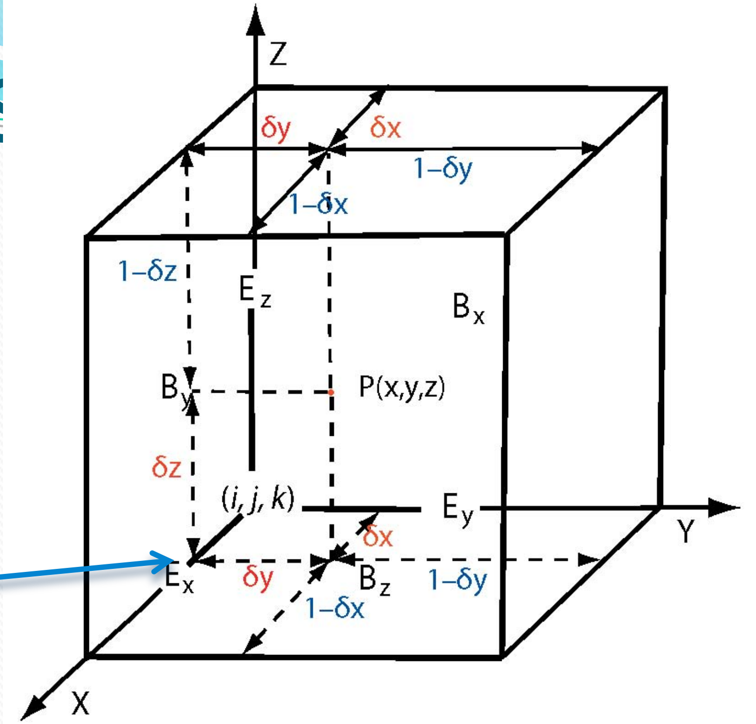
$$\left(\gamma^{n+1/2}\right)^2 = 1 + \left(\frac{u^{n+1/2}}{c}\right)^2$$

Force interpretations

"volume" weight

$$(i,j,k) \Leftarrow (1-\delta x)(1-\delta y)(1-\delta z) = cx \bullet cy \bullet cz$$

$$(i+1,j+1,k+1) \Leftarrow \delta x \bullet \delta y \bullet \delta z$$



$$\mathbf{F}_{e_x}^{(x,j,k)} = \overline{e}_x(i,j,k) + \left[\overline{e}_x(i+1,j,k) - \overline{e}_x(i,j,k)\right]\delta x$$

$$\overline{e}_x(i,j,k) = \frac{1}{2}\left\{e_x(i,j,k) + e_x(i-1,j,k)\right\} \qquad \overline{e}_x(i+1,j,k) = \frac{1}{2}\left\{e_x(i+1,j,k) + e_x(i,j,k)\right\}$$

on (*x*, *j*, *k*)

$$2\mathbf{F}_{e_x}^{(x,j,k)} = e_x(i,j,k) + e_x(i-1,j,k) + \left[e_x(i+1,j,k) - e_x(i-1,j,k)\right]\delta x$$

similarly on (x, *j*+1, *k*), (x, *j*, *k*+1), (x, *j*+1, *k*+1)

$$2\mathbf{F}_{e_x}^{(x,j+1,k)} = e_x(i,j+1,k) + e_x(i-1,j+1,k) + \left[e_x(i+1,j+1,k) - e_x(i-1,j+1,k)\right]\delta x$$

$$2\mathbf{F}_{e_x}^{(x,j,k+1)} = e_x(i,j,k+1) + e_x(i-1,j,k+1) + \left[e_x(i+1,j,k+1) - e_x(i-1,j,k+1)\right]\delta x$$

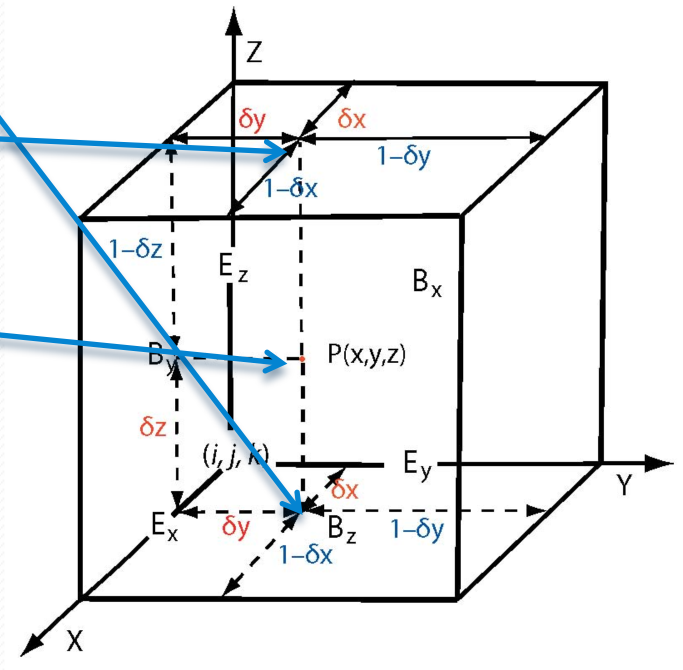
$$2\mathbf{F}_{e_x}^{(x,j+1,k+1)} = e_x(i,j+1,k+1) + e_x(i-1,j+1,k+1) + \left[e_x(i+1,j+1,k+1) - e_x(i-1,j+1,k+1)\right]\delta x$$

$$\mathbf{F}_{e_x}^{(x,y,k)} = \mathbf{F}_{e_x}^{(x,j,k)} + \left[\mathbf{F}_{e_x}^{(x,j+1,k)} - \mathbf{F}_{e_x}^{(x,j,k)}\right]\delta y$$

$$\mathbf{F}_{e_x}^{(x,y,k+1)} = \mathbf{F}_{e_x}^{(x,j,k+1)} + \left[\mathbf{F}_{e_x}^{(x,j+1,k+1)} - \mathbf{F}_{e_x}^{(x,j,k+1)}\right]\delta y$$

$$\mathbf{F}_{e_x}^{(x,y,z)} = \mathbf{F}_{e_x}^{(x,y,k)} + \left[\mathbf{F}_{e_x}^{(x,y,k+1)} - \mathbf{F}_{e_x}^{(x,y,k)}\right]\delta z$$

$$\mathbf{F}_{e_y}^{(x,y,z)}, \mathbf{F}_{e_z}^{(x,y,z)}, \mathbf{F}_{b_x}^{(x,y,z)}, \mathbf{F}_{b_y}^{(x,y,z)}, \mathbf{F}_{b_z}^{(x,y,z)}$$



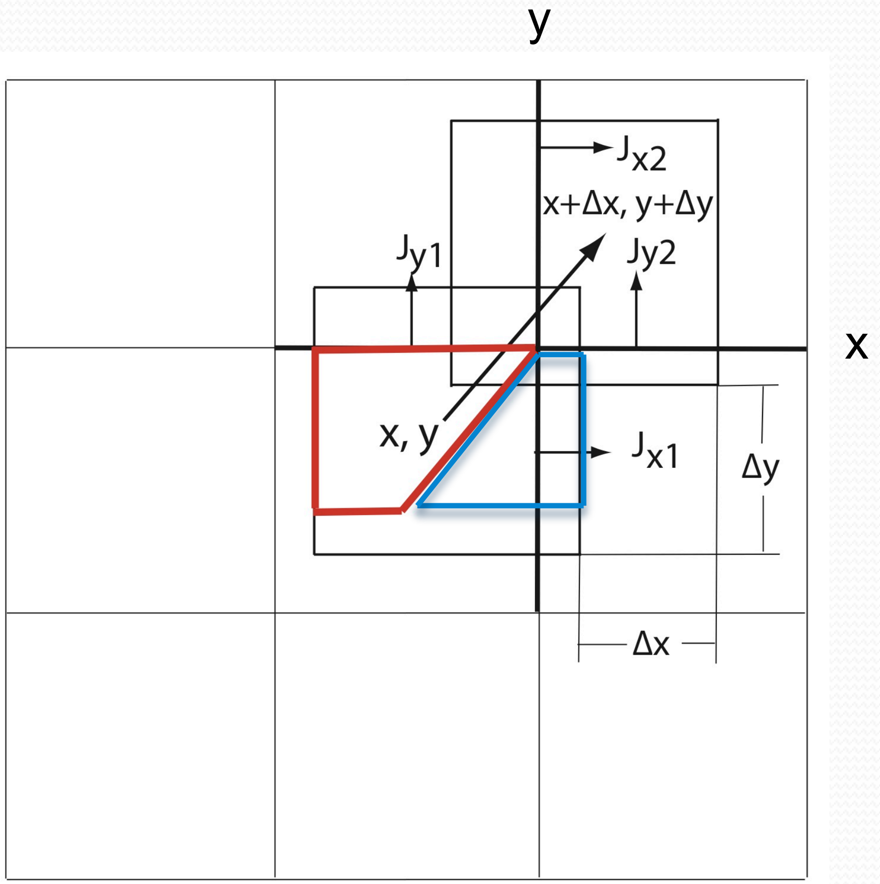
Current deposit scheme (2-D)

