21

Method of Undetermined Coefficients (aka: Method of Educated Guess)

In this chapter, we will discuss one particularly simple-minded, yet often effective, method for finding particular solutions to nonhomogeneous differential equations. As the above title suggests, the method is based on making "good guesses" regarding these particular solutions. And, as always, "good guessing" is usually aided by a thorough understanding of the problem (the 'education'), and usually works best if the problem is simple enough. Fortunately, you have had the necessary education, and a great many nonhomogeneous differential equations of interest are sufficiently simple.

As usual, we will start with second-order equations, and then observe that everything developed also applies, with little modification, to similar nonhomogeneous differential equations of any order.

21.1 Basic Ideas

Suppose we wish to find a particular solution to a nonhomogeneous second-order differential equation

$$ay'' + by' + cy = g \quad .$$

If g is a relatively simple function and the coefficients — a, b and c — are constants, then, after recalling what the derivatives of various basic functions look like, we might be able to make a good guess as to what sort of function $y_p(x)$ yields $g(x)$ after being plugged into the left side of the above equation. Typically, we won't be able to guess exactly what $y_p(x)$ should be, but we can often guess a formula for $y_p(x)$ involving specific functions and some constants that can then be determined by plugging the guessed formula for $y_p(x)$ into the differential equation and solving the resulting algebraic equation(s) for those constants (provided the initial 'guess' was good).

!►Example 21.1: *Consider*

$$y'' - 2y' - 3y = 36e^{5x} \quad .$$

Since all derivatives of e^{5x} *equal some constant multiple of* e^{5x} *, it should be clear that, if we let*

$$y(x) = \text{some multiple of } e^{5x} \quad ,$$

then

$$y'' - 2y' - 3y = \text{some other multiple of } e^{5x} \quad .$$

So let us let A *be some constant "to be determined", and try*

$$y_p(x) = Ae^{5x}$$

as a particular solution to our differential equation:

$$
\begin{aligned}
 & & y_p'' - 2y_p' - 3y_p &= 36e^{5x} \\
\Longrightarrow & & \left[Ae^{5x}\right]'' - 2\left[Ae^{5x}\right]' - 3\left[Ae^{5x}\right] &= 36e^{5x} \\
\Longrightarrow & & \left[25Ae^{5x}\right] - 2\left[5Ae^{5x}\right] - 3\left[Ae^{5x}\right] &= 36e^{5x} \\
\Longrightarrow & & 25Ae^{5x} - 10Ae^{5x} - 3Ae^{5x} &= 36e^{5x} \\
\Longrightarrow & & 12Ae^{5x} &= 36e^{5x} \\
\Longrightarrow & & A &= 3 \quad .
\end{aligned}
$$

So our "guess", $y_p(x) = Ae^{5x}$ *, satisfies the differential equation only if* $A = 3$*. Thus,*

$$y_p(x) = 3e^{5x}$$

is a particular solution to our nonhomogeneous differential equation.

In the next section, we will determine the appropriate "first guesses" for particular solutions corresponding to different choices of g in our differential equation. These guesses will involve specific functions and initially unknown constants that can be determined as we determined A in the last example. Unfortunately, as we will see, the first guesses will sometimes fail. So we will discuss appropriate second (and, when necessary, third) guesses, as well as when to expect the first (and second) guesses to fail.

Because all of the guesses will be linear combinations of functions in which the coefficients are "constants to be determined", this whole approach to finding particular solutions is formally called the *method of undetermined coefficients*. Less formally, it is also called the *method of (educated) guess*.

Keep in mind that this method only finds a particular solution for a differential equation. In practice, we really need the general solution, which (as we know from our discussion in the previous chapter) can be constructed from any particular solution along the general solution to the corresponding homogeneous equation (see theorem 20.1 and corollary 20.2 on page 411).

!►*Example 21.2:* *Consider finding the general solution to*

$$y'' - 2y' - 3y = 36e^{5x} \quad .$$

From the last example, we know

$$y_p(x) = 3e^{5x}$$

is a particular solution to the differential equation. The corresponding homogeneous equation is

$$y'' - 2y' - 3y = 0 \quad .$$

Its characteristic equation is

$$r^2 - 2r - 3 = 0 \quad ,$$

which factors as

$$(r+1)(r-3) = 0 \quad .$$

So $r = -1$ *and* $r = 3$ *are the possible values of* r *, and*

$$y_h(x) = c_1 e^{-x} + c_2 e^{3x}$$

is the general solution to the corresponding homogeneous differential equation.

As noted in corollary 20.2, it then follows that

$$y(x) = y_p(x) + y_h(x) = 3e^{5x} + c_1 e^{-x} + c_2 e^{3x} \quad .$$

is a general solution to our nonhomogeneous differential equation.

Also keep in mind that you may not just want the general solution, but also the one solution that satisfies some particular initial conditions.

