10

The Art and Science of Modeling with First-Order Equations

For some, “modeling” is the building of small plastic replicas of famous ships; for others, “modeling” means standing in front of cameras wearing silly clothing; for us, “modeling” is the process of developing sets of equations and formulas describing some process of interest. This process may be the falling of a frozen duck, the changes in a population over time, the consumption of fuel by a car traveling various distances, the accumulation of wealth by one individual or company, the cooling of a cup of coffee, the electronic transmission of sound and images from a television station to a home television, or any of a huge number of other processes affecting us. A major goal of modeling, of course, is to predict “how things will turn out” at some point of interest, be it a point of time in the future or a position along the road. Along with this, naturally, is often a desire to use the model to determine changes we can make to the process to force things to turn out as we desire.

Of course, some things are more easily modeled mathematically than others. For example, it will certainly be easier to mathematically describe the number of rabbits in a field than to mathematically describe the various emotions of these rabbits. Part of the art of modeling is the determination of which quantities the model will deal with (e.g., “number of rabbits” instead of “emotional states”).

Another part of modeling is the balancing between developing as complete a model as possible by taking into account all possible influences on the process as opposed to developing a simple and easy to use model by the use of simplifying assumptions and simple approximations. Attempting to accurately describe all possible influences usually leads to such a complicated set of equations and formulas that the model (i.e., the set of equations and formulas we’ve developed) is unusable. A model that is too simple, on the other hand, may lead to wildly inaccurate predictions, and, thus, would also not be a useful model.

Here, we will examine various aspects of modeling using first-order differential equations. This will be done mainly by looking at a few illustrative examples, though, in a few sections, we will also discuss how to go about developing and using models with first-order differential equations more generally.

10.1 Preliminaries

Suppose we have a situation in which some measurable quantity of interest (e.g.: velocity of a falling duck, number of rabbits in a field, gallons of fuel in a vehicle, amount of money in a bank, temperature of a cup of coffee) varies as some basic parameter (such as time or position) changes. For convenience, let's assume the parameter is time and denote that parameter by t, as is traditional. Recall that, if

$$Q(t) = \text{the } \textit{amount} \text{ of that measurable quantity at time } t \quad ,$$

then

$$\frac{dQ}{dt} = \text{the } \textit{rate} \text{ at which } Q \text{ varies as } t \text{ varies} \quad .$$

If we can identify what controls this rate, and can come up with a formula $F(t, Q)$ describing how this rate depends on t and the Q, then

$$\frac{dQ}{dt} = F(t, Q) \quad .$$

gives us a first-order differential equation for Q which, with a little luck, can be solved to obtain a general formula for Q in terms of t. At the very least, we will be able to construct this equation's slope field and sketch graphs of $Q(t)$.

Our development of the "improved falling object model" in chapter 1.2 is a good example of this sort of modeling. Go back to page 12 and take a quick look at it. There, the 'measurable quantity' is the velocity v (in meters/second); the rate at which it varies with time, ${}^{dv}/_{dt}$, is the acceleration, and we were able to determine a formula F for this acceleration by determining and adding together the accelerations due to gravity and air resistance,

$$\begin{aligned} F(t, v) &= \text{total acceleration} \\ &= \text{acceleration due to gravity} + \text{acceleration due to air resistance} \\ &= (-9.8) + (-\kappa v) \end{aligned}$$

where κ is some positive constant that would have to be determined by experiment. This gave us the first-order differential equation

$$\frac{dv}{dt} = F(t, v) = -9.8 - \kappa v \quad ,$$

which we were later able to solve and analyze.

In what follows, we will develop models for several other situations. We will also, in section 10.5, give further advice on developing your own models with first-order differential equations. Be sure to observe how we develop these models and to read the notes in section 10.5. You will be developing more models in the exercises and, maybe later, in 'real life'.

10.2 A Rabbit Ranch

The Situation to be Modeled

Pretend we've been given a breeding pair of rabbits along with acres and acres of prime rabbit range with no predators. Let us further assume this rabbit range is fenced in so that no rabbits can escape or come in, and so that no predators can come in. We release the rabbits, planning to return in a few years (say, five) to harvest rabbits for the Easter trade.