$$J_{x1} = q\Delta x(\frac{1}{2} - y - \frac{1}{2}\Delta y)$$

$$J_{x2} = q\Delta x(\frac{1}{2} + y + \frac{1}{2}\Delta y)$$

→ $$J_{y1} = q\Delta y(\frac{1}{2} - x - \frac{1}{2}\Delta x)$$

→ $$J_{y2} = q\Delta y(\frac{1}{2} + x + \frac{1}{2}\Delta x)$$

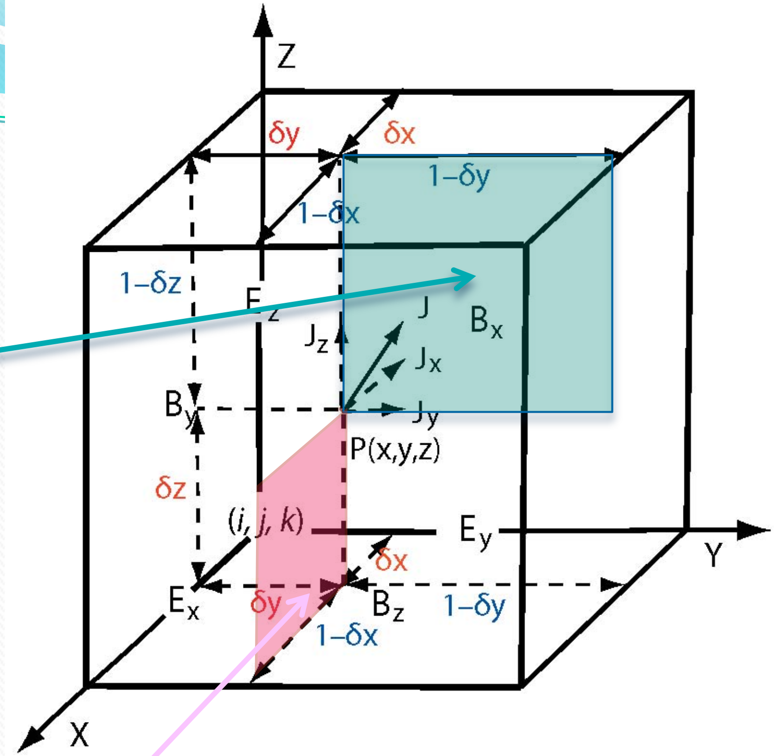


Current deposit

Charge conservation $\frac{\partial \rho}{\partial t} = -\nabla \bullet \mathbf{J}$

$$e_x(i,j,k) = e_x(i+.5,j,k)$$
$$= e_x(i,j,k) - J_x \bullet cy \bullet cz$$
$$e_x(i,j+1,k) = e_x(i+.5,j+1,k)$$
$$= e_x(i,j+1,k) - J_x \bullet \delta y \bullet cz$$
$$e_x(i,j,k+1) = e_x(i+.5,j,k+1)$$
$$= e_x(i,j,k+1) - J_x \bullet cy \bullet \delta z$$
$$e_x(i,j+1,k+1) = e_x(i+.5,j+1,k+1)$$
$$= e_x(i,j+1,k+1) - J_x \bullet \delta y \bullet \delta z$$

$$e_y(i,j,k) = e_y(i,j+.5,k)$$
$$= e_y(i,j,k) - J_y \bullet cx \bullet cz$$
$$e_y(i,j+1,k) = e_y(i+1,j+.5,k)$$
$$= e_y(i+1,j,k) - J_y \bullet \delta x \bullet cz$$
$$e_y(i,j,k+1) = e_y(i,j+.5,k+1)$$
$$= e_y(i,j,k+1) - J_y \bullet cx \bullet \delta z$$
$$e_y(i,j+1,k+1) = e_y(i+1,j+.5,k+1)$$
$$= e_y(i+1,j,k+1) - J_y \bullet \delta x \bullet \delta z$$