!►*Example 21.3:* *Consider the initial-value problem*

$$y'' - 2y' - 3y = 36e^{5x} \qquad \text{with} \quad y(0) = 9 \quad \text{and} \quad y'(0) = 25 \quad .$$

From above, we know the general solution to the differential equation is

$$y(x) = 3e^{5x} + c_1 e^{-x} + c_2 e^{3x} \quad .$$

Its derivative is

$$y'(x) = \left[3e^{5x} + c_1 e^{-x} + c_2 e^{3x}\right]' = 15e^{5x} - c_1 e^{-x} + 3c_2 e^{3x} \quad .$$

This, with our initial conditions, gives us

$$9 = y(0) = 3e^{5\cdot 0} + c_1 e^{-0} + c_2 e^{3\cdot 0} = 3 + c_1 + c_2$$

and

$$25 = y'(0) = 15e^{5\cdot 0} - c_1 e^{-0} + 3c_2 e^{3\cdot 0} = 15 - c_1 + 3c_2 \quad ,$$

which, after a little arithmetic, becomes the system

$$\begin{aligned} c_1 + c_2 &= 6 \\ -c_1 + 3c_2 &= 10 \end{aligned} \quad .$$

Solving this system by whatever means you prefer yields

$$c_1 = 2 \qquad \text{and} \qquad c_2 = 4 \quad .$$

So the solution to the given differential equation that also satisfies the given initial conditions is

$$y(x) = 3e^{5x} + c_1 e^{-x} + c_2 e^{3x} = 3e^{5x} + 2e^{-x} + 4e^{3x} \quad .$$

21.2 Good First Guesses For Various Choices of g

In all of the following, we are interested in finding a particular solution $y_p(x)$ to

$$ay'' + by' + cy = g \tag{21.1}$$

where a, b and c are constants and g is the indicated type of function. In each subsection, we will describe a class of functions for g and the corresponding 'first guess' as to the formula for a particular solution y_p. In each case, the formula will involve constants "to be determined". These constants are then determined by plugging the guessed formula for y_p into the differential equation and solving the system of algebraic equations that, with luck, results. Of course, if the resulting equations are not solvable for those constants, then the first guess is not adequate and you'll have to read the next section to learn a good 'second guess'.

Exponentials

As illustrated in example 21.1,

If, for some constants C and α,

$$g(x) = Ce^{\alpha x}$$

then a good first guess for a particular solution to differential equation (21.1) is

$$y_p(x) = Ae^{\alpha x}$$

where A is a constant to be determined.

Sines and Cosines

!▶*Example 21.4:* *Consider*

$$y'' - 2y' - 3y = 65\cos(2x) \quad .$$

A naive first guess for a particular solution might be

$$y_p(x) = A\cos(2x) \quad ,$$

where A is some constant to be determined. Unfortunately, here is what we get when plug this guess into the differential equation:

$$\begin{aligned}
& {y_p}'' - 2{y_p}' - 3y_p = 65\cos(2x) \\
\Longrightarrow \quad & [A\cos(2x)]'' - 2[A\cos(2x)]' - 3[A\cos(2x)] = 65\cos(2x) \\
\Longrightarrow \quad & -4A\cos(2x) + 4A\sin(2x) - 3A\cos(2x) = 65\cos(2x) \\
\Longrightarrow \quad & A[-7\cos(2x) + 4\sin(2x)] = 65\cos(2x) \quad .
\end{aligned}$$

But there is no constant A satisfying this last equation for all values of x. So our naive first guess will not work.

Since our naive first guess resulted in an equation involving both sines and cosines, let us add a sine term to the guess and see if we can get all the resulting sines and cosines in the resulting equation to balance. That is, assume

$$y_p(x) = A\cos(2x) + B\sin(2x)$$

where A and B are constants to be determined. Plugging this into the differential equation:

$$\begin{aligned}
& {y_p}'' - 2{y_p}' - 3y_p = 65\cos(2x) \\
\Longrightarrow\quad & [A\cos(2x) + B\sin(2x)]'' - 2[A\cos(2x) + B\sin(2x)]' \\
& \qquad - 3[A\cos(2x) + B\sin(2x)] = 65\cos(2x) \\
\Longrightarrow\quad & -4A\cos(2x) - 4B\sin(2x) - 2[-2A\sin(2x) + 2B\cos(2x)] \\
& \qquad - 3[A\cos(2x) + B\sin(2x)] = 65\cos(2x) \\
\Longrightarrow\quad & (-7A - 4B)\cos(2x) + (4A - 7B)\sin(2x) = 65\cos(2x) \quad .
\end{aligned}$$

For the cosine terms on the two sides of the last equation to balance, we need

$$-7A - 4B = 65 \quad ,$$

and for the sine terms to balance, we need

$$4A - 7B = 0 \quad .$$

This gives us a relatively simple system of two equations in two unknowns. Its solution is easily found. From the second equation, we have

$$B = \frac{4}{7}A \quad .$$

Combining this with the first equation yields

$$65 = -7A - 4\left[\frac{4}{7}A\right] = \left[-\frac{49}{7} - \frac{16}{7}\right]A = -\frac{65}{7}A \quad .$$

Thus,

$$A = -7 \qquad \text{and} \qquad B = \frac{4}{7}A = \frac{4}{7}(-7) = -4 \quad ,$$

and a particular solution to the differential equation is given by

$$y_p(x) = A\cos(2x) + B\sin(2x) = -7\cos(2x) - 4\sin(2x) \quad .$$

This example illustrates that, typically, if $g(x)$ is a sine or cosine function (or a linear combination of a sine and cosine function with the same frequency) then a linear combination of both the sine and cosine can be used for $y_p(x)$. Thus, we have the following rule:

If, for some constants C_c, C_s and ω,

$$g(x) = C_c\cos(\omega x) + C_s\sin(\omega x)$$

then a good first guess for a particular solution to differential equation (21.1) is

$$y_p(x) = A\cos(\omega x) + B\sin(\omega x)$$

where A and B are constants to be determined.

