23

Variation of Parameters (A Better Reduction of Order Method for Nonhomogeneous Equations)

"Variation of parameters" is another way to solve nonhomogeneous linear differential equations, be they second order,

$$ay'' + by' + cy = g \quad ,$$

or even higher order,

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y' + a_N y = g \quad .$$

One advantage of this method over the method of undetermined coefficients from chapter 21 is that the differential equation does not have to be simple enough that we can 'guess' the form for a particular solution. In theory, the method of variation of parameters will work whenever g and the coefficients are reasonably continuous functions. As you may expect, though, it is not quite as simple a method as the method of guess. So, for 'sufficiently simple' differential equations, you may still prefer using the guess method instead of what we'll develop here.

We will first develop the variation of parameters method for second-order equations. Then we will see how to extend it to deal with differential equations of even higher order.[1] As you will see, the method can be viewed as a very clever improvement on the reduction of order method for solving nonhomogeneous equations. What might not be so obvious is why the method is called "variation of parameters".

23.1 Second-Order Variation of Parameters

Derivation of the Method

Suppose we want to solve a second-order nonhomogeneous differential equation

$$ay'' + by' + cy = g$$

[1] It is possible to use a "variation of parameters" method to solve first-order nonhomogeneous linear equations, but that's just plain silly.

over some interval of interest, say,

$$x^2y'' - 2xy' + 2y = 3x^2 \qquad \text{for} \quad x > 0 \quad .$$

Let us also assume that the corresponding homogeneous equation,

$$ay'' + by' + cy = 0 \quad ,$$

has already been solved. That is, we already have an independent pair of functions $y_1 = y_1(x)$ and $y_2 = y_2(x)$ for which

$$y_h(x) = c_1y_1(x) + c_2y_2(x)$$

is a general solution to the homogeneous equation.

For our example,

$$x^2y'' - 2xy' + 2y = 3x^2 \quad ,$$

the corresponding homogeneous equation is the Euler equation

$$x^2y'' - 2xy' + 2y = 0 \quad .$$

You can easily verify that this homogeneous equation is satisfied if y *is either*

$$y_1 = x \qquad \text{or} \qquad y_2 = x^2 \quad .$$

Clearly, the set $\{x, x^2\}$ *is linearly independent, and, so, the general solution to the corresponding homogeneous homogeneous equation is*

$$y_h = c_1x + c_2x^2 \quad .$$

Now, in using reduction of order to solve our nonhomogeneous equation

$$ay'' + by' + cy = g \quad ,$$

we would first assume a solution of the form

$$y = y_0\,u$$

where $u = u(x)$ is an unknown function 'to be determined', and $y_0 = y_0(x)$ is any single solution to the corresponding homogeneous equation. However, we do not just have a single solution to the corresponding homogeneous equation — we have two: y_1 and y_2 (along with all linear combinations of these two). So why don't we use both of these solutions and assume, instead, a solution of the form

$$y = y_1u + y_2v$$

where y_1 and y_2 are the two solutions to the corresponding homogeneous equation already found, and $u = u(x)$ and $v = v(x)$ are two unknown functions to be determined.

For our example,

$$x^2y'' - 2xy' + 2y = 3x^2 \quad ,$$

we already have that

$$y_1 = x \qquad \text{and} \qquad y_2 = x^2$$

form a fundamental pair of solutions to the corresponding homogeneous differential equation. So, in this case, the assumption that

$$y = y_1 u + y_2 v$$

is

$$y = xu + x^2 v$$

where $u = u(x)$ *and* $v = v(x)$ *are two functions to be determined.*

To determine the two unknown functions $u(x)$ and $v(x)$, we will need two equations. One, of course, must be the original differential equation that we are trying to solve. The other equation can be chosen at our convenience (provided it doesn't contradict or simply repeat the original differential equation). Here is a remarkably clever choice for that other equation:

$$y_1 u' + y_2 v' = 0 \quad . \tag{23.1}$$

For our example,

$$y_1 = x \qquad \text{and} \qquad y_2 = x^2 \quad .$$

So we will require that

$$xu' + x^2 v' = 0 \quad .$$

To see why this is such a clever choice, let us now compute y' and y'', and see what the differential equation becomes in terms of u and v. We'll do this for the example first.

