29

Delta Functions

This chapter introduces mathematical entities commonly known as "delta functions". As we will see, a delta function is not really a function, at least not in the classical sense. Nonetheless, with a modicum of care, they can be treated like functions. More importantly, they are useful. They are valuable in modeling both "strong forces of brief duration" (such as the force of a baseball bat striking a ball) and "point masses". Moreover, their mathematical properties turn out to be remarkable, making them some of the simplest "functions" to deal with. After a little practice, you may rank them with the constant functions as some of your favorite functions to deal with. Indeed, the basic delta function has a relation with the constant function $f \equiv 1$ that will allow us to expand our discussion of Duhamel's principle.

29.1 Visualizing Delta Functions

What is commonly called "the delta function"— traditionally denoted by $\delta(t)$ — is best thought of as shorthand for a particular limiting process. One standard way to visualize $\delta(t)$ is as the limit

$$\delta(t) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \operatorname{rect}_{(0,\epsilon)}(t) \quad .$$

Look at the function we are taking the limit of,

$$\frac{1}{\epsilon} \operatorname{rect}_{(0,\epsilon)}(t) = \begin{cases} 0 & \text{if} \quad t < 0 \\ \frac{1}{\epsilon} & \text{if} \quad 0 < t < \epsilon \\ 0 & \text{if} \quad \epsilon < t \end{cases} \quad .$$

Graphs of this for various small positive values of ϵ have been sketched in figure 29.1a. Notice that, for each ϵ, the nonzero part of the graph forms a rectangle of width ϵ and height $1/\epsilon$. Consequently, the area of this rectangle is $\epsilon \cdot 1/\epsilon = 1$. Keep in mind that we are taking a limit as $\epsilon \to 0$; so ϵ is "small", which means that this rectangle is very narrow and very high, starts at $t = 0$, and is of unit area. As we let $\epsilon \to 0$ this "very narrow and very high rectangle starting at $t = 0$ and of unit area" becomes an "infinitesimally narrow and infinitely high 'spike' at $t = 0$ enclosing unit area".

Strictly speaking, there is no function whose graph is such a spike. The closest we can come is the function that is zero everywhere except at $t = 0$, where we pretend the function is infinite.

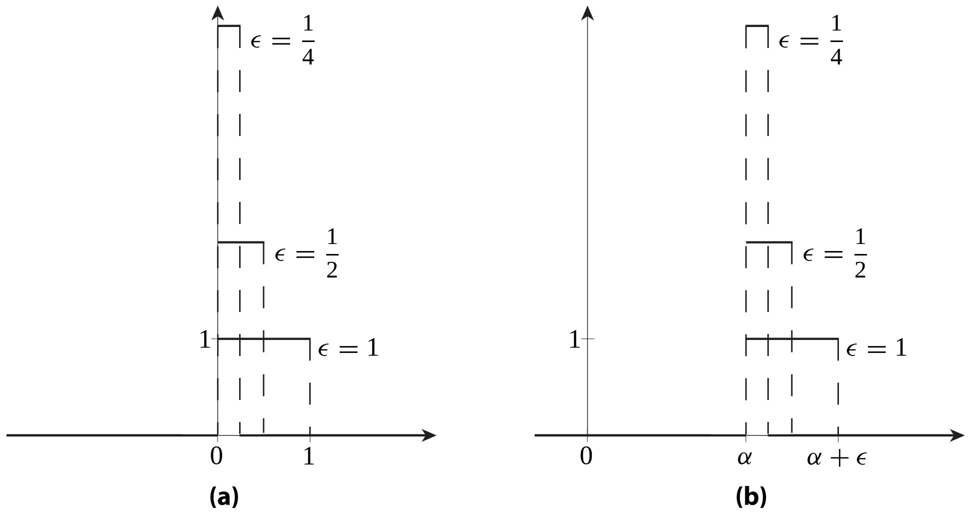

Figure 29.1: The graphs of **(a)** $\frac{1}{\epsilon}\,\mathrm{rect}_{(0,\epsilon)}(t)$ and **(b)** $\frac{1}{\epsilon}\,\mathrm{rect}_{(0,\epsilon)}(t-\alpha)$ (equivalently $\frac{1}{\epsilon}\,\mathrm{rect}_{(\alpha,\alpha+\epsilon)}(t)$) for $\epsilon = 1$, $\epsilon = 1/2$ and $\epsilon = 1/4$.

This sort of gives the infinite spike, but the “area enclosed” is not at all well defined. Still, the visualization of the delta function as “an infinite spike enclosing unit area” is useful, just as it is useful in physics to sometimes pretend that we can have a “point mass” (an infinitesimally small particle of nonzero mass).

The above describes “the” delta function. For any real number α , the delta function at α , $\delta_\alpha(t)$, is simply “the” delta function shifted by α ,

$$\delta_\alpha(t) \;=\; \delta(t-\alpha) \;=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,\mathrm{rect}_{(0,\epsilon)}(t-\alpha) \quad .$$

With a little thought (or a glance at figure 29.1b), you can see that the nonzero part of $\mathrm{rect}_{(0,\epsilon)}(t-\alpha)$ starts at $t = \alpha$ and ends at $t = \alpha + \epsilon$, and that

$$\delta_\alpha(t) \;=\; \delta(t-\alpha) \;=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\,\mathrm{rect}_{(\alpha,\alpha+\epsilon)}(t) \quad .$$

Do notice that

$$\delta_0(t) \;=\; \delta(t-0) \;=\; \delta(t) \quad .$$

This means that anything we derive concerning δ_α also holds for δ — just let $\alpha = 0$.

29.2 Delta Functions in Modeling

There are at least two general situations in which delta functions naturally arise when we attempt to describe “real world” phenomenon. One is when we attempt to model brief but strong forces. The other is when we imagine physical objects as “point masses”. In both, the delta functions appear in integrals. This will be significant, and is well worth observing in the models described below.

Since it will be especially useful to see how delta functions model “strong forces of brief duration”, we’ll start with that.