Polynomials

!►*Example 21.5:* *Let us find a particular solution to*

$$y'' - 2y' - 3y = 9x^2 + 1 \quad .$$

Now consider, if y is any polynomial of degree N, then y, y' and y'' are also polynomials of degree N or less. So the expression "$y'' - 2y' - 3y$" would then be a polynomial of degree N. Since we want this to match the right side of the above differential equation, which is a polynomial of degree 2, it seems reasonable to try a polynomial of degree N with $N = 2$. So we "guess"

$$y_p(x) = Ax^2 + Bx + C \quad .$$

In this case

$$y_p{}'(x) = 2Ax + B \qquad \text{and} \qquad y_p{}''(x) = 2A \quad .$$

Plugging these into the differential equation:

$$\begin{aligned} & y_p{}'' - 2y_p{}' - 3y_p = 9x^2 + 1 \\ \Longrightarrow \quad & 2A - 2[2Ax + B] - 3[Ax^2 + Bx + C] = 9x^2 + 1 \\ \Longrightarrow \quad & -3Ax^2 + [-4A - 3B]x + [2A - 2B - 3C] = 9x^2 + 1 \quad . \end{aligned}$$

For the last equation to hold, the corresponding coefficients to the polynomials on the two sides must equal, giving us the following system:

$$\begin{aligned} x^2 \text{ terms:} \qquad & -3A = 9 \\ x \text{ terms:} \qquad & -4A - 3B = 0 \quad . \\ \text{constant terms:} \qquad & 2A - 2B - 3C = 1 \end{aligned}$$

So,

$$A = -\frac{9}{3} = -3 \quad ,$$

$$B = -\frac{4A}{3} = -\frac{4(-3)}{3} = 4$$

and

$$C = \frac{1 - 2A + 2B}{-3} = \frac{1 - 2(-3) + 2(4)}{-3} = \frac{15}{-3} = -5 \quad .$$

And the particular solution is

$$y_p(x) = Ax^2 + Bx + C = -3x^2 + 4x - 5 \quad .$$

Generalizing from this example, we can see that the rule for the first guess for $y_p(x)$ when g is a polynomial is:

> *If*
>
> $$g(x) = \text{a polynomial of degree } K \quad ,$$
>
> *then a good first guess for a particular solution to differential equation (21.1) is a K^{th}-degree polynomial*
>
> $$y_p(x) = A_0x^K + A_1x^{K-1} + \cdots + A_{K-1}x + A_K$$
>
> *where the A_k's are constants to be determined.*

Products of Exponentials, Polynomials, and Sines and Cosines

If g is a product of the simple functions discussed above, then the guess for y_p must take into account everything discussed above. That leads to the following rule:

If, for some pair of polynomials $P(x)$ and $Q(x)$, and some pair of constants α and ω,

$$g(x) = P(x)e^{\alpha x}\cos(\omega x) + Q(x)e^{\alpha x}\sin(\omega x)$$

then a good first guess for a particular solution to differential equation (21.1) is

$$y_p(x) = \left[A_0x^K + A_1x^{K-1} + \cdots + A_{K-1}x + A_K\right]e^{\alpha x}\cos(\omega x) + \left[B_0x^K + B_1x^{K-1} + \cdots + B_{K-1}x + B_K\right]e^{\alpha x}\sin(\omega x)$$

where the A_k's and B_k's are constants to be determined and K is the highest power of x appearing in polynomial $P(x)$ or $Q(x)$.

(Note that the above include the cases where $\alpha = 0$ or $\omega = 0$. In these cases the formula for y_p simplifies a bit.)

!►*Example 21.6:* *To find a particular solution to*

$$y'' - 2y' - 3y = 65x\cos(2x) \quad ,$$

we should start by assuming it is of the form

$$y_p(x) = [A_0x + A_1]\cos(2x) + [B_0x + B_1]\sin(2x) \quad .$$

With a bit of work, you can verify yourself that, with $y = y_p(x)$, the above differential equation reduces to

$$[-2A_0 - 7A_1 + 4B_0 - 4B_1]\cos(2x) + [-7A_0 - 4B_0]x\cos(2x) + [-4A_0 + 4A_1 - 2B_0 - 7B_1]\sin(2x) + [4A_0 - 7B_0]x\sin(2x) = 65x\cos(2x) \quad .$$

Comparing the terms on either side of the last equation, we get the following system:

$$\begin{aligned}
\cos(2x) \textit{ terms:} \quad & -2A_0 - 7A_1 + 4B_0 - 4B_1 = 0 \\
x\cos(2x) \textit{ terms:} \quad & -7A_0 \qquad - 4B_0 \qquad = 65 \\
\sin(2x) \textit{ terms:} \quad & -4A_0 + 4A_1 - 2B_0 - 7B_1 = 0 \\
x\sin(2x) \textit{ terms:} \quad & 4A_0 \qquad - 7B_0 \qquad = 0
\end{aligned}$$

Solving this system yields

$$A_0 = -7 \quad , \quad A_1 = -\frac{158}{65} \quad , \quad B_0 = -4 \quad . \quad \textit{and} \quad B_1 = \frac{244}{65} \quad .$$

So a particular solution to the differential equation is given by

$$y_p(x) = \left[-7x - \frac{158}{65}\right]\cos(2x) + \left[-4x + \frac{244}{65}\right]\sin(2x) \quad .$$

21.3 When the First Guess Fails

!►*Example 21.7:* *Consider*

$$y'' - 2y' - 3y = 28e^{3x} \quad .$$

Our first guess is

$$y_p(x) = Ae^{3x} \quad .$$

Plugging it into the differential equation:

$$\begin{aligned} & & y_p{}'' - 2y_p{}' - 3y_p &= 28e^{3x} \\ &\Longrightarrow & \left[Ae^{3x}\right]'' - 2\left[Ae^{3x}\right]' - 3\left[Ae^{3x}\right] &= 28e^{3x} \\ &\Longrightarrow & \left[9Ae^{3x}\right] - 2\left[3Ae^{3x}\right] - 3\left[Ae^{3x}\right] &= 28e^{3x} \\ &\Longrightarrow & 9Ae^{3x} - 6Ae^{3x} - 3Ae^{3x} &= 28e^{3x} \quad . \end{aligned}$$

But when we add up the left side of the last equation, we get the impossible equation

$$0 = 28e^{3x} \quad !$$

No value for A can make this equation true! So our first guess fails.