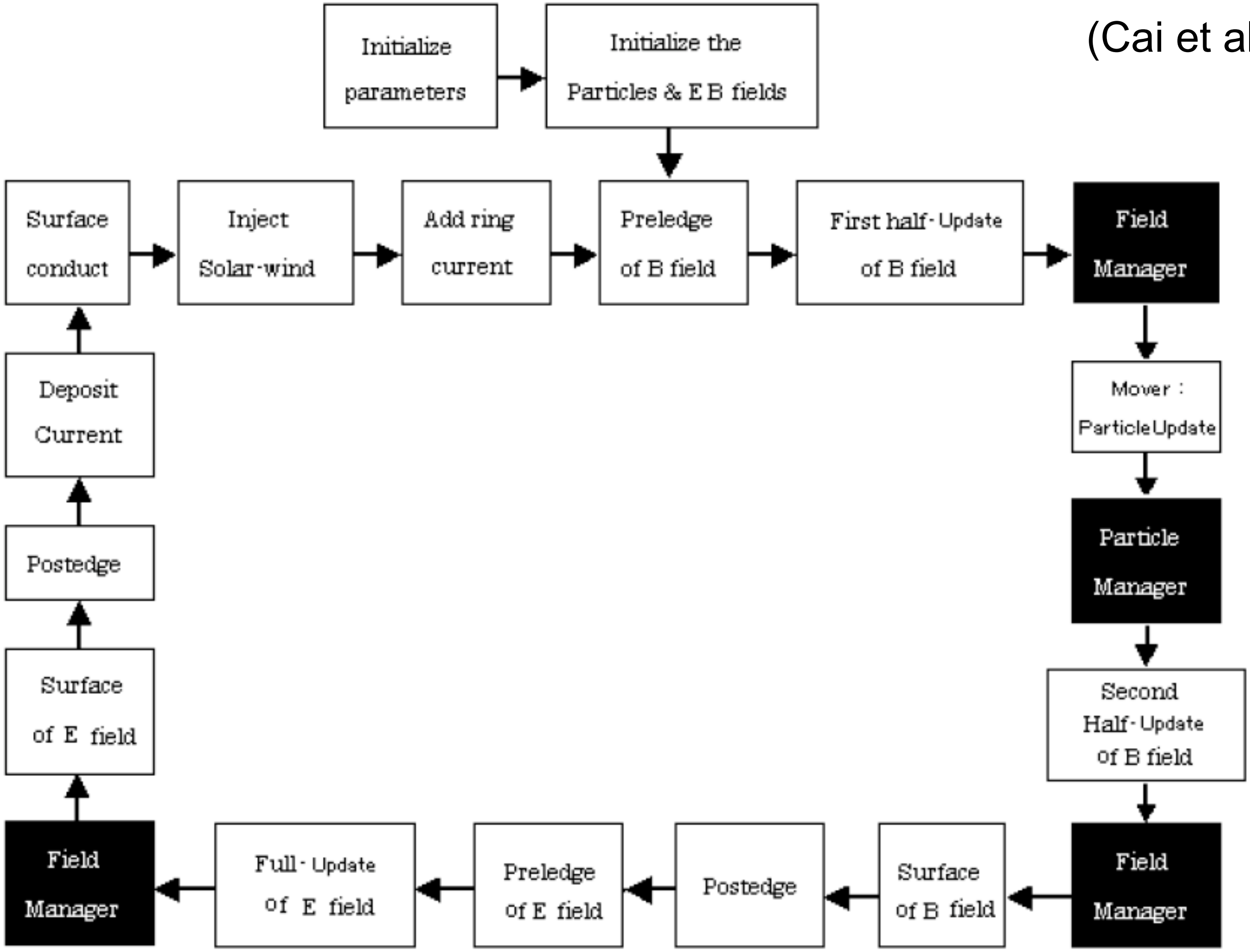


Villasenor and Buneman 1992

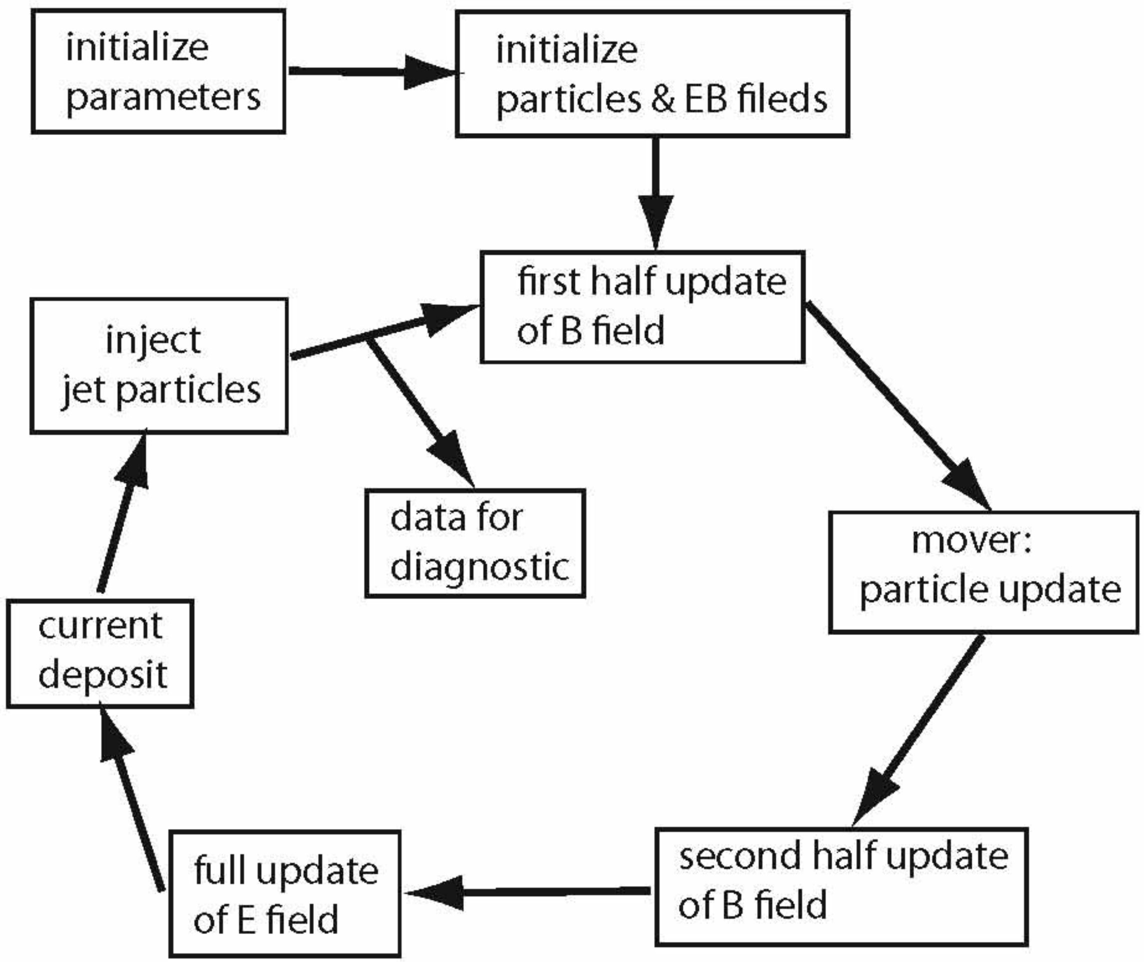
$$cx = 1 - \delta x$$
$$cy = 1 - \delta y$$
$$cz = 1 - \delta z$$

Schematic computational cycle

(Cai et al. 2006)



Time evolution of RPIC code



Code development

Combine these components

Set initial conditions for each problem you would like to investigate

Apply MPI for speed-up

Develop graphics using NCARGraphic, AVSExpress, IDL, etc

Analyze simulation results and compare with theory and other simulation results

Prepare report

Relativistic MHD Simulations of Relativistic Jets

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1. Development of 3D GRMHD Code "RAISHIN"

This section is only showing the differences between RPIC and RMHD (not covered in this course)

Mizuno et al. 2006a, Astro-ph/0609004

Numerical Approach to Relativistic MHD

- RHD: reviews Marti & Muller (2003) and Fonts (2003)

- SRMHD: many authors
  - Application: relativistic Riemann problems, relativistic jet propagation, jet stability, pulsar wind nebule, etc.
- GRMHD
  - Fixed spacetime (Koide, Shibata & Kudoh 1998; De Villiers & Hawley 2003; Gammie, McKinney & Toth 2003; Komissarov 2004; Anton et al. 2005; Annios, Fragile & Salmonson 2005; Del Zanna et al. 2007)
    - Application: The structure of accretion flows onto black hole and/or formation of jets, BZ process near rotating black hole, the formation of GRB jets in collapsars etc.
  - Dynamical spacetime (Duez et al. 2005; Shibata & Sekiguchi 2005; Anderson et al. 2006; Giacomazzo & Rezzolla 2007)

Propose to Make a New GRMHD Code

- The Koide's GRMHD Code (Koide, Shibata & Kudoh 1999; Koide 2003) has been applied to many high-energy astrophysical phenomena and showed pioneering results.
- However, the code can not perform calculation in highly relativistic (γ>5) or highly magnetized regimes.
- The critical problem of the Koide's GRMHD code is the schemes can not guarantee to maintain divergence free magnetic field.
- In order to improve these numerical difficulties, we have developed a new 3D GRMHD code RAISHIN (RelAtIviStic magnetoHydrodynamc sImulatioN, RAISHIN is the Japanese ancient god of lightning).

4D General Relativistic MHD Equation

- General relativistic equation of conservation laws and Maxwell equations:

$$\nabla_n(rU^n) = 0$$ (conservation law of particle-number)

$$\nabla_n T^{mn} = 0$$ (conservation law of energy-momentum)

$$\partial_m F_{nl} + \partial_n F_{lm} + \partial_l F_{mn} = 0$$

$$\nabla_m F^{mn} = -J^n$$

(Maxwell equations)

- Ideal MHD condition: $F_{nm}U^n = 0$
- metric : $ds^2 = g_{mn} dx^m dx^n$
- Equation of state : $p=(G-1)\, u$

r: rest-mass density. p : proper gas pressure. u: internal energy. c: speed of light.
h : specific enthalpy, $h = 1 + u + p/r$.
G: specific heat ratio.
U^{mu} : velocity four vector. J^{mu} : current density four vector.
∇^{mn} : covariant derivative. g_{mn} : 4-metric,
T^{mn} : energy momentum tensor, $T^{mn} = rh\, U^m U^n + p g^{mn} + F^{ms} F^n{}_s - g_{mn} F^{lk} F_{lk}/4$.
F_{mn} : field-strength tensor,

Conservative Form of GRMHD Equations (3+1 Form)