An obvious question is *How many rabbits will we have on our rabbit ranch in five years?*

Setting Up the Model

Experienced modelers typically begin by drawing a simple, enlightening picture of the process (if appropriate) and identifying the relevant variables (as we did for the "falling object model" — see page 9). Since the author could not think of a particularly appropriate and enlightening picture, we will skip the picture and go straight to identifying the obvious variables of interest. They are 'time' and 'the number of rabbits', which we will naturally denote using the symbols t and R, respectively. The time t can be measured in seconds, days, months, years, centuries, etc. We will use months, with $t = 0$ being the time the rabbits were released. So,

$$t = \text{number of months since the rabbits were released}$$

and

$$R = R(t) = \text{number of rabbits at time } t \quad .$$

Since we started with a pair of rabbits, the initial condition is

$$R(0) = 2 \quad . \tag{10.1}$$

Now, since t is being measured in months,

$$\begin{aligned}\frac{dR}{dt} &= \text{rate } R \text{ varies as } t \text{ varies}\\ &= \text{change in the number of rabbits per month} \quad .\end{aligned}$$

Because the fence prevents rabbits escaping or coming in, the change in the number of rabbits is due entirely to the number of births and deaths in our rabbit population. Thus,

$$\begin{aligned}\frac{dR}{dt} &= \text{change in the number of rabbits per month}\\ &= \text{number of births per month} - \text{number of deaths per month} \quad .\end{aligned} \tag{10.2}$$

Now we need to model the "number of births per month" and the "number of deaths per month". Starting with the birth process, and assuming that half the population are females, we note that

$$\begin{aligned}&\text{number of births per month}\\ &\quad = \text{number of births per female rabbit per month}\\ &\qquad \times \text{number of female rabbits that month}\\ &\quad = \text{number of births per female rabbit per month} \times \frac{1}{2}R \quad .\end{aligned}$$

(We are also assuming that all the females are capable of having babies, no matter what their age. Well, these are rabbits; they marry young.)

It seems reasonable to assume the average number of births per female rabbit per month is a constant. For future convenience, let

$$\beta = \frac{1}{2} \times \text{number of births per female rabbit per month} \quad .$$

This is the "monthly birthrate per rabbit" and allows us to write

$$\text{number of births per month} = \beta R \quad . \tag{10.3}$$

Checking a reliable reference on rabbits (any decent encyclopedia will do), it can be found that, on the average, each female rabbit has 6 litters per year with 5 bouncy baby bunnies in each litter. Hence, since there are 12 months in a year,

$$\begin{aligned}\beta &= \frac{1}{2} \times \text{number of births per female rabbit per month} \\ &= \frac{1}{2} \times \frac{1}{12} \times \text{number of births per female rabbit per year} \\ &= \frac{1}{2} \times \frac{1}{12} \times 6 \times 5 \quad .\end{aligned}$$

That is,

$$\beta = \frac{5}{4} \quad . \tag{10.4}$$

What about the death rate? Since there are no predators and plenty of food, it seems reasonable to assume old age is the main cause of death. Again checking a reliable reference on rabbits, it can be found that the average life span for a rabbit is 10 years. Clearly, then, the number of deaths per month will be negligible compared to the number of births. So we will assume

$$\text{number of deaths per month} = 0 \quad . \tag{10.5}$$

Combining equations (10.2), (10.3) and (10.5), we obtain

$$\begin{aligned}\frac{dR}{dt} &= \text{number of births per month} - \text{number of deaths per month} \\ &= \beta R - 0 \quad .\end{aligned}$$

That is,

$$\frac{dR}{dt} = \beta R \tag{10.6}$$

where β is the average monthly birthrate per rabbit.[1]

Of course, equation (10.6) does not just apply to the situation being considered here. The same equation would have been obtained for the changing population of any creature having zero death rate and a constant birthrate β per unit time per creature. But the problem at hand involves rabbits, and for rabbits, we derived $\beta = 5/4$. This, the above differential equation, and the fact that we started with two rabbits means that $R(t)$ must satisfy

$$\frac{dR}{dt} = \frac{5}{4}R \qquad \text{with} \quad R(0) = 2 \quad .$$

This is our "model".

[1] In developing this equation, we equated an "instantaneous rate of change", dR/dt, to a "change in the number of rabbits per month", and then found a formula for that "monthly change" based on the value of R "at time t" instead of over the entire month. If this bothers you, see appendix 10.8 on page 231.