Why did it fail? Because the guess, Ae^{3x} was already a solution to the corresponding homogeneous equation

$$y'' - 2y' - 3y = 0 \quad ,$$

which we would have realized if we had recalled the general solution to this homogeneous differential equation. So the left side of our differential equation will have to vanish when we plug in this guess, leaving us with an 'impossible' equation.

In general, whenever our first guess for a particular solution contains a term that is also a solution to the corresponding homogeneous differential equation, then the contribution of that term to

$$ay_p{}'' + by_p{}' + cy_p = g$$

vanishes, and we are left with an equation or a system of equations with no possible solution. In these cases, we can still attempt to solve the problem using the first guess with the reduction of order method mentioned in the previous chapter. To save time, though, I will tell you what would happen: You would discover that, if the first guess fails, then there is a particular solution of the form

$$x \times \text{"the first guess"}$$

unless this formula also contains a term satisfying the corresponding homogeneous differential equation, in which case there is a particular solution of the form

$$x^2 \times \text{"the first guess"} \quad .$$

Thus, instead of using reduction of order (or the method we'll learn in the next chapter), we can apply the following rules for generating the *appropriate guess* for the form for a particular solution $y_p(x)$ (given that we've already figured out the first guess using the rules in the previous section):

If the first guess for $y_p(x)$ contains a term that is also a solution to the corresponding homogeneous differential equation, then consider

$$x \times \text{“the first guess”}$$

as a “second guess”. If this (after multiplying through by the x) does not contain a term satisfying the corresponding homogeneous differential equation, then set

$$y_p(x) = \text{“second guess”} = x \times \text{“the first guess”} \quad .$$

If, however, the second guess also contains a term satisfying the corresponding homogeneous differential equation, then set

$$y_p(x) = \text{“the third guess”}$$

where

$$\text{“third guess”} = x \times \text{“the second guess”} = x^2 \times \text{“the first guess”} \quad .$$

I should emphasize that the second guess is used only if the first fails (i.e., has a term that satisfies the homogeneous equation). If the first guess works, then the second (and third) guesses will not work. Likewise, if the second guess works, the third guess is not only unnecessary, it will not work. If, however the first and second guesses fail, you can be sure that the third guess will work.

!►*Example 21.8:* *Again, consider*

$$y'' - 2y' - 3y = 28e^{3x} \quad .$$

Our first guess

$$Ae^{3x}$$

was a solution to the corresponding homogeneous differential equation. So we try a second guess of the form

$$x \times \text{“first guess”} = x \times Ae^{3x} = Axe^{3x} \quad .$$

Comparing this (our second guess) to the general solution to the general solution

$$y_h(x) = c_1e^{-x} + c_1e^{3x}$$

of the corresponding homogeneous equation (see exercise 21.2), we see that our second guess is not a solution to the corresponding homogeneous differential equation, and, so, we can find a particular solution to our nonhomogeneous differential equation by setting

$$y_p(x) = \text{“second guess”} = Axe^{3x} \quad .$$

The first two derivatives of this are

$$y_p{}'(x) = Ae^{3x} + 3Axe^{3x}$$

and

$$y_p{}''(x) = 3Ae^{3x} + 3Axe^{3x} + 9Axe^{3x} = 6Ae^{3x} + 9Axe^{3x} \quad .$$

Using this:

$$
\begin{aligned}
&& {y_p}'' - 2{y_p}' - 3y_p &= 28e^{3x} \\
\Longrightarrow && \left[Axe^{3x}\right]'' - 2\left[Axe^{3x}\right]' - 3\left[Axe^{3x}\right] &= 28e^{3x} \\
\Longrightarrow && \left[6Ae^{3x} + 9Axe^{3x}\right] - 2\left[Ae^{3x} + 3Axe^{3x}\right] - 3Axe^{3x} &= 28e^{3x} \\
\Longrightarrow && \underbrace{[9 - 2(3) - 3]}_{0} Axe^{3x} + [6 - 2]Ae^{3x} &= 28e^{3x} \\
\Longrightarrow && 4Ae^{3x} &= 28e^{3x} \quad .
\end{aligned}
$$

Thus,

$$A = \frac{28}{4} = 7$$

and

$$y_p(x) = 7xe^{3x} \quad .$$

21.4 Method of Guess in General

If you think about why the method of (educated) guess works with second-order equations, you will realize that this basic approach will work just as well with any linear differential equation with constant coefficients,

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y' + a_N y = g \quad ,$$

provided the $g(x)$ is any of the types of functions already discussed. The appropriate first guesses are exactly the same, and, if a term in one 'guess' happens to already satisfy the corresponding homogeneous differential equation, then x times that guess will be an appropriate 'next guess'. The only modification in our method is that, with higher order equations, we may have to go to a fourth guess or a fifth guess or $\ldots$.