Metric: $ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt)$

a: lapse function,
b^i: shift vector,
g_{ij}: 3-metric

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{\gamma}D) + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}(\sqrt{-g}D\tilde{v}^i) = 0,$$

(Particle number conservation)

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{\gamma}S_i) + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}(\sqrt{-g}T_i^j) = T^{\mu\nu}\left(\frac{\partial g_{\nu i}}{\partial x^\mu} - \Gamma^\sigma_{\nu\mu}g_{\sigma i}\right)$$

(Momentum conservation)

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{\gamma}\tau) + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}[\sqrt{-g}(\alpha T^{ti} - D\tilde{v}^i)] = \alpha\left(T^{\mu t}\frac{\partial \ln\alpha}{\partial x^\mu} - T^{\mu\nu}\Gamma^t_{\nu\mu}\right)$$

(Energy conservation)

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial t}(\sqrt{\gamma}B^i) + \frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^i}[\sqrt{-g}(\tilde{v}^j B^i - \tilde{v}^i B^j)] = 0.$$

(Induction equation)

U (conserved variables) F^i (numerical flux) S (source term)

$$D = \gamma\rho.$$

$$S_i = \alpha T_i^t = (\rho h + b^2)\gamma^2 v_j - \alpha b^0 b_j$$

$$\tau = \alpha^2 T^{tt} - D = (\rho h + b^2)\gamma^2 - (p + b^2/2) - \alpha^2(b^t)^2 - D.$$

$$\tilde{v}^i = v^i - \beta/\alpha$$

$\sqrt{-g}$: determinant of 4-metric
$\sqrt{g}$: determinant of 3-metric

Detail of derivation of GRMHD equations Anton et al. (2005) etc.

New 3D GRMHD Code "RAISHIN"

Mizuno et al. (2006)

- RAISHIN utilizes conservative, high-resolution shock capturing schemes (Godunov-type scheme) to solve the 3D general relativistic MHD equations *(metric is static)*

- Ability of RAISHIN code
  - Multi-dimension (1D, 2D, 3D)
  - Special (Minkowski spcetime) and General relativity (static metric; Schwarzschild or Kerr spacetime)
  - Different coordinates (RMHD: Cartesian, Cylindrical, Spherical and GRMHD: Boyer-Lindquist of non-rotating or rotating BH)
  - Use several numerical methods to solving each problem
  - Maintain divergence-free magnetic field by numerically
  - Use constant Gamma-law or variable equation of states
  - Parallelized by Open MP

Detailed Features of the Numerical Schemes

Mizuno et al. 2006a, astro-ph/0609004

- RAISHIN utilizes conservative, high-resolution shock capturing schemes (Godunov-type scheme) to solve the 3D GRMHD equations *(metric is static)*

* *Reconstruction:* PLM (Minmod & MC slope-limiter function), convex ENO, PPM

* *Riemann solver:* HLL, HLLC approximate Riemann solver

* *Constrained Transport:* Flux interpolated constrained transport scheme

* *Time evolution:* Multi-step Runge-Kutta method (2nd & 3rd-order)

* *Recovery step:* Koide 2 variable method, Noble 2 variable method, Mignore-McKinney 1 variable method

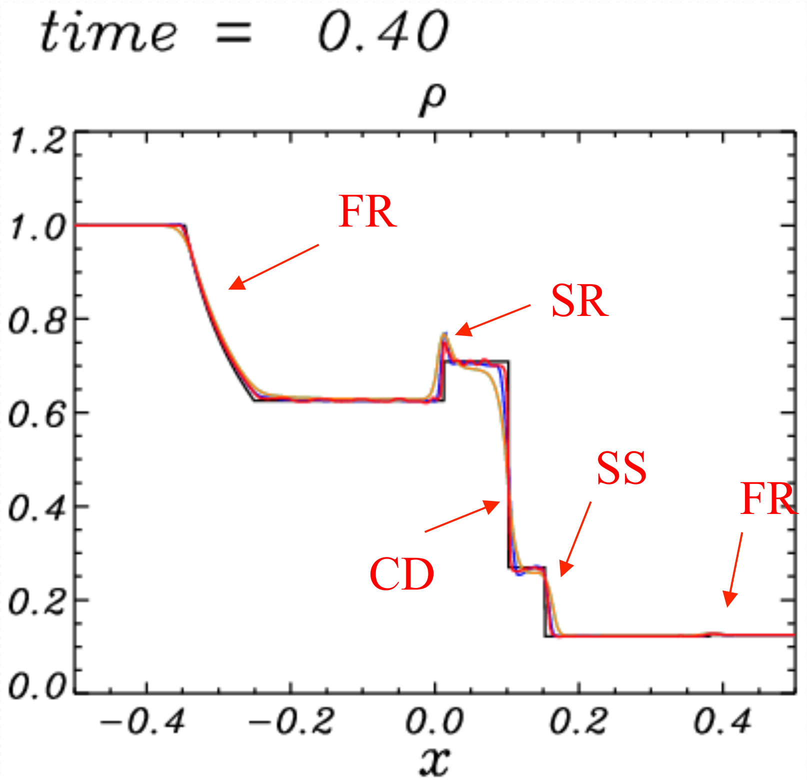
Relativistic MHD Shock-Tube Tests

Exact solution: Giacomazzo & Rezzolla (2006)

Test Type		ρ	p	v^x	v^y	v^z	B^x	B^y	B^z
Komissarov: Shock Tube Test1	$\Gamma = 4/3$								
	left state	1.0	1000.0	0.0	0.0	0.0	1.0	0.0	0.0
	right state	0.1	1.0	0.0	0.0	0.0	1.0	0.0	0.0
Komissarov: Collision Test	$\Gamma = 4/3$								
	left state	1.0	1.0	$5/\sqrt{26}$	0.0	0.0	10.0	10.0	0.0
	right state	1.0	1.0	$-5/\sqrt{26}$	0.0	0.0	10.0	-10.0	0.0
Barsara Test1 (Brio & Wo)	$\Gamma = 2$								
	left state	1.000	1.0	0.0	0.0	0.0	0.5	1.0	0.0
	right state	0.125	0.1	0.0	0.0	0.0	0.5	-1.0	0.0
Barsara Test2	$\Gamma = 5/3$								
	left state	1.0	30.0	0.0	0.0	0.0	5.0	6.0	6.0
	right state	1.0	1.0	0.0	0.0	0.0	5.0	0.7	0.7
Barsara Test3	$\Gamma = 5/3$								
	left state	1.0	1000.0	0.0	0.0	0.0	10.0	7.0	7.0
	right state	1.0	0.1	0.0	0.0	0.0	10.0	0.7	0.7
Barsara Test4	$\Gamma = 5/3$								
	left state	1.0	0.1	0.999	0.0	0.0	10.0	7.0	7.0
	right state	1.0	0.1	-0.999	0.0	0.0	10.0	-7.0	-7.0
Barsara Test5	$\Gamma = 5/3$								
	left state	1.08	0.95	0.40	0.3	0.2	2.0	0.3	0.3
	right state	1.00	1.0	-0.45	-0.2	0.2	2.0	-0.7	0.5
Generic Alfvén Test	$\Gamma = 5/3$								
	left state	1.0	5.0	0.0	0.3	0.4	1.0	6.0	2.0
	right state	0.9	5.3	0.0	0.0	0.0	1.0	5.0	2.0

Relativistic MHD Shock-Tube Tests

Balsara Test1 (Balsara 2001)



Black: exact solution, Blue: MC-limiter, Light blue: minmod-limiter, Orange: CENO, red: PPM

400 computational zones

- The results show good agreement of the exact solution calculated by Giacommazo & Rezzolla (2006).
- Minmod slope-limiter and CENO reconstructions are more diffusive than the MC slope-limiter and PPM reconstructions.
- Although MC slope limiter and PPM reconstructions can resolve the discontinuities sharply, some small oscillations are seen at the discontinuities.