Strong Forces of Brief Duration

Consider the motion of some object under a force that varies with time. We will assume the object's motion is one dimensional (say, along some X–axis), and, as usual, we'll let

$$m = \text{the mass (in kilograms) of the object (assumed constant)} \quad ,$$

$$t = \text{time (in seconds)} \quad ,$$

$$v(t) = \text{velocity (in meters/second) of the object at time } t \quad ,$$

and

$$F(t) = \text{force (in kilogram·meters/second}^2\text{) acting on the object at time } t \quad .$$

(Of course, any units for time, mass and distance can be used, as long as we are consistent.)

Newton's famous law of force gives

$$F(t) = m \times \text{acceleration} = m\frac{dv}{dt} \quad .$$

If we integrate this over a interval (t_0, t_1), we get

$$\int_{t_0}^{t_1} F(t)\,dt = \int_{t_0}^{t_1} m\frac{dv}{dt}\,dt = m\,[v(t_1) - v(t_0)] \quad .$$

So the integral of $F(t)$ from $t = t_0$ to $t = t_1$ is the object's mass times the change in the object's velocity over that period of time. This integral of F is sometimes called the *impulse* of the force over the interval (t_0, t_1) (with the total impulse being this integral with $t_0 = -\infty$ and $t_1 = \infty$).[1] Note that, following our above conventions for units, the units associated with the impulse is kilogram·meters/second.

Let's now restrict ourselves to situations in which the force is zero except for a very short period of time, during which the force is strong enough to significantly change the velocity of the object under question. We may be talking about the force of a baseball bat striking a baseball, or the force of some propellent (gunpowder, compressed air, etc.) forcing a bullet out of a gun, or even the force of a baseball in flight striking some unfortunate bat that fluttered out over the field to catch flies. For concreteness, let's pretend we are studying the force of a baseball bat hitting a baseball at "time $t = \alpha$". If we are very precise, we may let $t = \alpha$ be the first instant the bat comes into contact with the ball, and ϵ the length of time the bat remains in contact with the ball. Considering the situation this length of time, ϵ, must be positive, but very small.

Before and after the bat touches the ball, this force is zero. So our $F(t)$ must be some function, such as,

$$\frac{1}{\epsilon}\,\text{rect}_{(\alpha,\alpha+\epsilon)}(t) \quad ,$$

that satisfies

$$F(t) = 0 \qquad \text{if} \quad t < \alpha \quad \text{and if} \quad \alpha + \epsilon < t \quad .$$

Thus, if $t_0 < \alpha$ and $\alpha + \epsilon < t_1$, then

$$m\,[v(\alpha) - v(t_0)] = \int_{t_0}^{\alpha} F(t)\,dt = \int_{t_0}^{\alpha} 0\,dt = 0$$

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[1] Students of physics will observe that the impulse is actually equal to the change in the *momentum*, mv.

and

$$m\,[v(t_1) \;-\; v(\alpha+\epsilon)] \;=\; \int_{\alpha+\epsilon}^{t_1} F(t)\,dt \;=\; \int_{\alpha+\epsilon}^{t_1} 0\,dt \;=\; 0 \quad .$$

Since t_0 can be any value less than α , and t_1 can be any value greater than $\alpha+\epsilon$, the last two equations tell us that the velocity is one constant v_{before} before the bat hits the ball, and another constant v_{after} afterwards, with

$$v_{\text{before}} \;=\; v(\alpha) \qquad \text{and} \qquad v_{\text{after}} \;=\; v(\alpha+\epsilon) \quad .$$

(We are using v_{before} and v_{after} because the expressions $v(\alpha)$ and $v(\alpha+\epsilon)$ will become problematic when we let $\epsilon \to 0$.)

The precise formula for $F(t)$ while the bat is in contact with the ball is typically both difficult to determine and of little interest. All we usually care about is describing $F(t)$ well enough to get the correct change in the velocity of the ball, $v_{\text{after}} - v_{\text{before}}$. So let us pick

$$F(t) \;=\; \frac{1}{\epsilon}\,\text{rect}_{(\alpha,\alpha+\epsilon)}(t) \quad ,$$

and see what the resulting change of velocity is as t changes from t_0 to t_1 (with $t_0 < \alpha$ and $\alpha+\epsilon < t_1$):

$$\begin{aligned} m\,[v_{\text{after}} - v_{\text{before}}] \;&=\; m\,[v(t_1) - v(t_0)] \\ &=\; \int_{t_0}^{t_1} F(t)\,dt \\ &=\; \int_{t_0}^{t_1} \frac{1}{\epsilon}\,\text{rect}_{(\alpha,\alpha+\epsilon)}(t)\,dt \;=\; \frac{1}{\epsilon}\int_{\alpha}^{\alpha+\epsilon} dt \;=\; 1 \quad . \end{aligned}$$

In other words,

$$F(t) \;=\; \frac{1}{\epsilon}\,\text{rect}_{(\alpha,\alpha+\epsilon)}(t)$$

describes a force of duration ϵ starting at $t=\alpha$ with a total impulse of 1 . Obviously, if we, instead, wanted a force of duration ϵ starting at $t=\alpha$ with a total impulse of $\mathcal{I}$, we could just multiply the above by $\mathcal{I}$. The corresponding velocity of the ball is then given by

$$v(t) \;=\; \begin{cases} v_{\text{before}} & \text{if} \quad t < \alpha \\ v_{\text{after}} & \text{if} \quad \alpha+\epsilon < t \end{cases} \quad .$$

where

$$m\,[v_{\text{after}} \;-\; v_{\text{before}}] \;=\; \int_{t_0}^{t_1} \mathcal{I}\cdot\frac{1}{\epsilon}\,\text{rect}_{(\alpha,\alpha+\epsilon)}(t)\,dt \;=\; \mathcal{I} \quad .$$

There is just one little complication: determining ϵ , the length of time the bat is in contact with the ball. And, naturally, because this length of time is so close to being zero, we will simplify our computations by letting $\epsilon \to 0$. Thus, for some constant $\mathcal{I}$, we model the force by a *delta function force*

$$F(t) \;=\; \lim_{\epsilon\to 0^+} \mathcal{I}\cdot\frac{1}{\epsilon}\,\text{rect}_{(\alpha,\alpha+\epsilon)}(t) \;=\; \mathcal{I}\,\delta_\alpha(t) \quad .$$

The resulting velocity of the ball $v(t)$ is then given by two constants v_{before} and v_{after} , with

$$v(t) \;=\; \begin{cases} v_{\text{before}} & \text{if} \quad t < \alpha \\ v_{\text{after}} & \text{if} \quad \alpha < t \end{cases} \quad .$$

where

$$m\,[v_{\text{after}} - v_{\text{before}}] = \text{total impulse of } F = \mathcal{I} \quad .$$

Observe that using a delta function force leads to the velocity changing instantly from one constant to another. The velocity is no longer continuous, and the velocity right at $t = \alpha$ is no longer well defined. This is not physically possible, but is still a very good approximation of what really happens.