!►*Example 21.9:* *Consider the seventh-order nonhomogeneous differential equation*

$$y^{(7)} - 625y^{(3)} = 6e^{2x} \quad .$$

An appropriate first guess for a particular solution is still

$$y_p(x) = Ae^{2x} \quad .$$

Plugging this guess into the differential equation:

$$
\begin{aligned}
&& {y_p}^{(7)} - 625{y_p}^{(3)} &= 6e^{2x} \\
\Longrightarrow && \left[Ae^{2x}\right]^{(7)} - 625\left[Ae^{2x}\right]^{(3)} &= 6e^{2x} \\
\Longrightarrow && 2^7 Ae^{2x} - 625 \cdot 2^3 Ae^{2x} &= 6e^{2x}
\end{aligned}
$$

$$\Longrightarrow \qquad 128Ae^{2x} - 5{,}000Ae^{2x} = 6e^{2x}$$

$$\Longrightarrow \qquad -4{,}872Ae^{2x} = 6e^{2x}$$

$$\Longrightarrow \qquad A = -\frac{6}{4{,}872} = -\frac{1}{812} \quad .$$

So a particular solution to our differential equation is

$$y_p(x) = -\frac{1}{812}e^{2x} \quad .$$

Fortunately, we dealt with the corresponding homogeneous equation,

$$y^{(7)} - 625y^{(3)} = 0 \quad ,$$

in example 18.6 on page 382. Looking back at that example, we see that the general solution to this homogeneous differential equation is

$$y_h(x) = c_1 + c_2x + c_3x^2 + c_4e^{5x} + c_5e^{-5x} + c_6\cos(5x) + c_7\sin(5x) \quad . \tag{21.2}$$

Thus, the general solution to our nonhomogeneous equation,

$$y^{(7)} - 625y^{(3)} = 6e^{2x} \quad ,$$

is

$$\begin{aligned} y(x) &= y_p(x) + y_h(x) \\ &= -\frac{1}{809}e^{2x} + c_1 + c_2x + c_3x^2 + c_4e^{5x} + c_5e^{-5x} \\ &\qquad + c_6\cos(5x) + c_7\sin(5x) \quad . \end{aligned}$$

!►*Example 21.10:* *Now consider the nonhomogeneous equation*

$$y^{(7)} - 625y^{(3)} = 300x + 50 \quad .$$

Since the right side is a polynomial of degree one, the appropriate first guess for a particular solution is

$$y_p(x) = Ax + B \quad .$$

However, the general solution to the corresponding homogeneous equation (formula (21.2), above) contains both a constant term and an cx *term. So plugging this guess into the nonhomogeneous differential equation will yield the impossible equation*

$$0 = 300x + 50 \quad .$$

Likewise, both terms of the second guess,

$$x \times \text{“first guess”} = x \times (Ax + B) = Ax^2 + Bx \quad ,$$

and the last term of the third guess,

$$x \times \text{“second guess”} = x \times (Ax^2 + Bx) = Ax^3 + Bx^2 \quad ,$$

satisfy the corresponding homogeneous differential equation, and, thus, would fail. The fourth guess,

$$x \times \text{“third guess”} = x \times (Ax^3 + Bx^2) = Ax^4 + Bx^3 \quad ,$$

has no terms in common with the general solution to the corresponding homogeneous equation (formula (21.2), above). So the appropriate “guess” here is

$$y_p(x) = \text{“fourth guess”} = Ax^4 + Bx^3 \quad .$$

Using this:

$$y_p{}^{(7)} - 625 y_p{}^{(3)} = 300x + 50$$

$$\Longrightarrow \quad \left[Ax^4 + Bx^3\right]^{(7)} - 625\left[Ax^4 + Bx^3\right]^{(3)} = 300x + 50$$

$$\Longrightarrow \quad 0 - 625[A \cdot 4 \cdot 3 \cdot 2x + B \cdot 3 \cdot 2 \cdot 1] = 300x + 50$$

$$\Longrightarrow \quad -15{,}000Ax - 3{,}750B = 300x + 50 \quad .$$

Thus,

$$A = -\frac{300}{15{,}000} = -\frac{1}{50} \qquad \text{and} \qquad B = -\frac{50}{3{,}750} = -\frac{1}{75} \quad ,$$

and a particular solution to our nonhomogeneous differential equation is given by

$$y_p(x) = -\frac{x^4}{50} - \frac{x^3}{75} \quad .$$

For the sake of completeness, let me end our development of the method of (educated) guess (more properly called the method of undetermined coefficients) with a theorem that does two things:

1. It concisely summarizes the rules we’ve developed in this chapter. (But its conciseness may make it too dense to be easily used — so just remember the rules we’ve developed instead of memorizing this theorem.)

2. It assures us that the method we’ve just developed will always work.

Theorem 21.1

Suppose we have a nonhomogeneous linear differential equation with constant coefficients

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-1} y' + a_N y = g$$

where

$$g(x) = P(x)e^{\alpha x}\cos(\omega x) + Q(x)e^{\alpha x}\sin(\omega x)$$

for some pair of polynomials $P(x)$ and $Q(x)$, and some pair of constants α and ω. Let K be the highest power of x appearing in $P(x)$ or $Q(x)$, and let M be the smallest nonnegative integer such that

$$x^M e^{\alpha x}\cos(\omega x)$$

is not a solution to the corresponding homogeneous differential equation.