Relativistic MHD Shock-Tube Tests

Test	KO	MC	Min	CENO	PPM	
Komissarov: Shock Tube Test1	△	○	○	○	○	(large P)
Komissarov: Collision Test	×	○	○	○	○	(large g)
Balsara Test1(Brio & Wu)	○	○	○	○	○	
Balsara Test2	×	○	○	○	○	(large P & B)
Balsara Test3	×	○	○	○	○	(large g)
Balsara Test4	×	○	○	○	○	(large P & B)
Balsara Test5	○	○	○	○	○	
Generic Alfven Test	○	○	○	○	○	

Applicability of Hydrodynamic Approximation

- To apply hydrodynamic approximation, we need the condition:
  - Spatial scale >> mean free path
  - Time scale >> collision time
- These are not necessarily satisfied in many astrophysical plasmas
  - E.g., solar corona, galactic halo, cluster of galaxies etc.
- But in plasmas with magnetic field, the effective mean free path is given by the ion Larmor radius.
- Hence if the size of phenomenon is much larger than the ion Larmor radius, hydrodynamic approximation can be used.

Applicability of MHD Approximation

- MHD describe macroscopic behavior of plasmas if
  - Spatial scale >> ion Larmor radius
  - Time scale >> ion Larmor period

- But MHD can not treat
  - Particle acceleration
  - Origin of resistivity
  - Electromagnetic waves

Basics of Numerical RMHD Code

Non-conservative form (De Villier & Hawley (2003), Anninos et al.(2005))

$$\boldsymbol{B}(\boldsymbol{P})\frac{\partial \boldsymbol{P}}{\partial t}+\boldsymbol{C}(\boldsymbol{P})\frac{\partial \boldsymbol{P}}{\partial x}=0$$

U=U(P) - conserved variables,
P – primitive variables
F- numerical flux of U

where

$$B_{km}=\frac{\partial U_{(k)}}{\partial P_{(m)}}, \qquad C_{km}=\frac{\partial F_{(k)}}{\partial P_{(m)}},$$

Merit:

• they solve the internal energy equation rather than energy equation. → advantage in regions where the internal energy small compared to total energy (such as supersonic flow)

• Recover of primitive variables are fairly straightforward

Demerit:

• It can not applied high resolution shock-capturing method and artificial viscosity must be used for handling discontinuities

Basics of Numerical RMHD Code

Conservative form (Koide et al. (1999), Kommisarov (2001), Gammie et al (2003), Anton et al. (2004), Duez et al. (2005), Shibata & Sekiguchi (2005) etc)

System of Conservation Equations

$$\frac{\partial \boldsymbol{U}}{\partial t} + \frac{\partial \boldsymbol{F}^i(\boldsymbol{U})}{\partial x^i} = \boldsymbol{S}(\boldsymbol{U})$$

U=U(P) - conserved variables,
P – primitive variables
F- numerical flux of U,
S - source of U

Merit:

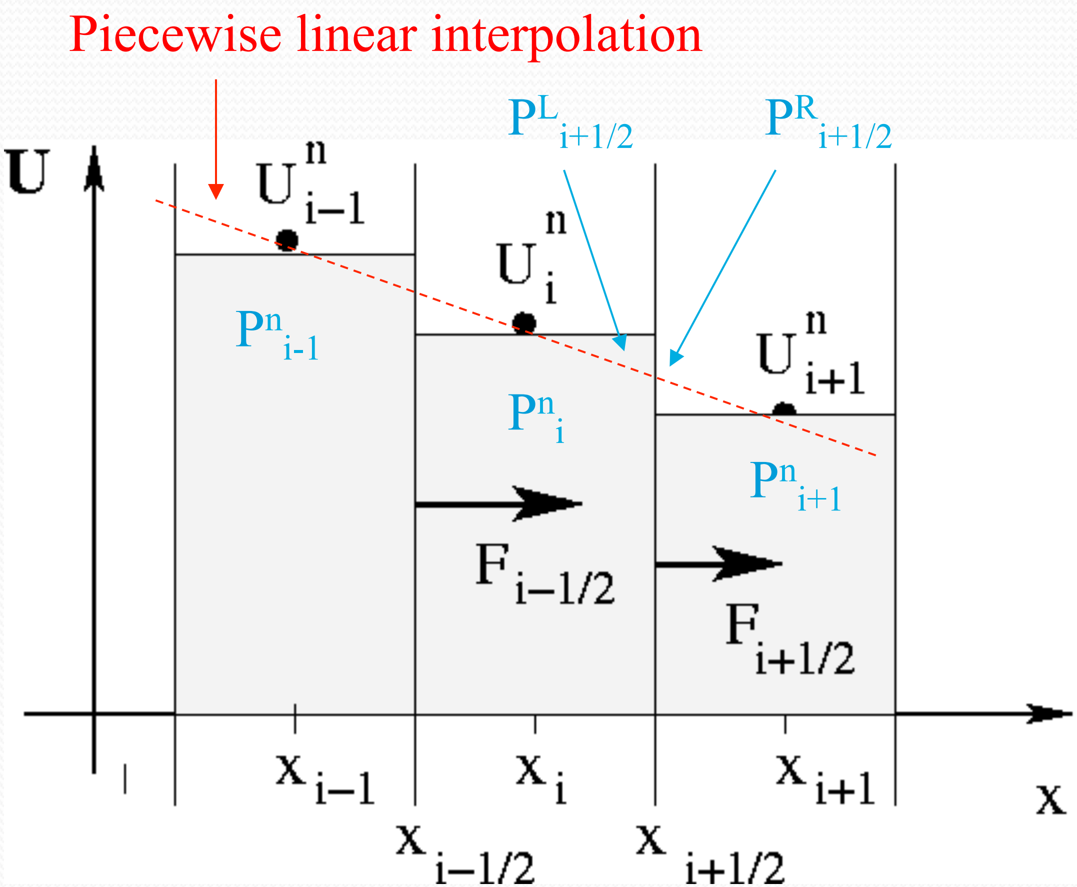
• High resolution shock-capturing method can be applied to GRMHD equations

Demerit:

• These schemes must recover primitive variables P by numerically solving the system of equations after each step (because the schemes evolve conservative variables U)

Reconstruction

Cell-centered variables (P_i)
→ right and left side of Cell-interface variables($P^L_{i+1/2}$, $P^R_{i+1/2}$)



- Minmod and MC Slope-limited Piecewise linear Method
  - 2nd-order 1st -order at local extrema
- Convex CENO (Liu & Osher 1998)
  - 3rd -order, 1st -order at local extrema
- Piecewise Parabolic Method (Marti & Muller 1996)
  - 4th -order, 1st -order at local extrema

Piecewise Linear Method

Reconstructed cell-interface variables

$$a_L = a_i + \nabla a_i \Delta x/2$$

$$a_R = a_{i+1} - \nabla a_{i+1} \Delta x/2$$

Slope-limiter flunction

Minmod function

$$\nabla a = \Delta x^{-1} minmod(a_{i+1} - a_i, a_i - a_{i-1}),$$

$$minmod(a,b) = \begin{cases} 0 & \text{if } ab \leq 0, \\ sign(a) min(|a|, |b|) & \text{otherwise.} \end{cases}$$

Monotonized Centeral (MC) function

$$\nabla a = \Delta x^{-1} MC(a_{i+1} - a_i, a_i - a_{i-1}),$$

$$MC(a,b) = \begin{cases} 0 & \text{if } ab \leq 0, \\ sign(a) min(2|a|, 2|b|, |a+b|/2) & \text{otherwise.} \end{cases}$$

Piecewise Parabolic Method

- 1st Step: interpolation
  - quartic polynomial interpolation determined by the five zone-averaged values.
- 2nd Step: contact steepening
  - slightly modified to produce narrower profiles in the vicinity of a contact discontinuity
- 3rd Step: Flattening
  - near strong shocks the order of the method is reduced locally to avoid spurious postshock oscillations
- 4th Step: Monotonization
  - monotonize by the interpolation parabola between smooth and shock region