Using Our Model

Our differential equation is

$$\frac{dR}{dt} = \beta R \qquad \text{with} \quad \beta = \frac{5}{4} \quad .$$

This is a simple separable and linear differential equation. You can easily show that its general solution is

$$R(t) = Ae^{\beta t} \quad .$$

Applying the initial condition,

$$2 = Ae^{\beta \cdot 0} = A \quad .$$

So the number of rabbits after t months is given by

$$R(t) = 2e^{\beta t} \qquad \text{with} \quad \beta = \frac{5}{4} \quad . \tag{10.7}$$

Five years is 60 months. Using a calculator, we find that the number of rabbits after 5 years is

$$R(60) = 2e^{\frac{5}{4} \cdot 60} = 2e^{75} \approx 7.47 \times 10^{32} \quad .$$

That is a lot of rabbits. At about 3 kilograms each, the mass of all the rabbits on the ranch will then be approximately

$$2.2 \times 10^{33} \text{ kilograms} \quad .$$

By comparison:

$$\text{the mass of the Earth} \approx 6 \times 10^{24} \text{ kilograms}$$

and

$$\text{the mass of the Sun} \approx 2 \times 10^{30} \text{ kilograms} \quad .$$

So our model predicts that, in five years, the total mass of our rabbits will be over a thousand times that of our nearest star.

This does not seem like a very realistic prediction. Later in this chapter, we will derive a more complete (but less simple) model.

But first, let us briefly discuss a few other modeling situations involving differential equations similar to the one derived here (equation (10.6)).

10.3 Exponential Growth and Decay

Whenever the rate of change of some quantity $Q(t)$ is directly proportional to that quantity, we automatically have

$$\frac{dQ}{dt} = \beta Q$$

with β being the constant of proportionality. Since this simple relationship is inherent in many processes of interest, it, along with with its general solution

$$Q(t) = Ae^{\beta t} \quad ,$$

arises in a large number of important applications, most of which do not involve rabbits. If $\beta > 0$, then $Q(t)$ increases rapidly as t increases, and we typically say we have *exponential growth*. If $\beta < 0$, then $Q(t)$ shrinks to 0 fairly rapidly as t increases, and we typically say we have *exponential decay*. And if $\beta = 0$, then $Q(t)$ remains constant — we just have equilibrium solutions.

Simple Population Models

Suppose we are interested in how the population of some set of creatures or plants varies with time. These may be rabbits on a ranch (as in our previous example) or yeast fermenting a vat of grape juice or the people in some city or the algae growing in a pond. They may even be the people in some country that are infected with and are helping spread some contagious disease. Whatever the individuals of this population happen to be, we can let $P(t)$ denote the total number of these individuals at time t, and ask how $P(t)$ varies with time (as we did in our "rabbit ranch" example). If we further assume (as we did in our previous "rabbit ranch" example) that

1. the change in $P(t)$ over a unit of time depends only on the number of "births" and "deaths" in the population;[2]

2. the "average birth rate per individual per unit of time" β_0 is constant,

and

3. the "average death rate per individual per unit of time" δ_0 is constant (i.e., a constant fraction δ_0 of the population dies off during each unit of time);

then

$$\begin{aligned}\frac{dP}{dt} &= \text{change in the number of individuals per unit time}\\ &= \text{number of births per unit time} - \text{number of deaths per unit time}\\ &= \beta_0 P(t) - \delta_0 P(t) \quad .\end{aligned}$$

Letting β be the "net birthrate per individual per unit time",

$$\beta = \beta_0 - \delta_0 \quad ,$$

this reduces to

$$\frac{dP}{dt} = \beta P(t) \quad , \tag{10.8}$$

the solution of which, as we already know, is

$$P(t) = P_0 e^{\beta t} = P_0 e^{(\beta_0 - \delta_0)t} \qquad \text{where} \quad P_0 = P(0) \quad .$$

If $\beta_0 > \delta_0$, then the model predicts that the population will grow exponentially. If $\beta_0 < \delta_0$, then the model predicts that the population will decline exponentially. And if $\beta_0 = \delta_0$, then the model predicts that the population remains static.

This is a simple model whose accuracy depends on the validity of the three basic assumptions made above. In many cases, these assumptions are often reasonably acceptable during the early

[2] Precisely what "birth" or "death" means may depend on the creatures/plants in the population. For a microbe, "birth" may be when a parent cell divides into two copies of itself. If the population is the set of people infected with a particular disease, then "birth" occurs when a person contracts the disease.

stages of the process, and, initially, we do see exponential growth of populations, say, of yeast added to grape juice or of a new species of plants or animals introduced to a region where it can thrive. As illustrated in our "rabbit ranch" example, however, this is too simple a model to describe the long-term growth of most biological populations.