For our example,

$$y = xu + x^2 v \quad ,$$

and we required that

$$xu' + x^2 v' = 0 \quad .$$

Computing the first derivative, rearranging a little, and applying the above requirement:

$$\begin{aligned} y' &= \left[xu + x^2 v\right]' \\ &= u + xu' + 2xv + x^2 v' \\ &= u + 2xv + \underbrace{xu' + x^2 v'}_{0} \quad . \end{aligned}$$

So

$$y' = u + 2xv \quad ,$$

and

$$y'' = [u + 2xv]' = u' + 2v + 2xv'$$

Notice that the formula for y'' *does not involve any second derivatives of* u *and* v. *Plugging the above formulas for* y, y' *and* y'' *into the left side of our original differential equation, we see that*

$$\begin{aligned} & x^2 y'' - 2xy' + 2y = 3x^2 \\ \Longrightarrow \quad & x^2\left[u' + 2v + 2xv'\right] - 2x\left[u + 2xv\right] + 2\left[xu + x^2 v\right] = 3x^2 \end{aligned}$$

$$\Longrightarrow \quad x^2u' + 2x^2v + 2x^3v' - 2xu - 4x^2v + 2xu + 2x^2v = 3x^2$$

$$\Longrightarrow \quad x^2u' + 2x^3v' + [\underbrace{2x^2 - 4x^2 + 2x^2}_{0}]v + [\underbrace{-2x + 2x}_{0}]u = 3x^2 \quad .$$

Hence, our original differential equation,

$$x^2y'' - 2xy' + 2y = 3x^2v' \quad ,$$

reduces to

$$x^2u' + 2x^3v' = 3x^2 \quad .$$

For reasons that will be clear in a little bit, let us divide this equation through by x^2 *, giving us*

$$u' + 2xv' = 3 \quad . \tag{23.2}$$

Keep in mind that this is what our differential equation reduces to if we start by letting

$$y = xu + x^2v$$

and requiring that

$$xu' + x^2v' = 0 \quad .$$

Now back to the general case, where our differential equation is

$$ay'' + by' + cy = g \quad .$$

If we set

$$y = y_1u + y_2v$$

(where y_1 and y_2 are solutions to the corresponding homogeneous equation), and require that

$$y_1u' + y_2v' = 0 \quad ,$$

then

$$\begin{aligned} y' &= [y_1u + y_2v]' \\ &= [y_1u]' + [y_2v]' \\ &= y_1'u + y_1u' + y_2'v + y_2v' \\ &= y_1'u + y_2'v + \underbrace{y_1u' + y_2v'}_{0} \quad . \end{aligned}$$

So

$$y' = y_1'u + y_2'v \quad ,$$

and

$$\begin{aligned} y'' &= [y_1'u + y_2'v]' \\ &= y_1''u + y_1'u' + y_2''v + y_2'v' \\ &= y_1'u' + y_2'v' + y_1''u + y_2''v \quad . \end{aligned}$$

Remember, y_1 and y_2 , being solutions to the corresponding homogeneous equation, satisfy

$$ay_1'' + by_1' + cy_1 = 0 \qquad \text{and} \qquad ay_2'' + by_2' + cy_2 = 0 \quad .$$

Using all the above, we have

$$
\begin{aligned}
& ay'' + by' + cy = g \\
\Longrightarrow\quad & a\big[y_1{}'u' + y_2{}'v' + y_1{}''u + y_2{}''v\big] + b\big[y_1{}'u + y_2{}'v\big] + c\big[y_1u + y_2v\big] = g \\
\Longrightarrow\quad & a\big[y_1{}'u' + y_2{}'v'\big] + \big[\underbrace{ay_1{}'' + by_1{}' + cy_1}_{0}\big]u + \big[\underbrace{ay_2{}'' + by_2{}' + cy_2}_{0}\big]v = g \quad .
\end{aligned}
$$

The vanishing of the u and v terms should not be surprising. A similar vanishing occurred in the original reduction of order method. What we also have here, thanks to the 'other equation' that we chose, is that no second-order derivatives of u or v occur either. Consequently, our original differential equation,

$$ay'' + by' + cy = g \quad ,$$

reduces to

$$a\big[y_1{}'u' + y_2{}'v'\big] = g \quad .$$

Dividing this by a then yields

$$y_1{}'u' + y_2{}'v' = \frac{g}{a} \quad .$$

Keep in mind what the last equation is. It is what our original differential equation reduces to after setting

$$y = y_1u + y_2v \tag{23.3}$$

(where y_1 and y_2 are solutions to the corresponding homogeneous equation), and requiring that

$$y_1u' + y_2v' = 0 \quad .$$

This means that the derivatives u' and v' of the unknown functions in formula (23.3) must satisfy the pair (or system) of equations

$$
\begin{aligned}
y_1u' + y_2v' &= 0 \\
y_1{}'u' + y_2{}'v' &= \frac{g}{a}
\end{aligned}
$$

This system can be easily solved for u' and v'. Integrating what we get for u' and v' then gives us the formulas for u and v which we can plug back into formula (23.3) for y, the solution to our nonhomogeneous differential equation.

