19

Euler Equations

We now know how to completely solve any equation of the form

$$ay'' + by' + cy = 0$$

or even

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

in which the coefficients are all constants (provided we can completely factor the corresponding characteristic polynomial).

Let us now consider some equations of the form

$$ay'' + by' + cy = 0$$

or even

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

when the coefficients are not all constants. In particular, let us consider the "Euler equations", described more completely in the next section, in which the coefficients happen to be particularly simple polynomials.[1]

As with the constant-coefficient equations, we will discuss the second-order Euler equations (and their solutions) first, and then note how those ideas extend to corresponding higher order Euler equations.

19.1 Second-Order Euler Equations

Basics

A second-order differential equation is called an *Euler equation* if it can be written as

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0$$

where α, β and γ are constants (in fact, we will assume they are real-valued constants). For example,

$$x^2 y'' - 6xy' + 10y = 0 \quad ,$$

[1] These differential equations are also called *Cauchy-Euler equations*

$$x^2y'' - 9xy' + 25y = 0 \quad ,$$

and

$$x^2y'' - 3xy' + 20y = 0$$

are the Euler equations we'll solve below. In these equations, the coefficients are not constants but are constants times the variable raised to the power equaling the order of the corresponding derivative. Notice, too, that the first coefficient, αx^2, vanishes at $x = 0$. This means we should not attempt to solve these equations over intervals containing 0. For convenience, we will use $(0, \infty)$ as the interval of interest. You can easily verify that the same formulas derived using this interval also work using the interval $(-\infty, 0)$ after replacing the x in these formulas with either $-x$ or $|x|$.

Euler equations are important for two or three good reasons:

1. They are easily solved.
2. They occasionally arise in applications, though not nearly as often as equations with constant coefficients.
3. They are especially simple cases of a broad class of differential equations for which infinite series solutions can be obtained using the "method of Frobenius".[2] (Whether or not this is a good reason may depend on your point of view.)

The basic approach to solving Euler equations is similar to the approach used to solve constant-coefficient equations: assume a particular form for the solution with one constant "to be determined", plug that form into the differential equation, simplify and solve the resulting equation for the constant, and then construct the general solution using the constants found and the basic theory already developed.

The appropriate form for the solution to an Euler equation is not the exponential assumed for a constant-coefficient equation. Instead, it is

$$y(x) = x^r$$

where r is a constant to be determined. This choice for $y(x)$ can be motivated by either first considering the solutions to the corresponding first-order equations

$$\alpha x \frac{dy}{dx} + \beta y = 0 \quad ,$$

or by just thinking about what happens when you compute

$$x^m \frac{d^m}{dx^m}\left[x^r\right] \quad .$$

We will outline the details of the method in a moment. Do not, however, bother memorizing anything except for the first assumption about the form of the solution and general outline of the method. The precise formulas that arise are not as easily memorized as the corresponding formulas for differential equations with constant coefficients. Moreover, you won't be using them enough in your work outside this class to justify memorizing these formulas.

[2] The method of Frobenious will be developed in a much much later chapter.

The Steps in Solving Second-Order Euler Equations

Here are the basic steps for finding a general solution to any second-order Euler equation

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0 \qquad \text{for} \quad x > 0 \quad .$$

Remember α , β and γ are real-valued constants. To illustrate the basic method, we will solve

$$x^2 y'' - 6xy' + 10y = 0 \qquad \text{for} \quad x > 0 \quad .$$

1. Assume a solution of the form
$$y = y(x) = x^r$$
where r is a constant to be determined.

2. Plug the assumed formula for y into the differential equation and simplify. Let's do the example first:

Replacing y with x^r gives

$$\begin{aligned} 0 &= x^2 y'' - 6xy' + 10y \\ &= x^2 \big[x^r\big]'' - 6x\big[x^r\big]' + 10\big[x^r\big] \\ &= x^2\big[r(r-1)x^{r-2}\big] - 6x\big[rx^{r-1}\big] + 10\big[x^r\big] \\ &= (r^2 - r)x^r - 6rx^r + 10x^r \\ &= \big[r^2 - r - 6r + 10\big]x^r \\ &= \big[r^2 - 7r + 10\big]x^r \quad . \end{aligned}$$