Natural Radioactive Decay

The effect of radioactive decay on the amount of some radioactive isotope can be described by a model completely analogous to the general population model just discussed. Assume we start with some amount (say, a kilogram) of some radioactive isotope of interest (say, uranium–235). During any given length of time, there is a certain probability that any given atom of that material will spontaneously decay into a smaller atom along with associated radiation and other atomic and subatomic particles. Thus, the amount we have of that particular radioactive isotope will decrease as more and more of the atoms decay (provided there is not some other material that decays into the isotope of interest.)

Let's assume we have some radioactive isotope of interest, and that there is no other radioactive material decaying into that isotope. For convenience, let $A(t)$ denote the amount of that radioactive material at time t, and let δ be the fraction of the material that decays per unit time. In essence, the decay of an atom is the death of that atom, and this δ is essentially the same as the δ_0 in the above population growth discussion. Virtually the same analysis done to obtain equation (10.8) (but using P instead of A, and noting that $\beta_0 = 0$ since no new atoms of the isotope are being "born") then yields

$$\frac{dA}{dt} = -\delta A(t) \quad .$$

Solving this differential equation then gives us

$$A(t) = A_0 e^{-\delta t} \qquad \text{with} \quad A_0 = A(0) \quad . \tag{10.9}$$

Because radioactive decay is a probabilistic event, and because there are typically huge numbers of atoms in any sample of radioactive material, the laws of probability and statistics ensure that this is usually a very accurate model over long periods of time (unlike the case with biological populations).

The positive constant δ, called the *decay rate*, is different for each different isotope. It is large if the isotope is very unstable and a large fraction of the atoms decay in a given time period, and it is small if the isotope is fairly stable and only a small fraction of the atoms decay in the same time period. In practice, the decay rate δ is usually described indirectly through the *half-life* $\tau_{1/2}$ of the isotope, which is the time it takes for half of the original amount to decay. Using the above formula for $A(t)$, you can easily verify that $\tau_{1/2}$ and δ are related by the equation

$$\delta \times \tau_{1/2} = \ln 2 \quad . \tag{10.10}$$

?►*Exercise 10.1:* *Derive equation (10.10). Use formula (10.9) for $A(t)$ and the fact that, by the definition of $\tau_{1/2}$,*

$$A(\tau_{1/2}) = \frac{1}{2} A(0) \quad .$$

!► *Example 10.1:* *Cobalt-60 is a radioactive isotope of cobalt with a half-life of approximately 5.27 years.*[3] *Using equation (10.10), we find that its (approximate) decay constant is given by*

$$\delta = \frac{\ln 2}{\tau_{1/2}} = \frac{\ln 2}{5.27 \text{ (years)}} \approx 0.1315 \quad \text{(per year)} \quad .$$

Combining this with formula (10.9) gives us

$$A(t) \approx A_0 e^{-0.1315t} \qquad \text{with} \quad A_0 = A(0) \quad .$$

as the formula for the amount of undecayed cobalt remaining after t years.

Suppose we initially have 10 grams of cobalt-60. At the end of one year, those 10 grams would have decayed to approximately

$$(10 \text{ gm.}) \times e^{-0.1315\times 1} \approx 8.77 \text{ grams of cobalt-60} \quad .$$

At the end of two years, those 10 grams would have decayed to approximately

$$(10 \text{ gm.}) \times e^{-0.1315\times 2} \approx 7.69 \text{ grams of cobalt-60} \quad .$$

And at the end of ten years, those 10 grams would have decayed to approximately

$$(10 \text{ gm.}) \times e^{-0.1315\times 10} \approx 2.68 \text{ grams of cobalt-60} \quad .$$

10.4 The Rabbit Ranch, Again

Back to wrangling rabbits.

The Situation (and Problem)

Recall that we imagined ourselves having a fenced-in ranch enclosing many acres of prime rabbit range. We start with a breeding pair of rabbits, and plan to return in five years. The question is *How many rabbits will we have then?*

In section 10.2, we attempted to answer this question using a fairly simple model we had just developed. However, the predicted number of rabbits after five years (which had a corresponding mass a thousand times greater than that of the Sun) was clearly absurd. That model did not account for the problems arising when a population of rabbits grows too large. Let us now see if we can derive a more realistic model.