Let's finish our example:

We have

$$y = xu + x^2v$$

where u' and v' satisfy the system

$$
\begin{aligned}
xu' + x^2v' &= 0 \\
u' + 2xv' &= 3
\end{aligned}
$$

(The first was the equation we chose to require; the second was what the differential equation reduced to.) From the first equation in this system, we have that

$$u' = -xv' \quad .$$

Combining this with the second equation:

$$\begin{aligned} && u' + 2xv' &= 3 \\ \Longrightarrow && -xv' + 2xv' &= 3 \\ \Longrightarrow && xv' &= 3 \\ \Longrightarrow && v' &= \frac{3}{x} \quad . \end{aligned}$$

Hence, also,

$$u' = -xv' = -x \cdot \frac{3}{x} = -3 \quad .$$

Remember, the primes denote differentiation with respect to x *. So we have*

$$\frac{du}{dx} = -3 \qquad \text{and} \qquad \frac{dv}{dx} = \frac{3}{x} \quad .$$

Integrating yields

$$u = \int \frac{du}{dx}\,dx = \int -3\,dx = -3x + c_1$$

and

$$v = \int \frac{dv}{dx}\,dx = \int \frac{3}{x}\,dx = 3\ln|x| + c_2 \quad .$$

Plugging these into the formula for y *, we get*

$$\begin{aligned} y &= xu + x^2 v \\ &= x\big[-3x + c_1\big] + x^2\big[3\ln|x| + c_2\big] \\ &= -3x^2 + c_1 x + 3x^2 \ln|x| + c_2 x^2 \\ &= 3x^2 \ln|x| + c_1 x + (c_2 - 3)x^2 \quad , \end{aligned}$$

which simplifies a little to

$$y = 3x^2 \ln|x| + C_1 x + C_2 x^2 \quad .$$

This, at long last, is our solution to

$$x^2 y'' - 2xy' + 2y = 3x^2 \quad .$$

Before summarizing our work (and reducing it to a fairly simple procedure) let us make two observations based on the above:

1. If we keep the arbitrary constants arising from the indefinite integrals of u' and v' , then the resulting formula for y is a general solution to the nonhomogeneous equation. If we drop the arbitrary constants (or use definite integrals), then we will end up with a particular solution.

2. After plugging the formulas for u and v into $y = y_1 u + y_2 v$, some of the resulting terms can often be absorbed into other terms. (In the above, for example, we absorbed the $-3x^2$ and $c_2 x^2$ terms into one $C_2 x^2$ term.)

Summary: How to Do It

If you look back over our derivation, you will see that we have the following:

> *To solve*
> $$ay'' + by' + cy = g \quad ,$$
> *first find a fundamental pair* $\{y_1, y_2\}$ *of solutions to the corresponding homogeneous equation*
> $$ay'' + by' + cy = 0 \quad .$$
> *Then set*
> $$y = y_1 u + y_2 v \tag{23.4}$$
> *assuming that* $u = u(x)$ *and* $v = v(x)$ *are unknown functions whose derivatives satisfy the system*
> $$y_1 u' + y_2 v' = 0 \tag{23.5a}$$
> $$y_1{}' u' + y_2{}' v' = \frac{g}{a} \tag{23.5b}$$
> *Solve the system for* u' *and* v'*; integrate to get the formulas for* u *and* v*, and plug the results back into (23.4). That formula for* y *is your solution.*

The above procedure is what we call (the method of) *variation of parameters* (for solving a second-order nonhomogeneous differential equation). Notice the similarity between the two equations in the system. That makes it relatively easy to remember the system of equations. This similarity is carried over to problems with equations of even higher order.

It is possible to reduce the procedure further to a single (not-so-simple) formula. We will discuss that in section 23.3. However, I recommend that you use the above method for most of your work, instead of that equation.

And remember:

1. To get a general solution to the nonhomogeneous equation, find u and v using indefinite integrals and keep the arbitrary constants arising from that integration. Otherwise, you get a particular solution.

2. After plugging the formulas for u and v into $y = y_1 u + y_2 v$, some of the resulting terms can often be absorbed into other terms. Go ahead and do so; it simplifies your final result.