Since we are solving on an interval where $x \neq 0$, we can divide out the x^r , leaving us with the algebraic equation

$$r^2 - 7r + 10 = 0 \quad .$$

In general, replacing y with x^r gives

$$\begin{aligned} 0 &= \alpha x^2 y'' + \beta xy' + \gamma y \\ &= \alpha x^2\big[x^r\big]'' + \beta x\big[x^r\big]' + \gamma\big[x^r\big] \\ &= \alpha x^2\big[r(r-1)x^{r-2}\big] + \beta x\big[rx^{r-1}\big] + \gamma\big[x^r\big] \\ &= \alpha(r^2 - r)x^r + \beta r x^r + \gamma x^r \\ &= \big[\alpha r^2 - \alpha r + \beta r + \gamma\big]x^r \\ &= \big[\alpha r^2 + (\beta - \alpha)r + \gamma\big]x^r \quad . \end{aligned}$$

Dividing out the x^r leaves us with the second-degree polynomial equation

$$\alpha r^2 + (\beta - \alpha)r + \gamma = 0 \quad .$$

This equation, which is sometimes called the *indicial equation* corresponding to the given Euler equation[3], is analogous to the characteristic equation for a second-order, homogeneous linear differential equation with constant coefficients. (Don't memorize this equation — it is easy enough to simply rederive it each time. Besides, analogous equations for higher-order Euler equations are significantly different.)

3. Solve the polynomial equation for r.

   In our example, we obtained the indicial equation

   $$r^2 - 7r + 10 = 0 \quad ,$$

   which factors to

   $$(r-2)(r-5) = 0 \quad .$$

   So $r = 2$ and $r = 5$ are the possible values of r.

4. Remember that, for each value of r obtained, x^r is a solution to the original Euler equation. If there are two distinct real values r_1 and r_2 for r, then

   $$\left\{ x^{r_1} , x^{r_2} \right\}$$

   is clearly a fundamental set of solutions to the differential equation, and

   $$y(x) = c_1 x^{r_1} + c_2 x^{r_2}$$

   is a general solution. If there is only one value for r, then

   $$y_1(x) = x^r$$

   is one solution to the differential equation and the general solution can be obtained via reduction of order. (The cases where there is only one value of r and where the two values of r are complex will be examined more closely in a little bit.)

   In our example, we obtained two values for r, 2 and 5. So

   $$\left\{ x^2 , x^5 \right\}$$

   is a fundamental set of solutions to the differential equation, and

   $$y(x) = c_1 x^2 + c_2 x^5$$

   is a general solution.

[3] usually, though, it's not called anything except "the equation we get for r".

19.2 The Special Cases

A Single Value for r

Let's do an example and then discuss what happens in general.

!►Example 19.1: *Consider*

$$x^2y'' - 9xy' + 25y = 0 \qquad \text{for} \quad x > 0 \quad .$$

Letting $y = x^r$ *, we get*

$$\begin{aligned} 0 &= x^2y'' - 9xy' + 25y \\ &= x^2\left[x^r\right]'' - 9x\left[x^r\right]' + 10\left[x^r\right] \\ &= x^2\left[r(r-1)x^{r-2}\right] - 9x\left[rx^{r-1}\right] + 25\left[x^r\right] \\ &= (r^2 - r)x^r - 9rx^r + 25x^r \\ &= \left[r^2 - r - 9r + 25\right]x^r \\ &= \left[r^2 - 10r + 25\right]x^r \quad . \end{aligned}$$