We should also note that $F(t) = \delta_\alpha(t)$ corresponds to a force acting instantaneously at $t = \alpha$ with total impulse of 1. For that reason, δ_α is also known as the *(instantaneous) unit impulse function at* α .

!► *Example 29.1:* *A baseball of mass* 0.145 *kilograms is thrown against a wall with a speed of* 40 *meters per second (about* 85 *miles per hour) and bounces off the wall with a speed of* 30 *meters per second (about* 64 *miles per hour). Its direction of travel before and after hitting the wall is along an imaginary* X*–axis perpendicular to the wall and pointing in the direction the ball is traveling after it bounces off the wall. So, (in meters/second)*

$$v_{before} = -40 \qquad \text{and} \qquad v_{after} = 30 \quad .$$

Letting α *be the time the ball hits the wall, we can model the force of the wall on the ball by*

$$F(t) = \mathcal{I}\,\delta_\alpha(t) \qquad \left(\frac{kg{\cdot}meter}{second^2}\right)$$

where the impulse of the force is

$$\mathcal{I} = m\,[v_{after} - v_{before}] = 0.145\,[30 + 40] = 10.15 \qquad \left(\frac{kg{\cdot}meter}{second}\right) \quad .$$

As Density Functions for Point Masses

Suppose we have some material spread out along the X–axis. Recall that the linear density of the material at position x , $\rho(x)$, is the “mass per unit length” of the material at point x . More precisely, it is the function such that, if $x_0 < x_1$, then

$$\int_{x_0}^{x_1} \rho(x)\,dx$$

gives the mass of the material between positions $x = x_0$ and $x = x_1$.

Now think about what it means to have a density function

$$\rho(x) = \frac{m}{\epsilon}\,\text{rect}_{(\alpha,\alpha+\epsilon)}(x) = \begin{cases} 0 & \text{if} \quad t < 0 \\ \frac{m}{\epsilon} & \text{if} \quad 0 < t < \epsilon \\ 0 & \text{if} \quad \epsilon < t \end{cases}$$

where α , m and ϵ real numbers with m and ϵ being positive. Here, all the mass is uniformly spread out in some object located between $x = \alpha$ and $x = \alpha + \epsilon$. Picking $x_0 < \alpha$ and

$\alpha + \epsilon < x_1$, we see that

$$\text{total mass of the object} = \int_{x_0}^{x_1} \rho(x)\,dx$$

$$= \int_{x_0}^{x_1} \frac{m}{\epsilon} \operatorname{rect}_{(\alpha,\alpha+\epsilon)}(x)\,dx = \frac{m}{\epsilon}\int_{\alpha}^{\alpha+\epsilon} 1\,dx = m \quad .$$

So we have an object of mass m occupying the X–axis from $x = \alpha$ to $x = \alpha + \epsilon$.

In many applications, the width of the object, ϵ , is much smaller than the other dimensions involved, and taking account of this width complicates computations without significantly affecting the results of the computations. In these cases, it is common to simplify the mathematics by letting $\epsilon \to 0$ and thereby converting

our object of mass m occupying the region between $x = \alpha$ and $x = \alpha + \epsilon$

to

an object of mass m occupying the point $x = \alpha$,

In doing so, we see that

$$\rho(x) = \lim_{\epsilon \to 0^+} \frac{m}{\epsilon} \operatorname{rect}_{(\alpha,\alpha+\epsilon)}(x) = m \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \operatorname{rect}_{(\alpha,\alpha+\epsilon)}(x) = m\delta_\alpha(x) \quad .$$

Thus, the delta function at α multiplied by m describes the linear density of a “point mass” at α of mass m .

29.3 The Mathematics of Delta Functions

Integrals with Delta Functions

While we used

$$\delta_\alpha(t) = \delta(t - \alpha) = \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \operatorname{rect}_{(\alpha,\alpha+\epsilon)}(t) \tag{29.1}$$

to visualize the delta function at α , it is mathematically better to view δ_α through the integral equation

$$\int_{t_0}^{t_1} g(t)\delta_\alpha(t)\,dt = \lim_{\epsilon \to 0^+} \int_{t_0}^{t_1} g(t)\frac{1}{\epsilon} \operatorname{rect}_{(\alpha,\alpha+\epsilon)}(t)\ dt \tag{29.2}$$

where (t_0, t_1) can be any interval and g can be any function on (t_0, t_1) continuous at α . This means we are really viewing “ $\delta_\alpha(t)$ ” as notation indicating a certain limiting process involving integration. Remember, that’s how we actually used delta functions in modeling strong brief forces and point masses.