!▶*Example 23.1:* *Using the above procedure, let us find the general solution to*

$$y'' + y = \tan(x) \quad .$$

The corresponding homogeneous equation is

$$y'' + y = 0 \quad .$$

We've already solved this equation a couple of times; its general solution is

$$y_h = c_1 \cos(x) + c_2 \sin(x) \quad .$$

So we will take

$$y_1 = \cos(x) \qquad \textit{and} \qquad y_2 = \sin(x)$$

as the independent pair of solutions to the corresponding homogeneous equation in the solution formula

$$y = y_1 u + y_2 v \quad .$$

That is, as the formula for the general solution to our nonhomogeneous differential equation, we will use

$$y = \cos(x)u + \sin(x)v$$

where $u = u(x)$ *and* $v = v(x)$ *are functions to be determined.*

For our differential equation, $a = 1$ *and* $g = \tan(x)$ *. Thus, with our choice of* y_1 *and* y_2 *, the system*

$$y_1 u' + y_2 v' = 0$$

$$y_1{}' u' + y_2{}' v' = \frac{g}{a}$$

is

$$\cos(x)u' + \sin(x)v' = 0$$

$$-\sin(x)u' + \cos(x)v' = \tan(x) \quad .$$

This system can be solved several different ways. Why don't we just observe that, if we solve the first equation for v' *, we get*

$$v' = -\frac{\cos(x)}{\sin(x)}u' \quad .$$

Combining this with the second equation (and recalling a little trigonometry) then yields

$$-\sin(x)u' + \cos(x)\left[-\frac{\cos(x)}{\sin(x)}u'\right] = \tan(x)$$

$$\Longrightarrow \qquad \left(-\sin(x) - \frac{\cos^2(x)}{\sin(x)}\right)u' = \tan(x)$$

$$\Longrightarrow \qquad -\left(\frac{\sin^2(x) + \cos^2(x)}{\sin(x)}\right)u' = \tan(x)$$

$$\Longrightarrow \qquad -\left(\frac{1}{\sin(x)}\right)u' = \tan(x) \quad .$$

Thus,

$$u' = -\tan(x)\sin(x) = -\frac{\sin^2(x)}{\cos(x)}$$

and

$$v' = -\frac{\cos(x)}{\sin(x)}u' = -\frac{\cos(x)}{\sin(x)} \times \left[-\frac{\sin^2(x)}{\cos(x)}\right] = \sin(x) \quad .$$

To integrate the formula for u' *, it may help to first observe that*

$$u'(x) = -\frac{\sin^2(x)}{\cos(x)} = -\frac{1 - \cos^2(x)}{\cos(x)} = -\sec(x) + \cos(x) \quad .$$

It may also help to review the the integration of the secant function in your old calculus text. After doing so, you'll see that

$$\begin{aligned} u(x) &= \int u'(x)\,dx \\ &= \int \big[\cos(x) - \sec(x)\big]\,dx \\ &= \sin(x) - \ln|\sec(x) + \tan(x)| + c_1 \end{aligned}$$

and

$$v(x) = \int \sin(x)\,dx = -\cos(x) + c_2 \quad .$$

Plugging these formulas for u *and* v *back into our solution formula*

$$y = y_1 u + y_2 v = \cos(x)u + \sin(x)v \quad ,$$

we get

$$\begin{aligned} y &= \cos(x)\big[\sin(x) - \ln|\sec(x) + \tan(x)| + c_1\big] + \sin(x)\big[-\cos(x) + c_2\big] \\ &= -\cos(x)\ln|\sec(x) + \tan(x)| + c_1\cos(x) + c_2\sin(x) \end{aligned}$$

as the general solution to the nonhomogeneous equation

$$y'' + y = \tan(x) \quad .$$

Possible Difficulties

The main 'work' in carrying out the the variation of parameters method to solve

$$ay'' + by' + cy = g$$

is in solving the system

$$\begin{aligned} y_1 u' + y_2 v' &= 0 \\ {y_1}' u' + {y_2}' v' &= \frac{g}{a} \end{aligned}$$

for u' and v', and then integrating the results to get the formulas for u and v. One can foresee two possible issues: (1) the 'solvability' of the system, and (2) the 'integrability' of the solutions to the system (assuming the solutions can be obtained).