Dividing out the x^r *, this becomes*

$$r^2 - 10r + 25 = 0 \quad ,$$

which factors to

$$(r-5)^2 = 0 \quad .$$

So $r = 5$*, and the corresponding solution to the differential equation is*

$$y_1(x) = x^5 \quad .$$

Since we only have one solution, we cannot just write out the general solution as we did in the previous example. But we can still use the reduction of order method. So let

$$y(x) = x^5u(x) \quad .$$

Computing the derivatives,

$$y'(x) = \left[x^5u\right]' = 5x^4u + x^5u'$$

and

$$y''(x) = \left[5x^4u + x^5u'\right]' = 20x^3u + 10x^4u' + x^5u'' \quad ,$$

and plugging into the differential equation yields

$$\begin{aligned} 0 &= x^2y'' - 9xy' + 25y \\ &= x^2\left[20x^3u + 10x^4u' + x^5u''\right] - 9x\left[5x^4u + x^5u'\right] + 25\left[x^5u\right] \\ &= 20x^5u + 10x^6u' + x^7u'' - 45x^5u - 9x^6u' + 25x^5u \end{aligned}$$

$$
\begin{aligned}
&= x^7 u'' \;+\; \left[10x^6 - 9x^6\right]u' \;+\; \left[20x^5 - 45x^5 + 25x^5\right]u \\
&= x^7 u'' \;+\; x^6 u' \quad .
\end{aligned}
$$

Letting $v = u'$*, this becomes*

$$
x^7 v' \;+\; x^6 v \;=\; 0 \quad ,
$$

a simple separable first-order equation. Solving it:

$$
\begin{aligned}
& & x^7 \frac{dv}{dx} \;+\; x^6 v \;&=\; 0 \\
&\Longrightarrow & \frac{1}{v}\frac{dv}{dx} \;&=\; -\frac{x^6}{x^7} \;=\; -\frac{1}{x} \\
&\Longrightarrow & \int \frac{1}{v}\frac{dv}{dx}\,dx \;&=\; -\int \frac{1}{x}\,dx \\
&\Longrightarrow & \ln|v| \;&=\; -\ln|x| + c_0 \\
&\Longrightarrow & v \;&=\; \pm e^{-\ln|x| + c_0} \;=\; \frac{c_2}{x} \quad .
\end{aligned}
$$

Thus,

$$
u(x) \;=\; \int u'(x)\,dx \;=\; \int v(x)\,dx \;=\; \int \frac{c_2}{x}\,dx \;=\; c_2 \ln|x| \;+\; c_1 \quad ,
$$

and the general solution to the differential equation is

$$
y(x) \;=\; x^5 u(x) \;=\; x^5[c_2 \ln|x| \;+\; c_1] \;=\; c_1 x^5 \;+\; c_2 x^5 \ln|x| \quad .
$$

While just using reduction of order is recommended, you can show that, if your indicial equation only has one solution r, then

$$
y_1(x) \;=\; x^r \qquad \text{and} \qquad y_2(x) \;=\; x^r \ln|x|
$$

will always be solutions to the differential equation (but why memorize something you won't use that much). Since they are clearly not constant multiples of each other, they form a fundamental set for the differential equation. Thus, in this case,

$$
y(x) \;=\; c_1 x^r \;+\; c_2 x^r \ln|x|
$$

will always be a general solution to the given Euler equation.

Verifying this claim is left to the interested reader (see exercise 19.3 on page 407).

Complex Values for r

Again, we start with an example.

!►*Example 19.2:* *Consider*

$$
x^2 y'' \;-\; 3xy' \;+\; 20y \;=\; 0 \qquad \textit{for} \quad x > 0 \quad .
$$

Using $y = x^r$ *, we get*

$$\begin{aligned} 0 &= x^2 y'' - 3xy' + 20y \\ &= x^2\left[x^r\right]'' - 3x\left[x^r\right]' + 20\left[x^r\right] \\ &= x^2\left[r(r-1)x^{r-2}\right] - 3x\left[rx^{r-1}\right] + 20\left[x^r\right] \\ &= x^r\left[r^2 - r - 3r + 20\right] \quad , \end{aligned}$$

which simplifies to

$$r^2 - 4r + 20 = 0 \quad .$$

The solution to this is

$$r = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(20)}}{2} = \frac{4 \pm \sqrt{-64}}{2} = 2 \pm i4 \quad .$$

Thus, we have two distinct values for r *,* $2 + i4$ *and* $2 - i4$*. Presumably, then, we could construct a general solution from*

$$x^{2+i4} \qquad \textit{and} \qquad x^{2-i4} \quad ,$$

provided we had some idea as to just what "x *to a complex power" meant.*

So let's figure out what " x to a complex power" means.