[3] Cobalt-60 has numerous medical applications, as well as having the potential as an ingredient in some particularly nasty nuclear weapons. It is produced by exposing cobalt-59 to "slow" neutrons, and decays to a stable nickel isotope after giving off one electron and two gamma rays.

A Better Model

Again, we let

$$R(t) = \text{number of rabbits after } t \text{ months}$$

with $R(0) = 2$. We still have

$$\begin{aligned} \frac{dR}{dt} &= \text{change in the number of rabbits per month} \\ &= \text{number of births per month} - \text{number of deaths per month} \quad . \end{aligned} \tag{10.11}$$

However, the assumptions that

$$\text{number of deaths per month} = 0 \quad ,$$

and

$$\text{number of births per month} = \beta R$$

where

$$\beta = \text{monthly birthrate per rabbit} = \frac{5}{4} \quad .$$

are too simplistic. As the population increases, the amount of range land (and, hence, food) per rabbit decreases. Eventually, the population may become too large for the available fields to support all the rabbits. Some will starve to death, and those female rabbits that survive will be malnourished and will give birth to fewer bunnies. In addition, overcrowding is conducive to the spread of diseases which, in a population already weakened by hunger, can be devastating.

Clearly, we at least need to correct our formula for the number of deaths per month, because, once overcrowding begins, we can expect a certain fraction of the population to die each month. Letting δ denote that fraction,

$$\begin{aligned} \text{number of deaths per month} &= \text{fraction of the population that dies each month} \\ &\qquad \times \text{number of rabbits} \\ &= \delta R \quad . \end{aligned}$$

Keep in mind that this fraction δ, which we can call the monthly death rate per rabbit, will not be constant. It will depend on just how overcrowded the rabbits are. In other words, δ will vary with R, and, thus, should be treated as a function of R, $\delta = \delta(R)$. Just how δ varies with R is yet unknown, but it should be clear that

> if R is small, then overcrowding is not a problem and $\delta(R)$ should be close to zero,

and

> as R increases, then overcrowding increases and more rabbits start dying. So, $\delta(R)$ should increase as R increases.

The simplest function of R for δ satisfying the two above conditions is

$$\delta = \delta(R) = \gamma_D R$$

where γ_D is some positive constant. This gives us

$$\text{number of deaths per month} = \delta R = \left[\gamma_D R\right] R = \gamma_D R^2 \quad .$$

What sort of "correction" should we now consider for

$$\text{number of births per month} \;=\; \beta R \quad ?$$

Well, as with the monthly death rate δ, above, we should expect the monthly birth rate per rabbit, β, to be a function of the number of rabbits, $\beta = \beta(R)$. Moreover:

> If R is small, then overcrowding is not a problem and $\beta(R)$ should be close to its ideal value $\beta_0 = 5/4$,

and

> as R increases, then more rabbits become malnourished, and females have fewer babies each month. So, $\beta(R)$ should decrease from its ideal value as R increases.

A simple formula describing this is obtained by subtracting from the ideal birth rate a simple correction term proportional to the number of rabbits,

$$\beta \;=\; \beta(R) \;=\; \beta_0 \;-\; \gamma_B R$$

where $\beta_0 = 5/4$ is the ideal monthly birthrate per rabbit and γ_B is some positive constant.[4] This then gives us

$$\text{number of births per month} \;=\; \beta R \;=\; \left[\beta_0 \;-\; \gamma_B R\right] R \;=\; \beta_0 R \;-\; \gamma_B R^2 \quad .$$

As with our simpler model, the one we are developing can be applied to populations of other organisms by using the appropriate value for the ideal birthrate per organism, β_0. For rabbits, we have $\beta_0 = 5/4$.

Combining the above formulas for the monthly number of births and deaths with the generic differential equation (10.11) yields

$$\begin{aligned}\frac{dR}{dt} &= \text{number of births per month} \;-\; \text{number of deaths per month}\\ &= \beta R \;-\; \delta R\\ &= \beta_0 R \;-\; \gamma_B R^2 \;-\; \gamma_D R^2 \quad ,\end{aligned}$$

which, letting $\gamma = \gamma_B - \gamma_D$, simplifies to

$$\frac{dR}{dt} \;=\; \beta_0 R \;-\; \gamma R^2 \tag{10.12}$$

where $\beta_0 = 5/4$ is the ideal monthly birthrate per rabbit and γ is some positive constant. Presumably, γ could be determined by observation (if this new model does accurately describe the situation).