Remember, some systems of equations are "degenerate" and not truly solvable, either having no solution or infinitely many solutions. For example,

$$\begin{aligned} u' + 3v' &= 0 \\ u' + 3v' &= 1 \end{aligned}$$

clearly has no solution. Fortunately, this not an issue in the variation of parameters method. As we will see in section 23.3 (when discussing a formula for this method), the requirement that

$\{y_1, y_2\}$ be a fundamental set for the corresponding homogeneous differential equation ensures that the above system is nondegenerate and can be solved uniquely for u' and v'.

The issue of the "integrability" of the formulas obtained for u' and v' is much more significant. In our discussion of a variation of parameters formula (again, in section 23.3), it will be noted that, in theory, the functions obtained for u' and v' will be integrable for any reasonable choice of a, b, c and g. In practice, though, it might not be possible to find usable formulas for the integrals of u' and v'. In these cases it may be necessary to use definite integrals instead of indefinite, numerically evaluating these integrals to obtain approximate values for specific choices of $u(x)$ and $v(x)$. We will discuss this further in section 23.3.

23.2 Variation of Parameters for Even Higher Order Equations

Take another quick look at part of our derivation in the previous section. In setting

$$y = y_1 u + y_2 v$$

and then requiring

$$y_1 u' + y_2 v' = 0 \quad ,$$

we ensured that the formula for y',

$$\begin{aligned} y' = [y_1 u + y_2 v]' &= y_1' u + y_1 u' + y_2' v + y_2 v' \\ &= y_1' u + y_2' v + y_1 u' + y_2 v' = y_1' u + y_2' v + 0 \quad , \end{aligned}$$

contains no derivatives of the unknown functions u and v.

Suppose, instead, that we have three known functions y_1, y_2 and y_3; and we set

$$y = y_1 u + y_2 v + y_3 w$$

where u, v and w are unknown functions. For the same reasons as before, requiring that

$$y_1 u' + y_2 v' + y_3 w' = 0 \tag{23.6}$$

will insure that the formula for y' contains no derivative of u, v and w; but will simply be

$$y' = y_1' u + y_2' v + y_3' w \quad .$$

Differentiating this yields

$$\begin{aligned} y'' &= y_1'' u + y_1' u' + y_2'' v + y_2' v' + y_3'' w + y_3' w' \\ &= [y_1'' u + y_2'' v + y_3'' w] + [y_1' u' + y_2' v' + y_3' w'] \quad , \end{aligned}$$

which reduces to

$$y'' = y_1'' u + y_2'' v + y_3'' w$$

provided we require that

$$y_1' u' + y_2' v' + y_3' w' = 0 \quad . \tag{23.7}$$

Thus, requiring equations (23.6) and (23.7) prevents derivatives of the unknown functions from appearing in the formulas for either y' or y''. As you can easily verify, differentiating the last formula for y'' and plugging the above formulas for y, y' and y'' into a third-order differential equation

$$a_0 y''' + a_1 y'' + a_2 y' + a_3 y = g$$

then yields

$$a_0 \left[{y_1}'' u' + {y_2}'' v' + {y_3}'' w' \right] + [\cdots] u + [\cdots] v + [\cdots] w = g \tag{23.8}$$

where the coefficients in the u, v and w terms will vanish if y_1, y_2 and y_3 are solutions to the corresponding homogeneous differential equation.

Together equations (23.6), (23.7) and (23.8) form a system of three equations in three unknown functions. If you look at this system, and recall the original formula for y, you'll see that we've derived the variation of parameters method for solving third-order nonhomogeneous linear differential equations:

> *To solve the nonhomogeneous differential equation*
>
> $$a_0 y''' + a_1 y'' + a_2 y' + a_3 y = g \quad ,$$
>
> *first find a fundamental set of solutions* $\{y_1, y_2, y_3\}$ *to the corresponding homogeneous equation*
>
> $$a_0 y''' + a_1 y'' + a_2 y' + a_3 y = 0 \quad .$$
>
> *Then set*
>
> $$y = y_1 u + y_2 v + y_3 w \tag{23.9}$$
>
> *assuming that* $u = u(x)$, $v = v(x)$ *and* $w = w(x)$ *are unknown functions whose derivatives satisfy the system*
>
> $$y_1 u' + y_2 v' + y_3 w' = 0 \tag{23.10a}$$
>
> $${y_1}' u' + {y_2}' v' + {y_3}' w' = 0 \tag{23.10b}$$
>
> $${y_1}'' u' + {y_2}'' v' + {y_3}'' w' = \frac{g}{a_0} \quad . \tag{23.10c}$$
>
> *Solve the system for* u', v' *and* w'; *integrate to get the formulas for* u, v *and* w, *and plug the results back into formula (23.9) for* y. *That formula is the solution to the original nonhomogeneous differential equation.*