For exactly the same reasons as when we were solving constant coefficient equations, the complex solutions to the indicial equation will occur as complex conjugate pairs

$$r_+ = \lambda + i\omega \qquad \text{and} \qquad r_- = \lambda - i\omega \quad ,$$

which, formally at least, yield

$$y_+(x) = x^{r_+} = x^{\lambda + i\omega} \qquad \text{and} \qquad y_-(x) = x^{r_-} = x^{\lambda - i\omega}$$

as solutions to the original Euler equation. Now, assuming the standard algebraic rules remain valid for complex powers[4],

$$x^{\lambda \pm i\omega} = x^\lambda x^{\pm i\omega} \quad ,$$

and, for $x > 0$,

$$x^{\pm i\omega} = e^{\ln|x|^{\pm i\omega}} = e^{\pm i\omega \ln|x|} = \cos(\omega \ln|x|) \pm i \sin(\omega \ln|x|) \quad .$$

So our two solutions can be written as

$$y_+(x) = x^\lambda\left[\cos(\omega \ln|x|) + i \sin(\omega \ln|x|)\right]$$

and

$$y_-(x) = x^\lambda\left[\cos(\omega \ln|x|) - i \sin(\omega \ln|x|)\right] \quad .$$

To get solutions in terms of only real-valued functions, essentially do what was done when we had complex-valued roots to characteristic equations for constant-coefficient equations: Use the fundamental set

$$\{ y_1, y_2 \}$$

[4] They do.

where

$$y_1(x) \;=\; \frac{1}{2}y_+(x) \;+\; \frac{1}{2}y_-(x) \;=\; \cdots \;=\; x^\lambda \cos(\omega \ln|x|)$$

and

$$y_2(x) \;=\; \frac{1}{2i}y_+(x) \;-\; \frac{1}{2i}y_-(x) \;=\; \cdots \;=\; x^\lambda \sin(\omega \ln|x|) \quad .$$

Note that these are just the real and the imaginary parts of the formulas for $y_\pm = x^{\lambda \pm i\omega}$.

If you really wish, you can memorize what we just derived, namely:

> *If you get*
>
> $$r \;=\; \lambda \;\pm\; i\omega$$
>
> *when assuming $y = x^r$ is a solution to an Euler equation, then*
>
> $$y_1(x) \;=\; x^\lambda \cos(\omega \ln|x|) \qquad \text{and} \qquad y_2(x) \;=\; x^\lambda \sin(\omega \ln|x|)$$
>
> *form a corresponding linearly independent pair of real-valued solutions to the differential equation, and*
>
> $$y(x) \;=\; c_1 x^\lambda \cos(\omega \ln|x|) \;+\; c_2 x^\lambda \sin(\omega \ln|x|)$$
>
> *is a general solution in terms of just real-valued functions.*

Memorizing these formulas is not recommended. It's easy enough (and safer) to simply re-derive the formulas for $x^{\lambda \pm i\omega}$ as needed, and then just take the real and imaginary parts as our the two real-valued solutions.

!►*Example 19.3:* *Let us finish solving*

$$x^2 y'' \;-\; 3xy' \;+\; 20y \;=\; 0 \qquad \text{for} \quad x > 0 \quad .$$

From above, we got the complex-power solutions

$$y_\pm(x) \;=\; x^{2 \,\pm\, i4} \quad .$$

Rewriting this using the corresponding complex exponential, we get

$$\begin{aligned} y_\pm(x) \;&=\; x^2 x^{\pm i4} \;=\; x^2 e^{\ln|x|^{\pm i4}} \\ &=\; x^2 e^{\pm i4 \ln|x|} \;=\; x^2\big[\cos(4\ln|x|) \;\pm\; i \sin(4\ln|x|)\big] \quad . \end{aligned}$$

Taking the real and imaginary parts of this then yields the corresponding linearly independent pair of real-valued solutions to the differential equation,

$$y_1(x) \;=\; x^2 \cos(4\ln|x|) \qquad \text{and} \qquad y_2(x) \;=\; x^2 \sin(4\ln|x|) \quad .$$

Thus,

$$y(x) \;=\; c_1 x^2 \cos(4\ln|x|) \;+\; c_2 x^2 \sin(4\ln|x|)$$

is a general solution in terms of just real-valued functions.