Since our interest is mainly in using delta functions with the Laplace transform, let us simplify matters a little and just consider the integral

$$\int_0^\infty g(t)\,\delta_\alpha(t)\,dt$$

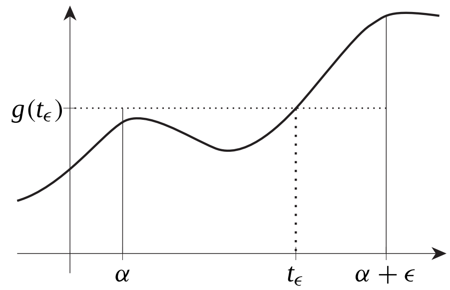

Figure 29.2: The rectangle with area equal to $\int_{\alpha}^{\alpha+\epsilon} g(t)\,dt$.

when $\alpha \geq 0$ and g is any function continuous at α and piecewise continuous on $[0,\infty)$. Before applying equation (29.2), observe that, because $0 \leq \alpha$ and

$$g(t)\,\mathrm{rect}_{(\alpha,\alpha+\epsilon)}(t) \;=\; \begin{cases} 0 & \text{if} \quad t < \alpha \\ g(t) & \text{if} \quad \alpha < t < \alpha+\epsilon \\ 0 & \text{if} \quad \alpha+\epsilon < t \end{cases} ,$$

we have

$$\int_0^\infty g(t) \cdot \frac{1}{\epsilon}\,\mathrm{rect}_{(\alpha,\alpha+\epsilon)}(t)\;dt \;=\; \frac{1}{\epsilon}\int_\alpha^{\alpha+\epsilon} g(t)\,dt \quad .$$

Because we will be taking the limit of the above as $\epsilon \to 0$, we can assume ϵ is small enough that g is continuous on the closed interval $[a,\alpha+\epsilon]$, and then apply the fact (illustrated in figure 29.2) that

$$\begin{aligned}\int_\alpha^{\alpha+\epsilon} g(t)\,dt \;&=\; \text{“(net) area between } T\text{–axis and graph of } y = g(t) \text{ with } \alpha \leq t \leq \alpha+\epsilon\text{”} \\ &=\; \text{“(net) area of rectangle with base } [\alpha,\alpha+\epsilon] \text{ and (signed) height } g(t_\epsilon) \\ &\qquad\quad \text{for some } t_\epsilon \text{ in the interval } [a,\alpha+\epsilon]\text{”} \\ &=\; \epsilon \times g(t_\epsilon) \qquad \text{for some } t_\epsilon \text{ in } [\alpha,\alpha+\epsilon] \quad .\end{aligned}$$

Combining the above and applying equation (29.2), we obtain

$$\begin{aligned}\int_0^\infty g(t)\,\delta_\alpha(t)\,dt \;&=\; \lim_{\epsilon\to 0}\int_0^\infty g(t)\cdot\frac{1}{\epsilon}\,\mathrm{rect}_{(\alpha,\alpha+\epsilon)}(t)\;dt \\ &=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_\alpha^{\alpha+\epsilon} g(t)\,dt \\ &=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\times\epsilon\times\bigl\{g(t_\epsilon) \quad \text{for some } t_\epsilon \text{ in } [\alpha,\alpha+\epsilon]\bigr\} \\ &=\; \lim_{\epsilon\to 0}\bigl\{g(t_\epsilon) \quad \text{for some } t_\epsilon \text{ in } [\alpha,\alpha+\epsilon]\bigr\} \\ &=\; g(t_\epsilon) \quad \text{for some } t_\epsilon \text{ in } [\alpha,\alpha+0] \quad .\end{aligned}$$

But, of course, the only t_ϵ in $[\alpha,\alpha+0]$ is $t_\epsilon = \alpha$. So the above reduces to a simple result that is important enough to place in a theorem.

Theorem 29.1
Let $\alpha \geq 0$ and let g be any piecewise continuous function on $[0, \infty)$ which is continuous at $t = \alpha$. Then

$$\int_0^\infty g(t)\,\delta_\alpha(t)\,dt = g(\alpha) \quad . \tag{29.3}$$

In particular, since $\delta = \delta_0$,

$$\int_0^\infty g(t)\,\delta(t)\,dt = g(0) \quad . \tag{29.4}$$

!►*Example 29.2:* *Actually, two examples:*

$$\int_0^\infty t^2\,\delta_3(t)\,dt = 3^2 = 9 \quad ,$$

and

$$\int_0^\infty (5-t)^3\ \delta(t)\,dt = (5-0)^3 = 125 \quad .$$

We derived the above theorem because it covers the cases of greatest interest to us. Still, it is worth noting that with just a little more work, you can verify that

$$\int_{t_0}^{t_1} g(t)\,\delta_\alpha(t)\,dt = \begin{cases} g(\alpha) & \text{if} \quad t_0 \leq \alpha < t_1 \\ 0 & \text{if} \quad \alpha < t_0 \text{ or } t_1 \leq \alpha \end{cases} \tag{29.5}$$

whenever g is a function continuous at α and piecewise continuous on $[t_0, t_1)$.

Equations (29.3) and (29.4) (and, more generally, equation (29.5)) are often used instead of equation (29.2) "fundamental descriptions" of the delta functions. Their simplicity belies their significance.

Laplace Transforms of Delta Functions

Finding the Laplace transform of a delta function is easy. Just use the integral formula for the Laplace transform along with an equation from theorem 29.1. Assuming $\alpha \geq 0$, we have

$$\mathcal{L}[\delta_\alpha(t)]|_s = \int_0^\infty \delta_\alpha(t) e^{-st}\,dt = e^{-s\alpha} \quad ,$$

which we usually prefer to write as

$$\mathcal{L}[\delta_\alpha(t)]|_s = e^{-\alpha s} \quad .$$

In particular,

$$\mathcal{L}[\delta(t)]|_s = \mathcal{L}[\delta_0(t)]|_s = e^{-0s} = 1 \quad .$$

These transforms are important enough to add to our table of common transforms, giving us table 29.1.

Table 29.1: Laplace Transforms of Common Functions (Version 2)

In the following, α and ω are real-valued constants, and, unless otherwise noted, $s > 0$.