Then there are constants A_0 , A_1 , ... *and* A_K *, and constants* B_0 , B_1 , ... *and* B_K *such that*

$$y_p(x) = x^M\big[A_0x^K + A_1x^{K-1} + \cdots + A_{K-1}x + A_K\big]e^{\alpha x}\cos(\omega x) + x^M\big[B_0x^K + B_1x^{K-1} + \cdots + B_{K-1}x + B_K\big]e^{\alpha x}\sin(\omega x)$$

is a particular solution to the given nonhomogeneous differential equation.

Proving this theorem is not that difficult, provided you have the right tools. Those who are interested can turn to section 21.6 for the details.

21.5 Using the Principle of Superposition

Suppose we have a nonhomogeneous linear differential equation with constant coefficients

$$a_0y^{(N)} + a_1y^{(N-1)} + \cdots + a_{N-1}y' + a_Ny = g$$

where g is the sum of functions,

$$g(x) = g_1(x) + g_2(x) + \cdots \quad .$$

with each of the g_k's requiring a different 'guess' for y_p . One approach to finding a particular solution $y_p(x)$ to this is to construct a big guess by adding together all the guesses suggested by the g_k's . This typically leads to rather lengthy formulas and requires keeping track of many undetermined constants, and that often leads to errors in computations — errors that, themselves, may be difficult to recognize or track down.

Another approach is to break down the differential equation to a collection of slightly simpler differential equations,

$$a_0y^{(N)} + a_1y^{(N-1)} + \cdots + a_{N-1}y' + a_Ny = g_1 \quad ,$$

$$a_0y^{(N)} + a_1y^{(N-1)} + \cdots + a_{N-1}y' + a_Ny = g_2 \quad ,$$

$$\vdots$$

and, for each g_k , find a particular solution $y = y_{pk}$ to

$$a_0y^{(N)} + a_1y^{(N-1)} + \cdots + a_{N-1}y' + a_Ny = g_k \quad .$$

By the principle of superposition, we know that a particular solution to the differential equation of interest,

$$a_0y^{(N)} + a_1y^{(N-1)} + \cdots + a_{N-1}y' + a_Ny = g_1 + g_2 + \cdots \quad ,$$

can then be constructed by simply adding up the y_{pk}'s ,

$$y_p(x) = y_{p1}(x) + y_{p2}(x) + \cdots \quad .$$

Typically, the total amount of computational work is essentially the same for either approach. Still the approach of breaking the problem into simpler problems and using superposition is usually considered to be easier to actually carry out since we are dealing with smaller formulas and fewer variables at each step.

!▶Example 21.11: *Consider*

$$y'' - 2y' - 3y = 65\cos(2x) + 9x^2 + 1 \quad .$$

Because

$$g_1(x) = 65\cos(2x) \qquad \text{and} \qquad g_2(x) = 9x^2 + 1$$

lead to different initial guesses for $y_p(x)$, we will break this problem into the separate problems of finding particular solutions to

$$y'' - 2y' - 3y = 65\cos(2x)$$

and

$$y'' - 2y' - 3y = 9x^2 + 1 \quad .$$

Fortunately, these happen to be differential equations considered in previous examples. From example 21.4 we know that a particular solution to the first of these two equations is

$$y_{p1}(x) = -7\cos(2x) - 4\sin(2x) \quad ,$$

and from example 21.5 we know that a particular solution to the second of these two equations is

$$y_{p2}(x) = -3x^2 + 4x - 5 \quad .$$

So, by the principle of superposition, we have that a particular solution to

$$y'' - 2y' - 3y = 65\cos(2x) + x^2 + 1$$

is given by

$$\begin{aligned} y_p(x) &= y_{p1}(x) + y_{p2}(x) \\ &= -7\cos(2x) - 4\sin(2x) - 3x^2 + 4x - 5 \quad . \end{aligned}$$

21.6 On Verifying Theorem 21.1

Theorem 21.1, which confirms our "method of guess", is the main theorem of this chapter. Its proof follows relatively easily using some of the ideas developed in sections 12.4 and 18.4 on multiplying and factoring linear differential operators.

A Useful Lemma

Rather than tackle the proof of theorem 21.1 directly, we will first prove the following lemma. This lemma contains much of our theorem, and its proof nicely illustrates the main ideas in the proof of the main theorem. After this lemma's proof, we'll see about proving our main theorem.

Lemma 21.2
Let L be a linear differential operator with constant coefficients, and assume y_p is a function satisfying

$$L[y_p] = g$$

where, for some nonnegative integer K and constants ρ, b_0, b_1, $\ldots$ and b_K,

$$g(x) = b_0 x^K e^{\rho x} + b_1 x^{K-1} e^{\rho x} + \cdots + b_{K-1} x e^{\rho x} + b_K e^{\rho x} \quad .$$

Then there are constants A_0, A_1, $\ldots$ and A_K such that

$$y_p(x) = x^M \big[A_0 x^K + A_1 x^{K-1} + \cdots + A_{K-1} x + A_K \big] e^{\rho x} \quad .$$

where

$$M = \begin{cases} \text{multiplicity of } \rho & \text{if } \rho \text{ is a root of } L\text{'s characteristic polynomial} \\ 0 & \text{if } \rho \text{ is not a root of } L\text{'s characteristic polynomial} \end{cases} \quad .$$