19.3 Euler Equations of Any Order

The definitions and ideas just described for second-order Euler equations are easily extended to analogous differential equations of any order. The natural extension of the concept of a second-order Euler differential equation is that of an n^{th}-order Euler equation, which is any differential equation that can be written as

$$\alpha_0 x^N y^{(N)} + \alpha_1 x^{N-1} y^{(N-1)} + \cdots + \alpha_{N-2} x^2 y'' + \alpha_{N-1} x y' + \alpha_N y = 0$$

where the α_k's are all constants (and $\alpha_0 \neq 0$). We will further assume they are all real constants.

The basic ideas used to find the general solution a n^{th}-order Euler equation over $(0, \infty)$ are pretty much the same as used to solve the second-order Euler equations:

1. Assume a solution of the form
$$y = y(x) = x^r$$
where r is a constant to be determined.

2. Plug the assumed formula for y into the differential equation and simplify. The result will be an N^{th} degree polynomial equation
$$A_0 r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N = 0 \quad .$$
We'll call this the *indicial equation* for the given Euler equation, and the polynomial on the left will be called the *indicial polynomial*. It is easily shown that the A_k's are all real (assuming the α_k's are real) and that $A_0 = \alpha_0$. However, the relation between the other A_k's and α_k's will depend on the order N of the original differential equation.

3. Solve the indicial equation. The same tricks used to help solve the characteristic equations in chapter 18 can be used here. And, as with those characteristic equations, we will obtain a list of all the different roots of the indicial polynomial,
$$r_1 \quad , \quad r_2 \quad , \quad r_3 \quad , \quad \ldots \quad \text{and} \quad r_K \quad ,$$
along with their corresponding multiplicities,
$$m_1 \quad , \quad m_2 \quad , \quad m_3 \quad , \quad \ldots \quad \text{and} \quad m_K \quad .$$
As noted in chapter 18,
$$m_1 + m_2 + m_3 + \cdots + m_K = N \quad .$$
What you do next with each r_k depends on whether r_k is real or complex, and on the multiplicity m_k of r_k.

4. If $r = r_k$ is real, then there will be a corresponding linearly independent set of $m = m_k$ solutions to the differential equation. One of these, of course, will be $y = x^r$. If this root's multiplicity m is greater than 1, then a second corresponding solution to the Euler equation is obtained by multiplying the first, x^r, by $\ln|x|$, just as in the second-order case. This — multiplying the last solution found by $\ln|x|$ — turns out to be the pattern

for generating the other solutions when $m = m_k > 2$. That is, the set of solutions to the differential equation corresponding to $r = r_k$ is

$$\left\{ x^r \,,\; x^r \ln|x| \,,\; x^r (\ln|x|)^2 \,,\; \dots \,,\; x^r (\ln|x|)^{m-1} \right\}$$

with $m = m_k$. (We'll verify this rigorously in the next section.)

5. If a root is complex, say, $r = \lambda + i\omega$, and has multiplicity m, then we know that this root's complex conjugate $r^* = \lambda - i\omega$ is another root of multiplicity m. By the same arguments given for real roots, we have that

$$\left\{ x^{\lambda+i\omega} \,,\; x^{\lambda+i\omega} \ln|x| \,,\; x^{\lambda+i\omega}(\ln|x|)^2 \,,\; \dots \,,\; x^{\lambda+i\omega}(\ln|x|)^{m-1} \right\}$$

with

$$\left\{ x^{\lambda-i\omega} \,,\; x^{\lambda-i\omega} \ln|x| \,,\; x^{\lambda-i\omega}(\ln|x|)^2 \,,\; \dots \,,\; x^{\lambda-i\omega}(\ln|x|)^{m-1} \right\}$$