Extending the method to nonhomogeneous linear equations of even higher order is straightforward. We simply continue to let y be given by formulas similar to formula (23.9) using fundamental sets of solutions to the corresponding homogeneous equations. Repeatedly imposing requirements patterned after equations (23.10a) and (23.10b) to ensure that no derivatives of unknown functions remain until we compute the highest order derivative in the differential equation, we eventually get the variation of parameters method for solving any nonhomogeneous linear differential equation:

> *To solve the* N^{th}*-order nonhomogeneous differential equation*
>
> $$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y' + a_N y = g \quad ,$$

first find a fundamental set of solutions $\{y_1, y_2, \ldots, y_N\}$ *to the corresponding homogeneous equation*

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y' + a_N y = 0 \quad .$$

Then set

$$y = y_1 u_1 + y_2 u_2 + \cdots + y_N u_N \tag{23.11}$$

assuming that the u_k*'s are unknown functions whose derivatives satisfy the system*

$$y_1 {u_1}' + y_2 {u_2}' + \cdots + y_N {u_N}' = 0 \tag{23.12a}$$

$${y_1}' {u_1}' + {y_2}' {u_2}' + \cdots + {y_N}' {u_N}' = 0 \tag{23.12b}$$

$${y_1}'' {u_1}' + {y_2}'' {u_2}' + \cdots + {y_N}'' {u_N}' = 0 \tag{23.12c}$$

$$\vdots \tag{23.12d}$$

$${y_1}^{(N-2)} {u_1}' + {y_2}^{(N-2)} {u_2}' + \cdots + {y_N}^{(N-2)} {u_N}' = 0 \tag{23.12e}$$

$${y_1}^{(N-1)} {u_1}' + {y_2}^{(N-1)} {u_2}' + \cdots + {y_N}^{(N-1)} {u_N}' = \frac{g}{a_0} \quad . \tag{23.12f}$$

Solve the system for ${u_1}'$, ${u_2}'$, $\ldots$ *and* ${u_N}'$; *integrate to get the formula for each* u_k, *and then plug the results back into formula (23.11) for* y. *That formula is the solution to the original nonhomogeneous differential equation.*

As with the second-order case, the above system can be shown to be nondegenerate, and the resulting formula for each ${u_k}'$ can be shown to be integrable, at least in some theoretical sense, as long as g and the a_k's are continuous functions with a_0 never being zero on the interval of interest.

23.3 The Variation of Parameters Formula Second-Order Version with Indefinite Integrals

By solving system (23.5) for u' and v' using generic y_1 and y_2, integrating, and then plugging the result back into formula (23.4)

$$y = y_1 u + y_2 v \quad ,$$

you can show that the solution to

$$ay'' + by' + cy = g \tag{23.13}$$

is given by

$$y(x) = -y_1(x) \int \frac{y_2(x) f(x)}{W(x)} \, dx + y_2(x) \int \frac{y_1(x) f(x)}{W(x)} \, dx \tag{23.14}$$

where

$$f(x) = \frac{g(x)}{a(x)} \quad , \quad W(x) = y_1(x) {y_2}'(x) - {y_1}'(x) y_2(x) \quad .$$

and $\{y_1, y_2\}$ is any fundamental set of solutions to the corresponding homogeneous equation. The details will be left as an exercise (exercise 23.5 on page 468).

A few observations should be made about the elements in formula 23.14:

1. Recall that we have been requiring solutions to second-order differential equations to be continuous and have continuous derivatives. Consequently, the above $W(x)$ will be a continuous function on the interval $\mathcal{I}$.

2. Moreover, if you recall the discussion about the "Wronskian" corresponding to the function set $\{y_1, y_2\}$ (see the discussion in chapter 14 starting on page 304), then you may have noticed that the $W(x)$ in the above formula is that very Wronskian. As noted in theorem 14.3 on page 305, $W(x)$ will be nonzero at every point in our interval of interest, provided a , b and c are continuous functions and a is never zero on that interval.

Consequently, the integrands of the integrals in formula (23.14) will, theoretically at least, be nice integrable functions over our interval of interest as long as a , b , c and g are continuous functions and a is never zero over this interval.[2] And this verifies that, in theory, the variation of parameters method does always yield the solution to a nonhomogeneous linear second-order differential equation over appropriate intervals.

In practice, this author discourages the use of formula (23.14), at least at first. For most, trying to memorize and effectively use this formula is more difficult than remembering the basic system from which it was derived. And the small savings in computational time gained by using this formula is hardly worth the effort unless you are going to be solving many of equations of the form

$$ay'' + by' + cy = g$$

in which the left side remains the same, but you have several different choices for g .