PROOF: Let y_q be any function on the real line satisfying

$$L[y_q] = g \quad .$$

(Theorem 11.4 on page 255 assures us that such a function exists.) Applying an equality from lemma 18.7 on page 388, you can easily verify that

$$\left(\frac{d}{dx} - \rho\right)^{K+1} [g] = 0 \quad ,$$

Hence,

$$\left(\frac{d}{dx} - \rho\right)^{K+1} \big[L[y_q]\big] = \left(\frac{d}{dx} - \rho\right)^{K+1} [g] = 0 \quad .$$

That is, y_q is a solution to the homogeneous linear differential equation with constant coefficients

$$\left(\frac{d}{dx} - \rho\right)^{K+1} L[y] = 0 \quad .$$

Letting r_1, r_2, $\ldots$ and r_L be all the roots other than ρ to the characteristic polynomial for L, we can factor the characteristic equation for the last differential equation above to

$$a(r-\rho)^{K+1}(r-r_1)^{m_1}(r-r_2)^{m_2}\cdots(r-r_L)^{m_L}(r-\rho)^M = 0 \quad .$$

Equivalently,

$$a(r-r_1)^{m_1}(r-r_2)^{m_2}\cdots(r-r_L)^{m_L}(r-\rho)^{M+K+1} = 0 \quad .$$

From what we learned about the general solutions to homogeneous linear differential equations with constant coefficients in chapter 18, we know that

$$y_q(x) = Y_1(x) + Y_\rho(x)$$

where Y_1 is a linear combination of the $x^k e^{rx}$'s arising from the roots other than ρ, and

$$Y_\rho(x) = C_0 e^{\rho x} + C_1 x e^{\rho x} + C_2 x^2 e^{\rho} + \cdots + C_{M+K} x^{M+K} e^{\rho x}$$

Now let $Y_{\rho,0}$ consist of the first M terms of Y_ρ, and set

$$y_p = y_q - Y_1 - Y_{\rho,0}$$

Observe that, while y_q is a solution to the nonhomogeneous differential equation $L[y] = g$, every term in $Y_1(x)$ and $Y_{\rho,0}(x)$ is a solution to the corresponding homogeneous differential equation $L[y] = 0$. Hence,

$$L[y_p] = L[y_q - Y_1 - Y_{\rho,0}] = L[y_q] - L[Y_1] - L[Y_{\rho,0}] = g - 0 - 0 \quad .$$

So y_p is a solution to the nonhomogeneous differential equation in the lemma. Moreover,

$$\begin{aligned} y_p(x) &= y_q - Y_1 - Y_{\rho,0} \\ &= Y_\rho(x) - \text{the first } M \text{ terms of } Y_\rho(x) \\ &= C_M x^M e^{\rho x} + C_{M+1} x^{M+1} x e^{\rho x} + C_{M+2} x^{M+2} e^{\rho} + \cdots + C_{M+K} x^{M+K} e^{\rho x} \\ &= x^M \left[C_M + C_{M+1} x + C_{M+2} x^2 + \cdots + C_{M+K} x^K \right] e^{\rho x} \quad , \end{aligned}$$

which, except for minor cosmetic differences, is the formula for y_p claimed in the lemma. ∎

Proving the Main Theorem

Take a look at theorem 21.1 on page 428. Observe that, if you set $\rho = \alpha$, then our lemma is just a restatement of that theorem with the additional assumption that $\omega = 0$. So the claims of theorem 21.1 follow immediately from our lemma when $\omega = 0$.

Verifying the claims of theorem 21.1 when $\omega \neq 0$ requires just a little more work. Simply let

$$\rho = \alpha + i\omega$$

and redo the lemma's proof (making the obvious modifications) with the double factor

$$\left(\frac{d}{dx} - \rho\right)\left(\frac{d}{dx} - \rho^*\right)$$

replacing the single factor

$$\left(\frac{d}{dx} - \rho\right) \quad ,$$

and keeping in mind what you know about the solutions to a homogeneous differential equation with constant coefficients corresponding to the complex roots of the characteristic polynomial. I'll leave the details to you.

Additional Exercises

21.1 a. *Find both a particular solution (via the method of educated guess) and a general solution to each of the following:*

i. $y'' + 9y = 52e^{2x}$ **ii.** $y'' - 6y' + 9y = 27e^{6x}$

iii. $y'' + 4y' - 5y = 30e^{-4x}$ **iv.** $y'' + 3y' = e^{x/2}$

b. *Solve the initial-value problem*

$$y'' - 3y' - 10y = -5e^{3x} \quad \text{with} \quad y(0) = 5 \quad \text{and} \quad y'(0) = 3 \quad .$$

21.2 a. *Find both a particular solution (via the method of educated guess) and a general solution to each of the following:*

i. $y'' + 9y = 10\cos(2x) + 15\sin(2x)$ **ii.** $y'' - 6y' + 9y = 25\sin(6x)$

iii. $y'' + 3y' = 26\cos\left(\frac{x}{3}\right) - 12\sin\left(\frac{x}{3}\right)$ **iv.** $y'' + 4y' - 5y = \cos(x)$

b. *Solve the initial-value problem*

$$y'' - 3y' - 10y = -4\cos(x) + 7\sin(x) \quad \text{with} \quad y(0) = 8 \quad \text{and} \quad y'(0) = -5 \quad .$$

21.3 a. *Find both a particular solution (via the method of educated guess) and a general solution to each of the following:*

i. $y'' - 3y' - 10y = -200$ **ii.** $y'' + 4y' - 5y = x^3$

iii. $y'' - 6y' + 9y = 18x^2 + 3x + 4$ **iv.** $y'' + 9y = 9x^4 - 9$

b. *Solve the initial-value problem*

$$y'' + 9y = x^3 \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0 \quad .$$