forms a linearly independent set of $2m$ solutions to the Euler equation. To obtain the corresponding set of real-valued solutions, we again use the fact that, for $x > 0$,

$$x^{\lambda\pm i\omega} \;=\; x^\lambda x^{\pm i\omega} \;=\; x^\lambda e^{\pm i\omega \ln|x|} \;=\; x^\lambda \big[\cos(\omega \ln|x|) \;\pm\; i \sin(\omega \ln|x|) \big] \tag{19.1}$$

to obtain the alternative solutions sets

$$\left\{ x^\lambda \cos(\omega \ln|x|) \,,\; x^\lambda \cos(\omega \ln|x|) \ln|x| \,,\, x^\lambda \cos(\omega \ln|x|)\,(\ln|x|)^2 \,, \right.$$
$$\left. \dots \,,\; x^\lambda \cos(\omega \ln|x|)\,(\ln|x|)^{m-1} \right\}$$

and

$$\left\{ x^\lambda \sin(\omega \ln|x|) \,,\; x^\lambda \sin(\omega \ln|x|) \ln|x| \,,\, x^\lambda \sin(\omega \ln|x|)\,(\ln|x|)^2 \,, \right.$$
$$\left. \dots \,,\; x^\lambda \sin(\omega \ln|x|)\,(\ln|x|)^{m-1} \right\}$$

for the Euler equation.

6. Now form the set of solutions to the Euler equation consisting of the m_k solutions described above for each real root r_k, and the $2m_k$ real-valued solutions described above for each conjugate pair of roots r_k and $r_k{}^*$. Since (as we saw in chapter 18) the sum of the multiplicities equals N, and since the r_k's are distinct, it will follow that the set of solutions to the Euler equation, this will be a fundamental set of solutions for our Euler equation. Thus, finally, a general solution to the given Euler equation can be written out as an arbitrary linear combination of the functions in this set.

We will do two examples (skipping some of the tedious algebra).

!►*Example 19.4:* *Consider the third-order Euler equation*

$$x^3 y''' \;-\; 6x^2 y'' \;+\; 19xy' \;-\; 27y \;=\; 0 \qquad \textit{for} \quad x > 0 \quad .$$

Plugging in $y = x^r$*, we get*

$$x^3 r(r-1)(r-2)x^{r-3} \;-\; 6x^2 r(r-1)x^{r-2} \;+\; 19xrx^{r-1} \;-\; 27x^r \;=\; 0 \quad ,$$

which, after a bit of algebra, reduces to

$$r^3 \;-\; 9r^2 \;+\; 27r \;-\; 2r \;=\; 0 \quad .$$

This is the indicial equation for our Euler equation. You can verify that its factored form is

$$(r-3)^3 = 0 \quad .$$

So the only root to our indicial polynomial is $r = 3$ *, and it has multiplicity* 3 *. As discussed above, the corresponding set of solutions to the Euler equation is*

$$\left\{ x^3 \,,\; x^3 \ln|x| \,,\; x^3 (\ln|x|)^2 \right\} \quad ,$$

and the corresponding general solution is

$$y = c_1 x^3 + c_2 x^3 \ln|x| + c_3 x^3 (\ln|x|)^2 \quad ,$$

!►*Example 19.5:* *Consider the fourth-order Euler equation*

$$x^4 y^{(4)} + 6x^3 y''' + 25x^2 y'' + 19xy' + 81y = 0 \qquad \text{for} \quad x > 0 \quad .$$

Plugging in $y = x^r$ *, we get*

$$x^4 r(r-1)(r-2)(r-3)x^{r-4} + 6x^3 r(r-1)(r-2)x^{r-3}$$
$$+ 25x^2 r(r-1)x^{r-2} + 19xrx^{r-1} + 81x^r = 0 \quad ,$$

which simplifies to

$$r^4 + 18r^2 + 81 = 0 \quad .$$

Solving this yields

$$r = \pm 3i \qquad \text{with multiplicity } 2 \quad .$$

This corresponds to the real-valued Euler equation solutions

$$\cos(3\ln|x|) \quad , \quad \sin(3\ln|x| \quad , \quad \cos(3\ln|x|)\ln|x| \quad \text{and} \quad \sin(3\ln|x|\ln|x| \quad .$$