$f(t)$	$F(s) = \mathcal{L}[f(t)]\vert_s$	Restrictions
1	$\frac{1}{s}$	
t	$\frac{1}{s^2}$	
t^n	$\frac{n!}{s^{n+1}}$	$n = 1, 2, 3, \ldots$
$\frac{1}{\sqrt{t}}$	$\frac{\sqrt{\pi}}{\sqrt{s}}$	
t^α	$\frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$	$-1 < \alpha$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$	$\alpha < s$
$e^{i\alpha t}$	$\frac{1}{s-i\alpha}$	
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$	
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$	
$\text{step}_\alpha(t)$, $\text{step}(t-\alpha)$	$\frac{e^{-\alpha s}}{s}$	$0 \leq \alpha$
$\delta(t)$	1	
$\delta_\alpha(t)$, $\delta(t-\alpha)$	$e^{-\alpha s}$	$0 \leq \alpha$

Differential Equations with Delta functions

Using the Laplace transform, it is relatively easy to solve many differential equations in which delta functions act as a forcing functions. Let us look at two examples.

!►*Example 29.3:* *Let's find the solution to*

$$\frac{dy}{dt} = \delta_\alpha(t) \qquad \text{with} \quad y(0) = 0$$

where α is any positive real number.

Taking the Laplace transform of both sides:

$$\mathcal{L}\left[\frac{dy}{dt}\right]\Bigg|_s = \mathcal{L}[\delta_\alpha(t)]|_s \tag{29.6}$$

$$\Longrightarrow \qquad sY(s) - y(0) = e^{-\alpha s} \tag{29.7}$$

$$\Longrightarrow \qquad sY(s) - 0 = e^{-\alpha s} \tag{29.8}$$

$$\Longrightarrow \qquad Y(s) = \frac{e^{-\alpha s}}{s} \quad . \tag{29.9}$$

Thus, the solution to our differential equation is

$$y(t) = \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}\left[\frac{e^{-\alpha s}}{s}\right]\Bigg|_t = \text{step}_\alpha(t) \quad .$$

According to the last example, $y(t) = \text{step}_\alpha(t)$ is a solution to $y'(t) = \delta_\alpha(t)$. In other words,

$$\frac{d}{dt}\text{step}_\alpha(t) = \delta_\alpha(t) \quad .$$

This interesting fact may also be a disturbing fact for those of you who realize that the step functions are not differentiable, at least not in the sense normally taught in calculus courses. The truth is that delta functions are somewhat exotic entities that are outside the classical theory of calculus. We will discuss this a little further in section 29.5. For now, let me just say that what we are calling 'delta functions' are really examples of things better referred to as "generalized functions", and that the above equation about the derivative of the step function, while not valid in a strict classical sense, is valid using a definition of differentiation appropriate for these generalized functions.

But enough worrying about technicalities. Let's solve another differential equation with a delta function.

!►*Example 29.4:* *Now consider*

$$y'' - 10y' + 21y = \delta(t) \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0 \quad .$$

Taking the Laplace transform of both sides:

$$\mathcal{L}\big[y'' - 10y' + 21y\big]\big|_s = \mathcal{L}[\delta(t)]|_s$$

$$\Longrightarrow \qquad \mathcal{L}\big[y''\big]\big|_s - 10\mathcal{L}\big[y'\big]\big|_s + 21\mathcal{L}[y]|_s = 1$$

$$\Longrightarrow \qquad s^2Y(s) - 10sY(s) + 21Y(s) = 1$$

$$\Longrightarrow \qquad \big[s^2 - 10s + 21\big]Y(s) = 1 \quad .$$

So,

$$Y(s) = \frac{1}{s^2 - 10s + 21} \quad ,$$

which just happens to be the function whose inverse transform was found in example 27.4 on page 540. Using the result of that example, we can just write out

$$y(t) = \mathcal{L}^{-1}[Y(s)]|_t = \frac{1}{4}\left[e^{7t} - e^{3t}\right] \quad .$$

29.4 Delta Functions and Duhamel's Principle

If you compare the results of the last example with the results of example 27.6 on page 542, you'll notice that the solution $y(t)$ to

$$y'' - 10y' + 21y = \delta(t) \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0$$

and the impulse response function $h(t)$ for

$$y'' - 10y' + 21y = f(t)$$

are one and the same. Is this an amazing coincidence?

No.

In section 27.3 we saw that, for any real constants a, b and c, and any Laplace transformable function f, the solution on $(0, \infty)$ to the generic initial-value problem

$$ay'' + by' + cy = f(t) \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0$$

is given by

$$y(t) = h * f(t)$$

where

$$h = \mathcal{L}^{-1}[H] \qquad \text{and} \qquad H(s) = \frac{1}{as^2 + bs + c} \quad .$$

Now consider the corresponding initial-value problem

$$ay'' + by' + cy = \delta(t) \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0 \quad ,$$

which is just the generic initial-value problem above with $f = \delta$. Taking the Laplace transform, we get

$$\begin{aligned}
& \mathcal{L}\big[ay'' + by' + cy\big]\big|_s = \mathcal{L}[\delta(t)]|_s \\
\Longrightarrow \quad & a\mathcal{L}\big[y''\big]\big|_s + b\mathcal{L}\big[y'\big]\big|_s + c\mathcal{L}[y]|_s = 1 \\
\Longrightarrow \quad & as^2 Y(s) + bsY(s) + cY(s) = 1 \\
\Longrightarrow \quad & \big[as^2 + bs + c\big]Y(s) = 1 \quad .
\end{aligned}$$

Dividing by the polynomial and comparing the result with the above formula for H, we see that

$$Y(s) = \frac{1}{as^2 + bs + c} = H(s) \quad .$$

Thus,

$$y(t) = \mathcal{L}^{-1}[Y(s)]|_t = \mathcal{L}^{-1}[H(s)]|_t = h(t) \quad .$$

In other words, $h(t)$ is the solution to the particular initial-value problem

$$ah'' + bh' + ch = \delta(t) \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0 \quad .$$

This explains why h is commonly referred to as the "impulse response function" — well, almost explains. Here's a little background: In many applications, the solution to the initial-value problem

$$ay'' + by' + cy = f(t) \qquad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0$$

describes how some physical system responds to an applied "force" $f(t)$ (actually, f might not be an actual force). With this interpretation, $h(t)$ does give the response of the system to a delta function force, and, as noted earlier, the delta function is also known as an impulse function. Hence the term "impulse response function" for $h(t)$.