21.4 a. *Find both a particular solution (via the method of educated guess) and a general solution to each of the following:*

i. $y'' + 9y = 25x\cos(2x)$ **ii.** $y'' - 6y' + 9y = e^{2x}\sin(x)$

iii. $y'' + 9y = 54x^2e^{3x}$ **iv.** $y'' = 6xe^x\sin(x)$

b. *Solve the initial-value problem*

$$y'' + 9y = 39xe^{2x} \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 0 \quad .$$

21.5. *Find both a particular solution (via the method of educated guess) and a general solution to each of the following:*

a. $y'' - 3y' - 10y = -3e^{-2x}$ **b.** $y'' + 4y' = 20$

c. $y'' + 4y' = x^2$ **d.** $y'' + 9y = 3\sin(3x)$

e. $y'' - 6y' + 9y = 10e^{3x}$ **f.** $y'' + 4y' = 4xe^{-4x}$

21.6. *Find a general solution to each of the following, using the method of educated guess to find a particular solution.*

a. $y'' - 3y' - 10y = \left(72x^2 - 1\right)e^{2x}$ **b.** $y'' - 3y' - 10y = 4xe^{6x}$

c. $y'' - 10y' + 25y = 6e^{-5x}$ **d.** $y'' - 10y' + 25y = 6e^{5x}$

e. $y'' + 4y' + 5y = 24\sin(3x)$ **f.** $y'' + 4y' + 5y = 8e^{-3x}$

g. $y'' - 4y' + 5y = e^{2x}\sin(x)$ **h.** $y'' - 4y' + 5y = e^{-x}\sin(x)$

i. $y'' - 4y' + 5y = 39xe^{-x}\sin(x) + 47e^{-x}\sin(x)$

j. $y'' - 4y' + 5y = 100$ **k.** $y'' - 4y' + 5y = e^{-x}$

l. $y'' - 4y' + 5y = 10x^2 + 4x + 8$ **m.** $y'' + 9y = e^{2x}\sin(x)$

n. $y'' + y = 6\cos(x) - 3\sin(x)$ **o.** $y'' + y = 6\cos(2x) - 3\sin(2x)$

21.7. *For each of the following, state the appropriate guess for the form for a particular solution $y_p(x)$. This 'guess' should be the one that works, not necessarily the first. Leave the coefficients 'undetermined' — do NOT attempt to find the values of the coefficients.*

a. $y'' - 4y' + 5y = x^3e^{-x}\sin(x)$ **b.** $y'' - 4y' + 5y = x^3e^{2x}\sin(x)$

c. $y'' - 5y' + 6y = x^2e^{-7x} + 2e^{-7x}$ **d.** $y'' - 5y' + 6y = x^2$

e. $y'' - 5y' + 6y = 4e^{-8x}$ **f.** $y'' - 5y' + 6y = 4e^{3x}$

g. $y'' - 5y' + 6y = x^2e^{3x}$ **h.** $y'' - 5y' + 6y = x^2\cos(2x)$

i. $y'' - 5y' + 6y = x^2e^{3x}\sin(2x)$ **j.** $y'' - 4y' + 20y = e^{4x}\sin(2x)$

k. $y'' - 4y' + 20y = e^{2x}\sin(4x)$ **l.** $y'' - 4y' + 20y = x^3\sin(4x)$

21.8. *Find particular solutions to the following problems. (Note: The corresponding homogeneous equations were solved in the exercises for chapter 18.)*

a. $y^{(4)} - 4y^{(3)} = 12e^{-2x}$ **b.** $y^{(4)} - 4y^{(3)} = 32e^{4x}$

c. $y^{(4)} - 4y^{(3)} = 10\sin(2x)$ **d.** $y^{(4)} - 4y^{(3)} = 32x$

e. $y^{(3)} - y'' + y' - y = x^2$ **f.** $y^{(3)} - y'' + y' - y = 30\cos(2x)$

g. $y^{(3)} - y'' + y' - y = 6e^x$

21.9. *For each of the following, state the appropriate guess for the form for a particular solution $y_p(x)$. Leave the coefficients undetermined; do NOT actually determine the values of the coefficients. (Again, the corresponding homogeneous equations were solved in the the exercises for chapter 18.)*

a. $y^{(5)} + 18y^{(3)} + 81y' = x^2e^{3x}$ **b.** $y^{(5)} + 18y^{(3)} + 81y' = x^2\sin(3x)$

c. $y^{(5)} + 18y^{(3)} + 81y' = x^2e^{3x}\sin(3x)$ **d.** $y^{(3)} - y'' + y' - y = 30x\cos(2x)$

e. $y^{(3)} - y'' + y' - y = 3x\cos(x)$ **f.** $y^{(3)} - y'' + y' - y = 3xe^x\cos(x)$

g. $y^{(3)} - y'' + y' - y = 3x^5e^{2x}$

21.10. *Find particular solutions to the following. Use superposition and/or answers to previous exercises, if practical.*

a. $y'' - 6y' + 9y = 27e^{6x} + 25\sin(6x)$

b. $y'' + 9y = 25x\cos(2x) + 3\sin(3x)$

c. $y'' - 4y' + 5y = 5\sin^2(x)$ *(Hint: Use a trig. identity to rewrite the* $\sin^2(x)$ *in a form we've already discussed.)*

d. $y'' - 4y' + 5y = 20\sinh(x)$