The general solution, then, is

$$y = c_1 \cos(3\ln|x|) + c_2 \sin(3\ln|x| + c_4 \cos(3\ln|x|)\ln|x| + c_4 \sin(3\ln|x|\ln|x| \quad .$$

19.4 The Relation Between Euler and Constant Coefficient Equations

Let us suppose that

$$A_0 r^N + A_1 r^{N-1} + \cdots + A_{N-1} r + A_N = 0 \tag{19.2}$$

is the indicial equation for some N^{th}-order Euler equation

$$\alpha_0 x^N \frac{d^N y}{dx^N} + \alpha_1 x^{N-1} \frac{d^{N-1} y}{dx^{N-1}} + \cdots + \alpha_{N-2} x^2 \frac{d^2 y}{dx^2} + \alpha_N y = 0 \quad . \tag{19.3}$$

Observe that polynomial equation (19.2) is also the characteristic equation for the N^{th}-order constant coefficient equation

$$A_0 \frac{d^N Y}{dt^N} + A_1 \frac{d^{N-1} Y}{dt^{N-1}} + \cdots + A_{N-1} \frac{dY}{dt} + A_N Y = 0 \quad . \tag{19.4}$$

(We've changed notation a little to avoid confusion.)

This means that, if r is a solution to polynomial equation (19.2), then

$$x^r \qquad \text{and} \qquad e^{rt}$$

are solutions, respectively, to above Euler equation and the above constant coefficient equation. This suggests that these two differential equations are related to each other, possibly through a substitution of the form

$$x^r = e^{rt} \quad .$$

Taking the r^{th} root of both sides, this simplifies to

$$x = e^t \qquad \text{or, equivalently,} \qquad \ln|x| = t \quad .$$

Exploring this possibility further eventually leads to the following lemma about the solutions to the above differential equations:

Lemma 19.1
Let $y(x)$ and $Y(t)$ be two functions, with y defined on $(0,\infty)$, and $Y(t)$ defined on $(-\infty,\infty)$. Assume they are related by the substitution $x = e^t$ (equivalently, $\ln|x| = t$); that is,

$$y(x) = Y(t) \qquad \text{where} \quad x = e^t \quad \text{and} \quad t = \ln|x| \quad .$$

Then y is a solution to Euler equation (19.3) if and only if Y is a solution to constant coefficient equation (19.3).

The proof of this lemma involves repeated chain rule computations such as

$$\frac{dy}{dx} = \frac{d}{dx}Y(t) = \frac{dt}{dx}\frac{d}{dt}Y(t) = \frac{d\ln|x|}{dx}\frac{d}{dt}Y(t) = \frac{1}{x}\frac{dY}{dt} = e^{-t}\frac{dY}{dt} \quad . \tag{19.5}$$

We'll leave the details to the adventurous (see exercises 19.5, 19.6 and 19.7).

There are two noteworthy consequences of this lemma:

1. It gives us another way to solve Euler equations. To be specific: we can use the substitution in the lemma to convert the Euler equation into a constant coefficient equation (with t as the variable); solve that coefficient equation for its general solution (in terms of functions of t), and then use the substitution backwards to get the general solution to the Euler equation (in terms of functions of x).[5]

2. We can now confirm the claim made (and used) in the previous section about solutions to the Euler equation corresponding to a root r of multiplicity m to the indicial equation. After all if r is a solution of multiplicity m to equation (19.2), then we know that

$$\left\{ e^{rt} \,,\, te^{rt} \,,\, t^2 e^{rt} \,,\, \dots \,,\, t^{m-1} e^{rt} \right\}$$

[5] It may be argued that this method, requiring the repeated use of the chair rule, is more tedious and error-prone than the one developed earlier, which only requires algebra and differentiation of x^r . That would be a good argument.

is a set of solutions to constant coefficient equation (19.4). The lemma then assures us that this set, with $t = \ln|x|$, is the corresponding set of solutions to Euler equation (19.3). But, using this substitution,

$$t^k e^{rt} = \left(e^t\right)^r t^k = x^r (\ln|x|)^k \quad .$$

So the set of solutions obtained to the Euler equation is

$$\left\{ x^r \, , \, x^r \ln|x| \, , \, x^r (\ln|x|)^2 \, , \, \dots \, , \, x^r (\ln|x|)^{m-1} \right\} \quad ,$$

just as claimed in the previous section.