Of course, the generic computations just done can be done with higher-order differential equations. Combining this with theorem 27.2 on page 545 yields

Theorem 29.2 (Duhamel's principle, version 2)
Let N be any positive integer, let a_0, a_1, ... and a_N be any collection of constants, and let $f(t)$ be any Laplace transformable function. Then, the solution to the initial-value problem

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = f(t)$$

with

$$y(0) = 0 \quad , \quad y'(0) = 0 \quad , \quad y''(0) = 0 \quad , \quad \ldots \quad \text{and} \quad y^{N-1}(0) = 0 \quad ,$$

is given by

$$y(t) = h * f(t) = \int_0^t h(x) f(t-x)\, dx$$

where $h(t)$ is the solution to

$$a_0 h^{(N)} + a_1 h^{(N-1)} + \cdots + a_{N-2} h'' + a_{N-1} h' + a_N h = \delta(t)$$

with

$$h(0) = 0 \quad , \quad h'(0) = 0 \quad , \quad h''(0) = 0 \quad , \quad \ldots \quad \text{and} \quad h^{N-1}(0) = 0 \quad .$$

There is a practical consequence to h being the impulse response function. Suppose you have a physical system in which you know the 'output' $y(t)$ is related to an 'input' $f(t)$ through a differential equation of the form given in the above theorem. Suppose, further, that you do not know exactly what that differential equation is. Maybe, for example, you have a mass/spring system some of whose basic parameters — mass, spring constant or damping constant — are unknown and cannot be easily measured. The above theorem tells us that, if we input the physical equivalent of a delta function (say, we provide a unit impulse to the mass/spring system by carefully hitting the mass with a hammer), then measuring the output over time will yield a description of the impulse response function, $h(t)$. Save those values for $h(t)$ over time in a computer, and you can then numerically evaluate the output $y(t)$ corresponding to any other input $f(t)$ through the formula

$$y(t) = f * h(t) \quad .$$

In practice, generating and inputting the physical equivalent of $\delta(t)$ is usually impossible. What is often possible is to generate and input a good approximation to the delta function, say,

$$\frac{1}{\epsilon} \operatorname{rect}_{(0,\epsilon)}(t)$$

for some small value of ϵ. The resulting measured output will not be $h(t)$ exactly, but, if the errors in measurement aren't too bad, it will be a close approximation.

29.5 Some "Issues" with Delta Functions

The astute reader may have noticed that we've glossed over a few troublesome issues in our discussion of delta functions. Let's deal with a few of these now.

Defining the Delta Functions

You may have noticed that we have not yet *defined* the delta function. In particular, I've not given you any formula for computing the values of $\delta(t)$ or $\delta_\alpha(t)$ for different values of t. Instead, I've only told you to *visualize* $\delta_\alpha(t)$ in terms of either the limit

$$\delta_\alpha(t) \;=\; \delta(t-\alpha) \;=\; \lim_{\epsilon\to 0}\frac{1}{\epsilon}\operatorname{rect}_{(\alpha,\alpha+\epsilon)}(t) \quad , \tag{29.10}$$

or the limit

$$\int_{t_0}^{t_1} g(t)\delta_\alpha(t)\,dt \;=\; \lim_{\epsilon\to 0^+}\int_{t_0}^{t_1} g(t)\frac{1}{\epsilon}\operatorname{rect}_{(\alpha,\alpha+\epsilon)}(t)\;dt \quad . \tag{29.11}$$

If you check other texts, you'll often find δ_a (with $\alpha \geq 0$) "defined" either as the limit in (29.10) or as the 'function' such that

$$\int_0^\infty g(t)\,\delta_\alpha(t)\,dt \;=\; g(\alpha) \tag{29.12}$$

whenever g is a function continuous at α. (This, recall, was something we derived from equation (29.11).) Both of these are good 'working' definitions in that, properly interpreted, they tell you how you should use the symbol δ_α in computations (provided you interpret the limit in (29.10) as really meaning the limit in (29.11)).

Unfortunately, if you treat either as a rigorous definition for a classical function δ_α, then you can then rigorously derive

$$\delta_\alpha(t) \;=\; 0 \qquad \text{whenever} \quad t \neq \alpha \quad .$$

Rigorously applying the classical theory of integration normally developed in undergraduate mathematics, you then find that

$$\begin{aligned}\int_0^\infty g(t)\,\delta_\alpha(t)\,dt \;&=\; \int_0^\alpha g(t)\,\delta_\alpha(t)\,dt \;+\; \int_\alpha^\infty g(t)\,\delta_\alpha(t)\,dt \\ &=\; \int_0^\alpha g(t)\cdot 0\,dt \;+\; \int_\alpha^\infty g(t)\cdot 0\,dt \;=\; 0 \quad .\end{aligned}$$

In particular, using $g(t) = t^2$, $\alpha = 1$ and both equation (29.11) and the last equation above, we get

$$1 \;=\; 1^2 \;=\; \int_0^\infty t^2\,\delta_2(t)\,dt \;=\; 0 \quad !$$

The problem is that there is no classical function that satisfies either definition. Fortunately, there is a way to 'generalize' the classical notion of 'functions' yielding a class of things called "generalized functions". Delta functions are members of this class. Unfortunately, a proper development of "generalized functions" goes beyond the scope of this text. I will tell you that, if f is a generalized function, then, for every sufficiently smooth and integrable function g and suitable interval (t_0, t_1), then

$$\int_{t_0}^{t_1} g(t) f(t)\, dt$$

"makes sense" in some generalized sense. For $f = \delta_\alpha$, this integral can be defined by equation (29.11).[2] Using the theory of generalized functions, along with the corresponding generalization of the theory of calculus, everything developed in this chapter can be rigorously defined or derived, including the observation that, "in a generalized sense",

$$\delta_\alpha(t) \;=\; \frac{d}{dt}\,\mathrm{step}_\alpha(t) \quad .$$

For now, however, it may best to view the computations we are doing with δ_α as shorthand for doing the same computations with

$$\frac{1}{\epsilon}\,\mathrm{rect}_{(\alpha,\alpha+\epsilon)} \quad ,$$

and then letting $\epsilon \to 0^+$ in the final result.