Second-Order Version with Definite Integrals

If you fix a point x_0 in the interval of interest, and rederive formula (23.14) using definite integrals instead of the indefinite ones used just above, you get that a particular solution to

$$ay'' + by' + cy = g$$

is given by

$$y_p(x) = -y_1(x)\int_{x_0}^{x} \frac{y_2(s)\,f(s)}{W(s)}\,ds + y_2(x)\int_{x_0}^{x} \frac{y_1(s)\,f(s)}{W(s)}\,ds \tag{23.15}$$

where, again,

$$f(s) = \frac{g(s)}{a(s)} \quad , \quad W(s) = y_1(s){y_2}'(s) - {y_1}'(s)y_2(s) \quad .$$

and $\{y_1, y_2\}$ is any fundamental set of solutions to the corresponding homogeneous equation.

Then, of course,

$$y(x) = y_p(x) + c_1 y(x) + c_2(x) \quad . \tag{23.16}$$

is the corresponding general solution to original nonhomogeneous differential equation

There are two practical advantages to using definite integral formula (23.15) instead of the corresponding indefinite integral formula, formula (23.14):

[2] In fact, f does not have to even be continuous. It just cannot have particularly bad discontinuities.

1. Often, it is virtually impossible to find usable, explicit formulas for the integrals of
$$\frac{y_2(x)f(x)}{W(x)} \qquad \text{and} \qquad \frac{y_1(x)f(x)}{W(x)} \quad .$$
In these cases, formula (23.14), with its impossible to compute indefinite integrals, is of very little practical value. However, the definite integrals in formula (23.15) can still be accurately approximated for specific values of x using any decent numerical integration method. Thus, while we may not be able to obtain a nice formula for $y_p(x)$, we can still evaluate it at desired points on any reasonable interval of interest, possibly using these values to generate a table for $y_p(x)$ or to sketch its graph.

2. As you can easily show (exercise 23.6), the y_p given by formula (23.15) satisfies the initial conditions
$$y(x_0) = 0 \qquad \text{and} \qquad y'(x_0) = 0 \quad .$$
This makes it a little easier to find the constants c_1 and c_2 such that
$$y(x) = y_p(x) + c_1 y(x) + c_2(x)$$
satisfies initial conditions
$$y(x_0) = A \qquad \text{and} \qquad y'(x_0) = B$$
for some values A and B (especially, if we cannot explicitly compute the integrals in formulas (23.14) and (23.15)).

For Arbitrary Orders

In using variation of parameters to solve the more general N^{th}-order nonhomogeneous differential equation
$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y' + a_N y = g \quad ,$$
we need to solve system (23.12) for $u_1{}'$, $u_2{}'$, ... and $u_N{}'$. If you carefully solve this system for arbitrary y_k's, or simply apply the procedure known as "Cramers rule" (see almost any introductory text in linear algebra), you will discover that
$$u_k{}'(x) = (-1)^{N+k} \frac{W_k(x) f(x)}{W(x)} \qquad \text{for} \quad k = 1, 2, 3, \ldots, n$$
where $f = {}^{g}/_{a_0}$, W is the determinant of the matrix
$$\mathbf{M} = \begin{bmatrix} y_1 & y_2 & y_3 & \cdots & y_N \\ y_1{}' & y_2{}' & y_3{}' & \cdots & y_N{}' \\ y_1{}'' & y_2{}'' & y_3{}'' & \cdots & y_N{}'' \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ y_1{}^{(N-1)} & y_2{}^{(N-1)} & y_3{}^{(N-1)} & \cdots & y_N{}^{(N-1)} \end{bmatrix}$$
and W_k is the determinant of the submatrix of $\mathbf{M}$ obtained by deleting the last row and k^{th} column.

Integrating and plugging into formula (23.11) for the differential equation's solution y, we obtain either

$$y(x) = \sum_{k=1}^{n} (-1)^{N+k} y_k(x) \int \frac{W_k(x) f(x)}{W(x)} \, dx \tag{23.17a}$$

or

$$y_p(x) = \sum_{k=1}^{n} (-1)^{N+k} y_k(x) \int_{x_0}^{x} \frac{W_k(s) f(s)}{W(s)} \, ds \tag{23.17b}$$

depending on whether indefinite or definite integrals are used.