Additional Exercises

19.1. *Find the general solution to each of the following Euler equations on* $(0, \infty)$ *:*

a. $x^2 y'' - 5xy' + 8y = 0$

b. $x^2 y'' - 2y = 0$

c. $x^2 y'' - 2xy' = 0$

d. $2x^2 y'' - xy' + y = 0$

e. $x^2 y'' - 5xy' + 9y = 0$

f. $x^2 y'' + 5xy' + 4y = 0$

g. $4x^2 y'' + y = 0$

h. $x^2 y'' - xy' + 10y = 0$

i. $x^2 y'' + 5xy' + 29y = 0$

j. $x^2 y'' + xy' + y = 0$

19.2. *Solve the following initial-value problems:*

a. $x^2 y'' - 6xy' + 10y = 0$ *with* $y(1) = -1$ *and* $y'(1) = 7$

b. $4x^2 y'' + 4xy' - y = 0$ *with* $y(4) = 0$ *and* $y'(4) = 2$

c. $x^2 y'' - 11xy' + 36y = 0$ *with* $y(1) = 1/2$ *and* $y'(1) = 2$

d. $x^2 y'' - 3xy' + 13y = 0$ *with* $y(1) = 9$ *and* $y'(1) = 3$

19.3. *Suppose that the indicial equation for a second-order Euler equation only has one solution* r *. Using reduction of order (or any other approach you think appropriate) show that both*

$$y_1(x) = x^r \qquad \textit{and} \qquad y_2(x) = x^r \ln|x|$$

are solutions to the differential equation on $(0, \infty)$ *.*

19.4. *Find the general solution to each of the following third- and fourth-order Euler equations on* $(0, \infty)$ *:*

a. $x^3 y''' + 2x^2 y'' - 4xy' + 4y = 0$

b. $x^3 y''' + 2x^2 y'' + xy' - y = 0$

c. $x^3 y''' - 5x^2 y'' + 14xy' - 18y = 0$

d. $x^4 y^{(4)} + 6x^3 y''' - 3x^2 y'' - 9xy' + 9y = 0$

e. $x^4 y^{(4)} + 2x^3 y''' + x^2 y'' - xy' + y = 0$

19.5. *Confirm that the claim of lemma 19.1 holds when* $N = 2$ *by considering the general second-order Euler equation*

$$\alpha x^2 \frac{d^2y}{dx^2} + \beta x \frac{dy}{dx} + \gamma y = 0$$

and doing the following:

a. *Find the corresponding indicial equation.*

b. *Convert the above Euler equation to a second-order, constant coefficient differential equation using the substitution* $x = e^t$ *. Remember, this is equivalent to* $t = \ln|x|$ *. (You may want to glance back at the chain rule computations in line (19.5).)*

c. *Confirm (by inspection!) that the characteristic equation for the constant coefficient equation just obtained is identical to the indicial equation for the above Euler equation.*

19.6. *Confirm that the claim of lemma 19.1 holds when* $N = 3$ *by considering the general third-order Euler equation*

$$\alpha_0 x^3 \frac{d^3y}{dx^3} + \alpha_1 x^2 \frac{d^2y}{dx^2} + \alpha_2 x \frac{dy}{dx} + \alpha_3 y = 0$$

and doing the following:

a. *Find the corresponding indicial equation.*

b. *Convert the above Euler equation to a second-order, constant coefficient differential equation using the substitution* $x = e^t$ *.*

c. *Confirm that the characteristic equation for the constant coefficient equation just obtained is identical to the indicial equation for the above Euler equation.*

19.7. *Confirm that the claim of lemma 19.1 holds when* N *is any positive integer.*