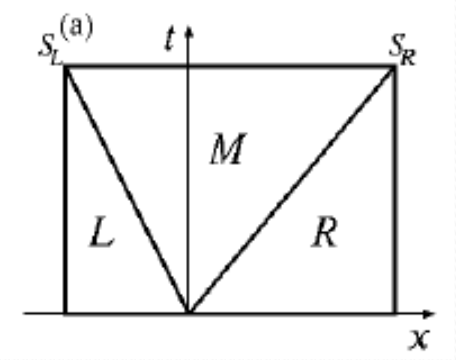
HLL Approximate Riemann Solver

- Calculate numerical flux at cell-inteface from reconstructed cell-interface variables based on Riemann problem
- We use HLL approximate Riemann solver
  - Need only the maximum left- and right- going wave speeds (in MHD case, fast mode)

HLL flux

$$\mathbf{F}_{\mathrm{HLL}} = \frac{S_R \mathbf{U}_L - S_L \mathbf{U}_R + S_R S_L \left(\mathbf{U}_R - \mathbf{U}_L \right)}{S_R - S_L},$$

$F_R=F(P_R)$, $F_L=F(P_L)$; $U_R=U(P_R)$, $U_L=U(P_L)$
$S_R=\max(0,c_{+R}, c_{+L})$; $S_L=\max(0,c_{-R},c_{-L})$

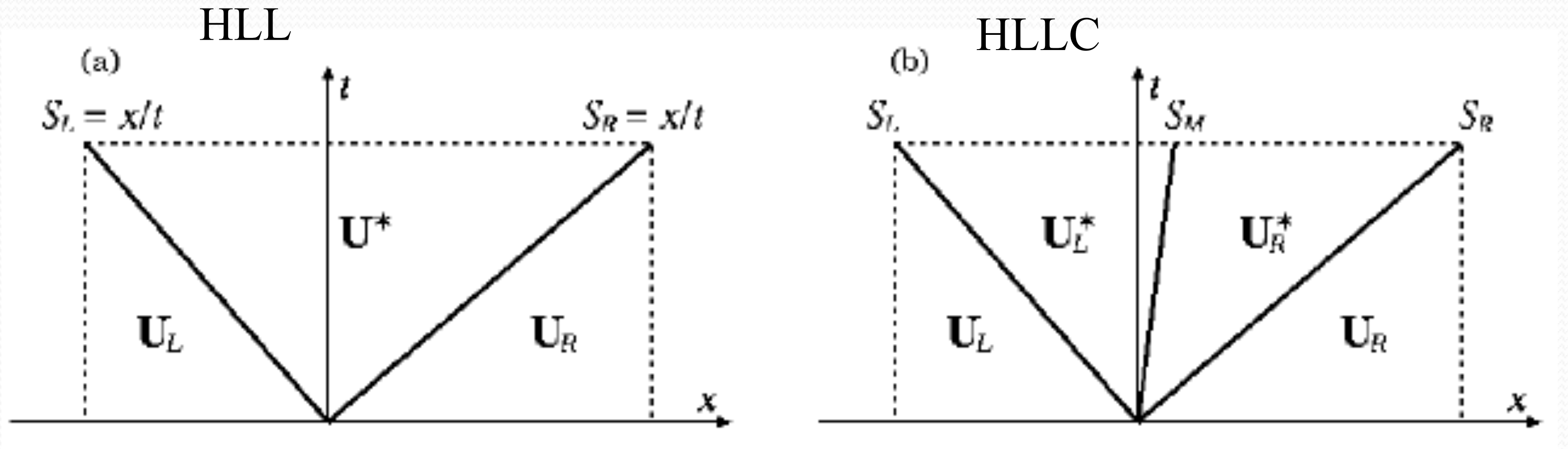


If $S_L > 0$ $\quad F_{HLL}=F_L$
$S_L < 0 < S_R$, $\quad F_{HLL}=F_M$
$S_R < 0$ $\quad F_{HLL}=F_R$

HLLC Approximate Riemann Solver

Mignore & Bodo (2006)

- HLL Approximate Riemann solver: single state in Riemann fan
- HLLC Approximate Riemann solver: two-state in Riemann fan



$$
\boldsymbol{U}(0,t) = \begin{cases} \boldsymbol{U}_L & \text{if} \quad \lambda_L \geqslant 0\,, \\ \boldsymbol{U}_L^* & \text{if} \quad \lambda_L \leqslant 0 \leqslant \lambda^*\,, \\ \boldsymbol{U}_R^* & \text{if} \quad \lambda^* \leqslant 0 \leqslant \lambda_R\,, \\ \boldsymbol{U}_R & \text{if} \quad \lambda_R \leqslant 0\,, \end{cases}
\qquad
\boldsymbol{f} = \begin{cases} \boldsymbol{F}_L & \text{if} \quad \lambda_L \geqslant 0\,, \\ \boldsymbol{F}_L^* & \text{if} \quad \lambda_L \leqslant 0 \leqslant \lambda^*\,, \\ \boldsymbol{F}_R^* & \text{if} \quad \lambda^* \leqslant 0 \leqslant \lambda_R\,, \\ \boldsymbol{F}_R & \text{if} \quad \lambda_R \leqslant 0\,. \end{cases}
$$

Constrained Transport

Differential Equations

$$\frac{1}{c}\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\longrightarrow \quad \frac{\partial (\vec{\nabla} \cdot \vec{B})}{\partial t} = 0$$

- The evolution equation can keep divergence free magnetic field

• If treat the induction equation as all other conservation laws, it can not maintain divergence free magnetic field

→ We need spatial treatment for magnetic field evolution

Constrained transport method

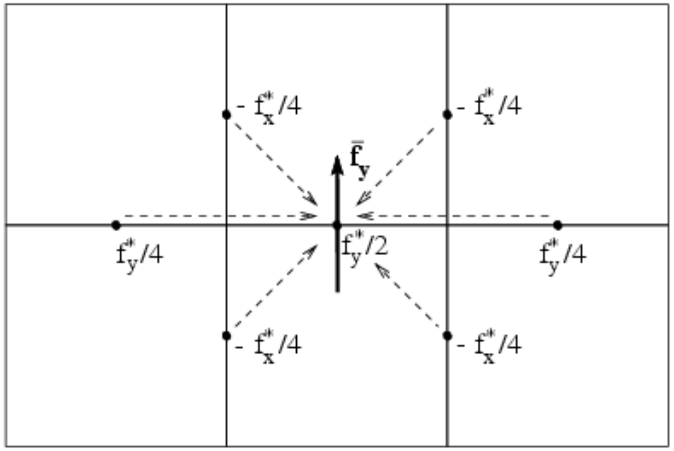
• Evans & Hawley's Constrained Transport (Komissarov (1999,2002,2004), de Villiers & Hawley (2003), Del Zanna et al.(2003), Anton et al.(2005))

• Toth's constrained transport (Gammie et al.(2003), Duez et al.(2005))

• Diffusive cleaning (Annios et al.(2005))

Constrained Transport (Toth 2000)

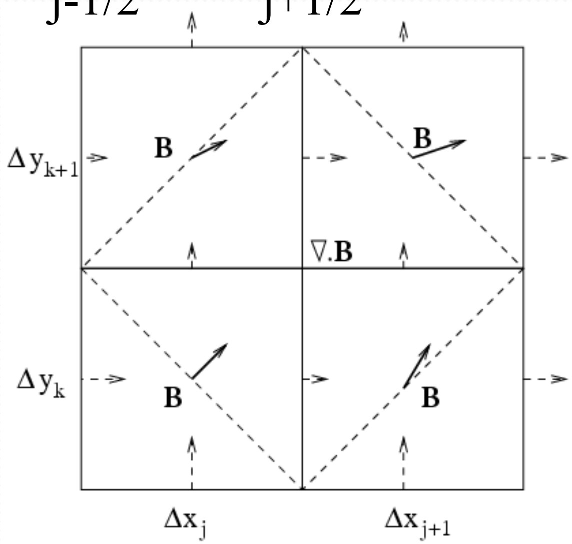
2D case



k+1/2

k-1/2

j-1/2 j+1/2



Use the "modified flux" f that is such a linear combination of normal fluxes at neighbouring interfaces that the "corner-centred" numerical representation of div*B* is kept invariant during integration.