Again, it should be noted that the W in these formulas is the Wronskian of the fundamental set $\{y_1, y_2, \ldots, y_N\}$, and, from the corresponding theory developed for these Wronskians (see, in particular, theorem 14.3 on page 305), it follows that the above integrands will, theoretically at least, be nice integrable functions over our interval of interest as long as g and the a_k's are continuous functions and a_0 is never zero over this interval.[3]

Additional Exercises

23.1. *Find the general solution to each of the following nonhomogeneous differential equations. Use variation of parameters even if another method might seem easier. For your convenience, each equation is accompanied by a general solution to the corresponding homogeneous equation.*

a. $x^2 y'' - 2xy' + 2y = 3\sqrt{x} \quad , \quad y_h = c_1 x + c_2 x^2$

b. $y'' + y = \cot(x) \quad , \quad y_h = c_1 \cos(x) + c_2 \sin(x)$

c. $y'' + 4y = \csc(2x) \quad , \quad y_h = c_1 \cos(2x) + c_2 \sin(2x)$

d. $y'' - 7y' + 10y = 6e^{3x} \quad , \quad y_h = c_1 e^{2x} + c_2 e^{5x}$

e. $y'' - 4y' + 4y = \left[24x^2 + 2\right] e^{2x} \quad , \quad y_h = c_1 e^{2x} + c_2 x e^{2x}$

f. $x^2 y'' + xy' - 9y = 12x^3 \quad , \quad y_h = c_1 x^{-3} + c_2 x^3$

g. $xy'' - y' - 4x^3 y = x^3 e^{x^2} \quad , \quad y_h = c_1 e^{x^2} + c_2 e^{-x^2}$

h. $x^2 y'' - 3xy' + 4y = x^2 \quad , \quad y_h = c_1 x^2 + c_2 x^2 \ln|x|$

i. $xy'' + (2 + 2x)y' + 2y = 8e^{2x} \quad , \quad y_h = c_1 x^{-1} + c_2 x^{-1} e^{-2x}$

j. $(x + 1)y'' + xy' - y = (x + 1)^2 \quad , \quad y_h = c_1 x + c_2 e^{-x}$

23.2. *Solve the following initial-value problems using variation of parameters to find the general solution to the given differential equations.*

a. $x^2 y'' - 2xy' - 4y = \dfrac{10}{x}$ *with* $y(1) = 3$ *and* $y'(1) = -15$

[3] And, again, g does not have to even be continuous. It just cannot have particularly bad discontinuities.

b. $y'' - y' - 6y = 12e^{2x}$ *with* $y(0) = 0$ *and* $y'(0) = 8$

23.3. *Find the general solution to each of the following nonhomogeneous differential equations. Use variation of parameters even if another method might seem easier. For your convenience, each equation is accompanied by a general solution to the corresponding homogeneous equation.*

a. $y''' - 4y' = 30e^{3x}$, $y_h = c_1 + c_2e^{2x} + c_3e^{-2x}$

b. $x^3y''' - 3x^2y'' + 6xy' - 6y = x^3$, $y_h = c_1x + c_2x^2 + c_3x^3$

23.4. *For each of the following, set up the system of equations (corresponding to system 23.5, 23.10 or 23.12) arising in solving the equations via variation of parameters.*

a. $x^3y''' - 3x^2y'' + 6xy' - 6y = e^{-x^2}$, $y_h = c_1x + c_2x^2 + c_3x^3$

b. $y''' - y'' + y' - y = \tan(x)$, $y_h = c_1e^x + c_2\cos(x) + c_3\sin(x)$

c. $y^{(4)} - 81y = \sinh(x)$, $y_h = c_1e^{3x} + c_2e^{-3x} + c_3\cos(3x) + c_4\sin(3x)$

d. $x^4y^{(4)} + 6x^3y''' - 3x^2y'' - 9xy' + 9y = 12x\sin(x^2)$,
$y_h = c_1x + c_2x^{-1} + c_3x^3 + c_4x^{-3}$

23.5. *Derive formula (23.14) on page 464 for the solution* y *to*

$$ay'' + by' + cy = g$$

from the fact that $y = y_1u + y_2v$ *where*

$$y_1u' + y_2v' = 0 \qquad \text{and} \qquad {y_1}'u' + {y_2}'v' = \frac{g}{a} \quad .$$

(Hint: Start by solving the system for u' *and* v' *.)*

23.6. *Show that* y_p *given by formula (23.15) on page 465 satisfies the initial conditions*

$$y(x_0) = 0 \qquad \text{and} \qquad y'(x_0) = 0 \quad .$$