!►*Example 29.5:* *Let's reconsider solving*

$$\frac{dy_\epsilon}{dt} \;=\; \delta_\alpha(t) \qquad \text{with} \quad y(0) = 0$$

where α is any positive real number. Doing the replacement suggested above, we'll first solve

$$\frac{dy_\epsilon}{dt} \;=\; \frac{1}{\epsilon}\,\mathrm{rect}_{(\alpha,\alpha+\epsilon)} \qquad \text{with} \quad y(0) = 0 \tag{29.13}$$

assuming $\epsilon > 0$, and then take the limit of the result as $\epsilon \to 0$.

Taking the Laplace transform of both sides of the last equation:

$$\mathcal{L}\left[\frac{dy_\epsilon}{dt}\right]\bigg|_s \;=\; \mathcal{L}\left[\frac{1}{\epsilon}\,\mathrm{rect}_{(\alpha,\alpha+\epsilon)}\right]\bigg|_s$$

$$\Longrightarrow \qquad sY(s) - y(0) \;=\; \frac{1}{\epsilon}\mathcal{L}\big[\mathrm{rect}_{(\alpha,\alpha+\epsilon)}\big]\big|_s$$

$$\Longrightarrow \qquad sY_\epsilon(s) - 0 \;=\; \frac{1}{\epsilon}\left[\frac{1}{s}e^{-\alpha s} - \frac{1}{s}e^{-(\alpha+\epsilon)s}\right]$$

$$\Longrightarrow \qquad Y_\epsilon(s) \;=\; \frac{1}{\epsilon}\left[\frac{1}{s^2}e^{-\alpha s} - \frac{1}{s^2}e^{-(\alpha+\epsilon)s}\right] \quad .$$

So,

$$\begin{aligned} y_\epsilon(t) \;&=\; \mathcal{L}^{-1}\left[\frac{1}{\epsilon}\left[\frac{1}{s^2}e^{-\alpha s} - \frac{1}{s^2}e^{-(\alpha+\epsilon)s}\right]\right]\bigg|_t \\ &=\; \frac{1}{\epsilon}\left\{\mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-\alpha s}\right]\bigg|_t \;-\; \mathcal{L}^{-1}\left[\frac{1}{s^2}e^{-(\alpha+\epsilon)}\right]\bigg|_t\right\} \\ &=\; \frac{1}{\epsilon}\left\{[t-\alpha]\,\mathrm{step}_\alpha(t) - [t-(\alpha+\epsilon)]\,\mathrm{step}_{\alpha+\epsilon}(t)\right\} \end{aligned}$$

[2] If you must know, "generalized functions" are actually "continuous linear functionals on a suitable space of test functions", and if you want to find out what that means, see part IV of the author's *Principles of Fourier Analysis*, or go to the library and look up books on either generalized functions or distributional theory.

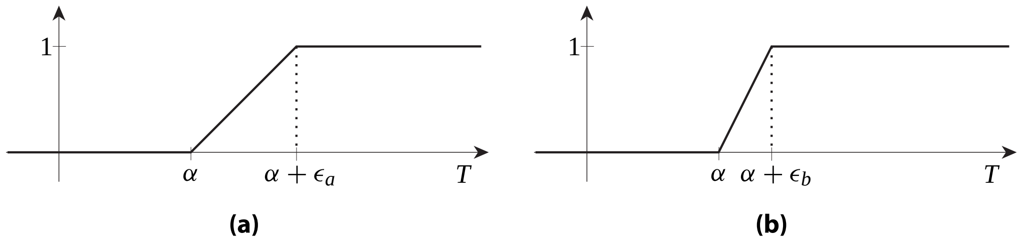

Figure 29.3: The graph of the solution to initial-value problem (29.13) **(a)** when $\epsilon = \epsilon_a$ and **(b)** when $\epsilon = \epsilon_b$ with $0 < \epsilon_b < \epsilon_a$.

$$= \frac{1}{\epsilon}\begin{cases} 0 - 0 & \text{if} \quad t < \alpha \\ [t-\alpha] - 0 & \text{if} \quad \alpha < t < \alpha + \epsilon \\ [t-\alpha] - [t - (\alpha+\epsilon)] & \text{if} \quad \alpha + \epsilon < t \end{cases}$$

$$= \begin{cases} 0 & \text{if} \quad t < \alpha \\ \dfrac{t-\alpha}{\epsilon} & \text{if} \quad \alpha < t < \alpha + \epsilon \\ 1 & \text{if} \quad \alpha + \epsilon < t \end{cases} .$$

Graphs of this function for two different values of ϵ *are sketched in figure 29.3.*

Finally, taking the limit, we get

$$y(t) = \lim_{\epsilon \to 0^+} y_\epsilon(t) = \lim_{\epsilon \to 0^+} \left\{ \begin{array}{ll} 0 & \text{if} \quad t < \alpha \\ \dfrac{t-\alpha}{\epsilon} & \text{if} \quad \alpha < t < \alpha + \epsilon \\ 1 & \text{if} \quad \alpha + \epsilon < t \end{array} \right\} = \begin{cases} 0 & \text{if} \quad t < \alpha \\ 1 & \text{if} \quad \alpha < t \end{cases} .$$

That is,

$$y(t) = \text{step}_\alpha(t) \quad ,$$

just as obtained (with much less work!) in example 29.3.

Continuity of Solutions and Problems with Initial Values

Early in this text, it was stated that solutions to first-order differential equations had to be continuous, and solutions to second-order differential equations had to be continuous and have continuous derivatives. But $y = \text{step}_\alpha(t)$, the solution to

$$\frac{dy}{dt} = \delta_\alpha(t) \qquad \text{with} \quad y(0) = 0$$

obtained in exercises 29.3 and 29.5, is clearly not continuous. If you think about it, this may not be so surprising. Our original insistence on the continuity of solutions assumed we were using classical functions. The exotic nature of the delta functions takes us outside the classical theory to the idealized cases where instantaneous change can occur.