$$B^{x,n+1}_{j,k} = B^{x,n}_{j,k} - \Delta t \frac{\bar{f}^y_{j,k+1/2} - \bar{f}^y_{j,k-1/2}}{\Delta y}$$

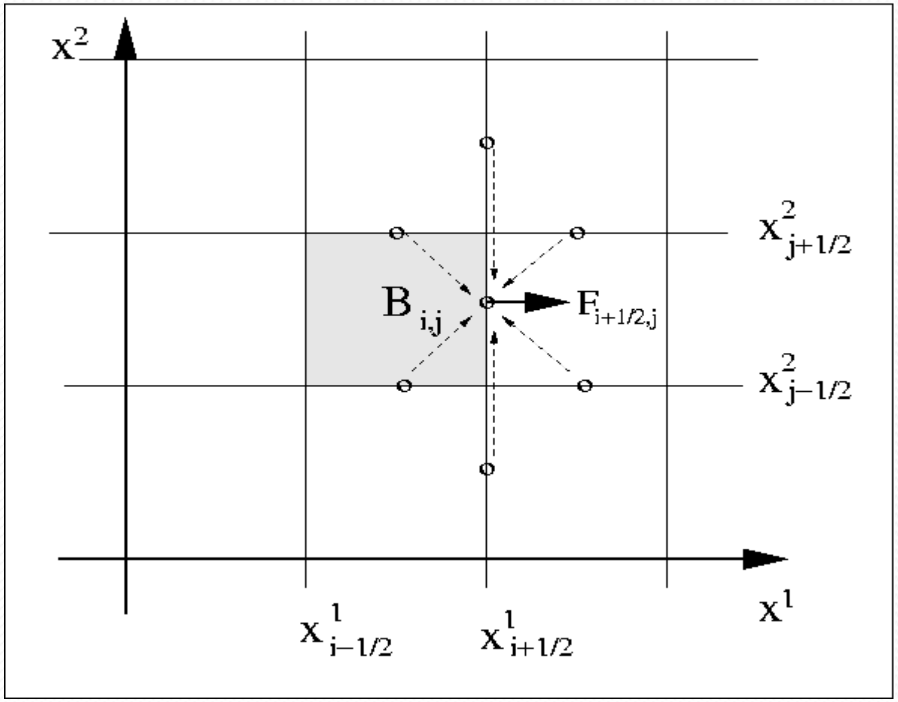
$$B^{y,n+1}_{j,k} = B^{y,n}_{j,k} - \Delta t \frac{\bar{f}^x_{j+1/2,k} - \bar{f}^x_{j-1/2,k}}{\Delta x}$$

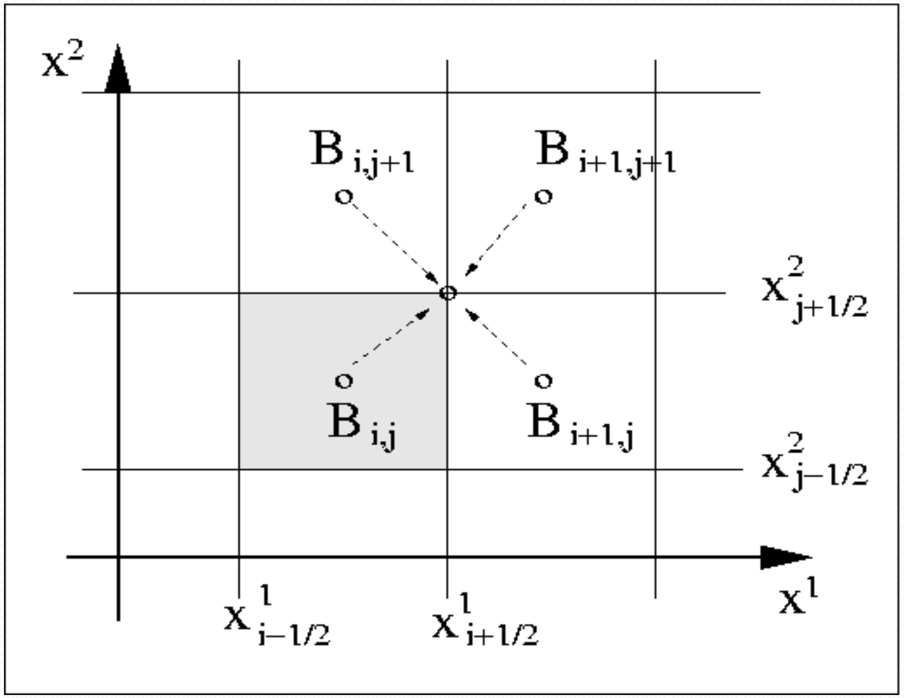
$$\bar{f}^x_{j+1/2,k} = \frac{1}{8}\big(2\, f^{x,*}_{j+1/2,k} + f^{x,*}_{j+1/2,k+1} + f^{x,*}_{j+1/2,k-1} - f^{y,*}_{j,k+1/2} - f^{y,*}_{j+1,k+1/2} - f^{y,*}_{j,k-1/2} - f^{y,*}_{j+1,k-1/2} \big)$$

$$\bar{f}^y_{j,k+1/2} = \frac{1}{8}\big(2\, f^{y,*}_{j,k+1/2} + f^{y,*}_{j+1,k+1/2} + f^{y,*}_{j-1,k+1/2} - f^{x,*}_{j+1/2,k} - f^{x,*}_{j+1/2,k+1} - f^{x,*}_{j-1/2,k} - f^{x,*}_{j-1/2,k+1} \big)$$

$$(\nabla \cdot \boldsymbol{B})_{j,k} = \frac{B^x_{j+1,k} - B^x_{j-1,k}}{2\Delta x} + \frac{B^y_{j,k+1} - B^y_{j,k-1}}{2\Delta y}$$

Constrained Transport (Toth 2000)

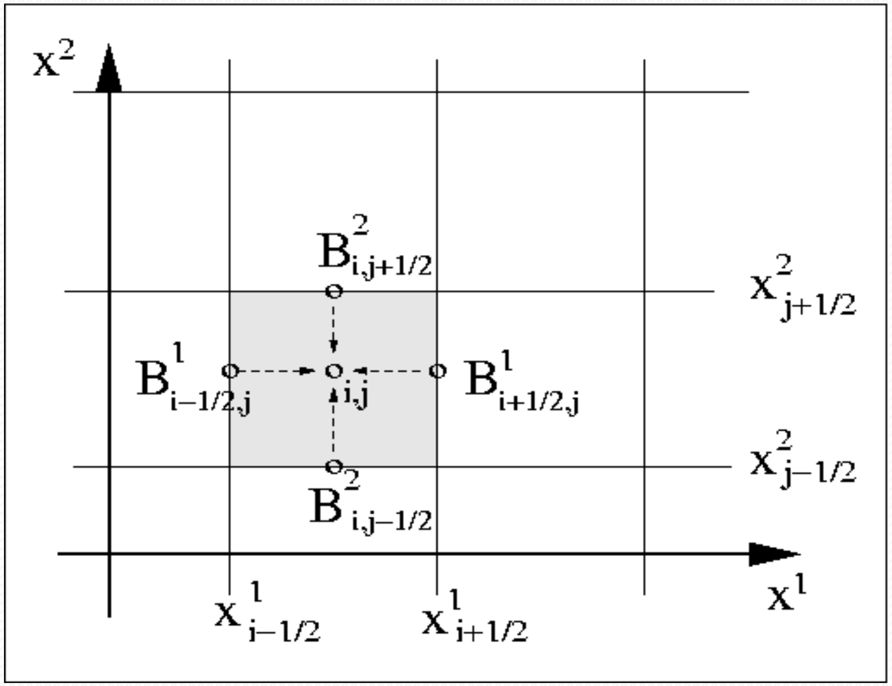




Evans & Hawley's Constrained Transport

Use staggered grid (with B defined at the cell-interfaces) and evolve magnetic fluxes through the cell interfaces using the electric field evaluated at the cell-edges.

This keeps the following "cell-centred" numerical representation of div$B$ invariant



$$\vec{\nabla}\cdot\vec{B}_{i,j} = \frac{(B^1_{i+1/2,j} - B^1_{i-1/2,j})}{\Delta x^1} + \frac{(B^2_{i,j+1/2} - B^2_{i,j-1/2})}{\Delta x^2}$$

Time evolution

System of Conservation Equations

$$\partial_t \mathbf{U} = -\nabla \cdot \mathbf{F} + \mathbf{S} \equiv \mathbf{L}(\mathbf{U}).$$

We use multistep Runge-Kutta method for time advance of conservation equations (RK2: 2nd-order, RK3: 3rd-order in time)

RK2, RK3: first step $\mathrm{U}^{(1)} = \mathrm{U}^n + \Delta t L(\mathrm{U}^n).$

RK2: second step (a=2, b=1)

$$\mathrm{U}^{n+1} = \frac{1}{\alpha}[\beta \mathrm{U}^n + \mathrm{U}^{(1)} + \Delta t L(\mathrm{U}^{(1)})],$$

RK3: second and third step (a=4, b=3)

$$\mathrm{U}^{(2)} = \frac{1}{\alpha}[\beta \mathrm{U}^n + \mathrm{U}^{(1)} + \Delta t L(\mathrm{U}^{(1)})],$$

$$\mathrm{U}^{n+1} = \frac{1}{\beta}[\beta \mathrm{U}^n + 2\mathrm{U}^{(2)} + 2\Delta t L(\mathrm{U}^{(2)})],$$

Recovery step

- The GRMHD code require a calculation of primitive variables from conservative variables.
- The forward transformation (primitive → conserved) has a close-form solution, but the inverse transformation (conserved → primitive) requires the solution of a set of five nonlinear equations

$$D = \gamma\rho.$$
$$S_i = \alpha T_i^t = (\rho h + b^2)\gamma^2 v_j - \alpha b^0 b_j$$
$$\tau = \alpha^2 T^{tt} - D = (\rho h + b^2)\gamma^2 - (p + b^2/2) - \alpha^2 (b^t)^2 - D.$$

Method

- Koide's 2D method (Koide, Shibata & Kudoh 1999)
- Noble's 2D method (Noble et al. 2005)

Recovery step (Koide's 2D method)

Conserved quantities(D,**P**,e,**B**) → primitive variables (r,p,**v**,**B**)
2-variable Newton-Raphson iteration method

$$x \equiv \gamma - 1 \text{ and } y \equiv \gamma(\hat{\mathbf{v}} \cdot \hat{\mathbf{B}})/c^2$$

$$x(x+2)\left[\Gamma R x^2 + (2\Gamma R - d)x + \Gamma R - d + u + \frac{\Gamma}{2}y^2\right]^2 = (\Gamma x^2 + 2\Gamma x + 1)^2 [f^2(x+1)^2 + 2\sigma y + 2\sigma x y + b^2 y^2], \quad (80)$$

$$\left[\Gamma(R - b^2)x^2 + (2\Gamma R - 2\Gamma b^2 - d)x + \Gamma R - d + u - b^2 + \frac{\Gamma}{2}y^2\right] y = \sigma(x+1)(\Gamma x^2 + 2\Gamma x + 1), \quad (81)$$

$R = D + \epsilon/c^2$, $d = (\Gamma - 1)D$, $u = (1 - \Gamma/2)\hat{B}^2/c^2$, $f = \hat{P}/c$, $b = \hat{B}/c$, and $\sigma = \hat{\mathbf{B}} \cdot \hat{\mathbf{P}}/c^2$

$$\gamma = 1 + x,$$

$$p = \frac{(\Gamma - 1)[\epsilon - xDc^2 - (2 - 1/\gamma^2)B^2/2 + (cy/\gamma)^2/2]}{[\Gamma x(x+2) + 1]}$$

$$\hat{\mathbf{v}} = \frac{\mathbf{P} + (y/\gamma)\mathbf{B}}{D + \{\epsilon + p + B^2/2\gamma^2 + (cy/\gamma)^2/2\}/c^2}.$$

Noble's 2D method

- Conserved quantities(D,**S**,t,**B**) → primitive variables (**r**,p,**v**,**B**)
- Solve two-algebraic equations for two independent variables $W \equiv h g^2$ and v^2 by using 2-variable Newton-Raphson iteration method

$$\mathbf{S} = (W + B^2)\mathbf{v} - \frac{(B^2 + 2W)(\mathbf{S}\cdot\mathrm{B})^2}{W^2},$$

$$\tau = \frac{B^2}{2}(1 + v^2) + \frac{\mathbf{S}\cdot\mathrm{B}}{2W} + W - D - p(u, \rho).$$

W and v^2 →primitive variables r p, and v

Summary (cont.)

- We have investigated stability properties of magnetized spine-sheath relativistic jets by the theoretical work and 3D RMHD simulations.
- The most important result is that destructive KH modes can be stabilized even when the jet Lorentz factor exceeds the Alfven Lorentz factor. Even in the absence of stabilization, spatial growth of destructive KH modes can be reduced by the presence of magnetically sheath flow (~0.5c) around a relativistic jet spine (>0.9c)

Summary (cont.)

- We performed relativistic magnetohydrodynamic simulations of the hydrodynamic boosting mechanism for relativistic jets explored by Aloy & Rezzolla (2006) using the RAISHIN code.
- We find that magnetic fields can lead to more efficient acceleration of the jet, in comparison to the pure-hydrodynamic case.
- The presence and relative orientation of a magnetic field in relativistic jets can significant modify the hydrodynamic boost mechanism studied by Aloy & Rezzolla (2006).

Future Work

- Code Development
  - Kerr-Schild Coordinates: long-term simulation in GRMHD
  - Resistivity: extension to non-ideal MHD; (e.g., Watanabe & Yokoyama 2007; Komissarov 2007)
  - Couple with radiation transfer: link to observation
- Research of Jet Formation and Propagation
  - Dependence on Magnetic field structure, BH spin parameter, disk structure and perturbation etc.
- Research of Jet Stability
  - Dependence on EoS
  - Current-Driven instability

- Apply to astrophysical phenomena in which relativistic outflows and/or GR essential (AGNs, microquasars, neutron stars, and GRBs etc.)

Current Driven instability

- From MHD model of jet formation, jets launch with twisted magnetic field from the black hole magnetosphere
- In such configuration, the most dangerous instability is current driven (CD) kink mode
- This instability excites large-scale helical motions and it disrupts the system
- Relativistic CD kink instability was studied only in linear approximation (e.g., Lyubarskii 1999)
- The investigation of nonlinear behavior of CD kink instability is important for the stability and structure of relativistic jets
    - Need 3D relativistic MHD simulation

Plan of Operation

- Study numerically non-linear evolution of different kink-unstable configurations
    - When the instability saturates, disrupts, and final state
- Coupling KH and CD instabilities
    - What is fastest growth modes in magnetized relativistic jets?
    - Take into account rotation, velocity shear, and expansion
    - Take into account several jet configuration
        - Top-hat jets (Mizuno et al. 2007)
        - Spine-sheath jet configuration (Mizuno et al. 2007)
        - Boosted boundary layer jets (Mizuno et al. 2008) etc

Future Long-Term Plans

- I apply to astrophysical phenomena in which relativistic outflows and/or GR essential (AGNs, microquasars, neutron stars, and GRBs etc.)
- I apply my future works on jet study to results obtained at very high-energy particle observations by HESS, MAGIC and VERITAS will require to link large scale processes to the microphysics of the plasma, efficient particle acceleration processes and radiation mechanisms