Normally, this is not a problem. Indeed, it can desirable, especially if you are modeling "brief strong forces". One place where this can case some confusion is when the discontinuities occur where initial data is given. In these cases, the confusion can be somewhat abated by remembering that a delta function really indicate a limiting process.

!►Example 29.6: *Letting $\alpha = 0$, we see that the solution to*

$$\frac{dy}{dt} = \delta(t) \qquad \text{with} \quad y(0) = 0$$

is

$$y(t) = \text{step}(t) \quad .$$

However, $\text{step}(t)$ *has a jump at* $t = 0$*, and its limit from the right at this point is* 1*. So how can we say this step function satisfies the given initial condition,* $y(0) = 0$*? By going back to exercise 29.5, which showed that the above solution should be viewed as the limit as* $\epsilon \to 0$ *of the function* $y_\epsilon(t)$ *graphed in figure 29.3 with* $\alpha = 0$*. For each* $\epsilon > 0$*,* $y_\epsilon(t)$ *is continuous at* $t = 0$ *and satisfies* $y_\epsilon(0) = 0$*. As* ϵ *becomes smaller, the values of* $y_\epsilon(t)$ *increase more rapidly to* 1 *for positive values of* t*. So what we end up with after taking* $\epsilon \to 0$ *is that the* left-hand limit *of* $y(t)$ *at* $t = 0$ *is* 0*, but* $y(t)$ *"immediately" increases from* 0 *to* 1 *as* t *switches from negative values to positive values.*

What this last example demonstrates is that, when the differential equation has a $\delta(t)$ in its forcing function, then initial conditions naively written as

$$y(0) = y_0 \quad , \quad y'(0) = y_1 \quad , \quad \ldots$$

are, well, naive. What is really meant is that these values give the left-hand limits,

$$\lim_{t \to 0^-} y(t) = y_0 \quad , \quad \lim_{t \to 0^-} y'(0) = y_1 \quad , \quad \ldots \quad .$$

Additional Exercises

29.1. *For the following, assume an object of mass* m *kilograms is initially moving along the* X*–axis with constant velocity* v_{before} *meters/second until its velocity is changed to* v_{after} *meters/second by a delta function force with impulse* $\mathcal{I}$ *at time* $t = \alpha$ *seconds.*

a. *Find* v_{after} *assuming* $m = 2$*,* $v_{\text{before}} = -10$ *and*

i. $\mathcal{I} = 60$ **ii.** $\mathcal{I} = 100$ **iii.** $\mathcal{I} = 20$

b. *Assume* $m = 0.2$ *and* $v_{\text{before}} = -40$*. What impulse* $\mathcal{I}$ *is needed to obtain*

i. $v_{\text{after}} = 50$ **ii.** $v_{\text{after}} = 100$ **iii.** $v_{\text{after}} = 0$

c. *Assume* $\mathcal{I} = 30$*, and that the velocity of the object before and after* $t = \alpha$ *is determined by radar. What is the mass of the object if*

i. $v_{\text{before}} = -10$ *and* $v_{\text{after}} = 50$ **ii.** $v_{\text{before}} = 0$ *and* $v_{\text{after}} = 15$

29.2. *Using the results given in theorem 29.1, compute the following integrals*

a. $\displaystyle\int_0^\infty t^2 \delta_4(t)\, dt$ **b.** $\displaystyle\int_0^\infty t^2 \delta(t)\, dt$

c. $\displaystyle\int_0^{\infty} \cos(t)\,\delta(t)\,dt$

d. $\displaystyle\int_0^{\infty} \sin(t)\,\delta_{\pi/6}(t)\,dt$

e. $\displaystyle\int_0^{\infty} t^2\,\text{rect}_{(1,4)}(t)\delta_3(t)\,dt$

f. $\displaystyle\int_0^{\infty} t^2\,\text{rect}_{(1,4)}(t)\delta_6(t)\,dt$

29.3. *Prove/derive equation (29.5) on page 588.*

29.4. *Show that*

$$g * \delta_\alpha(t) \;=\; g(t-\alpha)\,\text{step}_\alpha(t)$$

whenever $\alpha \geq 0$ *and* g *is a piecewise continuous function on* $(0, \infty)$.

29.5. *Find and sketch the solution over* $[0, \infty)$ *to each of the following:*

a. $y' \;=\; 3\delta_2(t)$ *with* $y(0) = 0$

b. $y' \;=\; \delta_2(t) \;-\; \delta_4(t)$ *with* $y(0) = 0$

c. $y'' \;=\; \delta_3(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

d. $y'' \;=\; \delta_1(t) - \delta_4(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

e. $y' \;+\; 2y \;=\; 4\delta_1(t)$ *with* $y(0) = 0$

f. $y'' \;+\; y \;=\; \delta(t) \;+\; \delta_\pi(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

g. $y'' \;+\; y \;=\; -2\delta_{\pi/2}(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

29.6. *Find the solution on* $t > 0$ *to each of the following initial-value problems:*

a. $y' \;+\; 3y \;=\; \delta_2(t)$ *with* $y(0) = 2$

b. $y'' \;+\; 3y' \;=\; \delta(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

c. $y'' \;+\; 3y' \;=\; \delta_1(t)$ *with* $y(0) = 0$ *and* $y'(0) = 1$

d. $y'' \;+\; 16y \;=\; \delta_2(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

e. $y'' \;-\; 16y \;=\; \delta_{10}(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

f. $y'' \;+\; y \;=\; \delta(t)$ *with* $y(0) = 0$ *and* $y'(0) = -1$

g. $y'' \;+\; 4y' \;-\; 12y \;=\; \delta(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

h. $y'' \;+\; 4y' \;-\; 12y \;=\; \delta_3(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

i. $y'' \;+\; 6y' \;+\; 9y \;=\; \delta_4(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

j. $y'' \;-\; 12y' \;+\; 45y \;=\; \delta(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

k. $y''' \;+\; 9y' \;=\; \delta_1(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$

l. $y'''' \;-\; 16y \;=\; \delta(t)$ *with* $y(0) = 0$ *and* $y'(0) = 0$