[4] Yes, the birthrate becomes negative if R becomes large enough, and negative birthrates are not realistic. But we still trying for as simple a model as feasible — with luck R will not get large enough that the birthrate becomes negative.

Using the Better Model

Equation (10.12) is called the *logistic equation* and was an important development in the study of population dynamics. It is a relatively simple separable equation that can be solved without too much difficulty. But let's not, at least, not yet. Instead, let us first rewrite this equation by factoring out γR,

$$\frac{dR}{dt} = \gamma R \left(\frac{\beta_0}{\gamma} - R \right) \quad .$$

From this it is obvious that our differential equation has two constant solutions,

$$R = 0 \qquad \text{and} \qquad R = \frac{\beta_0}{\gamma} \quad .$$

The first tells us that, if we start with no rabbits, then we get no rabbits in the future (no surprise there). The second is more interesting. For convenience, let $\kappa = {}^{\beta_0}/_{\gamma}$. Our differential equation can then be written as

$$\frac{dR}{dt} = \gamma R (\kappa - R) \qquad \text{with} \quad \gamma = \frac{\beta_0}{\kappa} \quad , \tag{10.13}$$

and the two constant solutions are

$$R = 0 \qquad \text{and} \qquad R = \kappa \quad .$$

(We probably should note that κ, as a ratio of positive constants, is a positive constant.)

While we are at it, let's further observe that, if $0 < R < \kappa$, then

$$\frac{dR}{dt} = \underbrace{\gamma R}_{>0} \underbrace{(\kappa - R)}_{>0} > 0 \quad .$$

In other words, if $0 < R < \kappa$, then the population is increasing.

On the other hand, if $\kappa < R$, then

$$\frac{dR}{dt} = \underbrace{\gamma R}_{>0} \underbrace{(\kappa - R)}_{<0} < 0 \quad .$$

That is, the population will be decreasing if $\kappa < R$.

We can graphically represent these observations using the crude slope field sketched in figure 10.1a. This figure suggests that, over time, the number of rabbits will stabilize around κ. If there are initially fewer than κ rabbits (but at least some), then the rabbit population will increase towards a total of κ rabbits. If there are initially more than κ rabbits, then the population will decrease towards a total of κ rabbits. This prediction is reflected in the more carefully constructed slope field in figure 10.1b. Because κ is the maximum number of rabbits that can exist in the long term given the resources available, κ is often called the *carrying capacity* of the 'system' consisting of the rabbits and their environment. (Of course, if the carrying capacity is too low, say, $\kappa \approx 0.5$ then, realistically, all the rabbits will die.)

Finding the precise formula for $R(t)$ will be left to you (exercise 10.7). What you will show is that, in terms of the carrying capacity κ, ideal birthrate β_0, and the initial population $R_0 = R(0)$,

$$R(t) = \frac{\kappa R_0}{R_0 + (\kappa - R_0) e^{-\beta_0 t}} \quad . \tag{10.14}$$

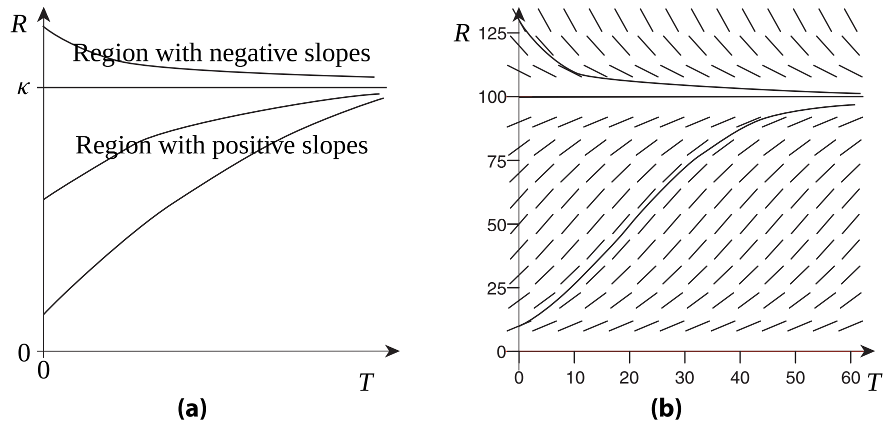

Figure 10.1: Slope Fields for Logistic Equation (10.13): **(a)** A minimal field for a generic logistic equation and **(b)** a slope field for the logistic equation with $\kappa = 100$ and $\beta_0 = {}^1\!/_{10}$. The graphs of a few solutions have been roughly sketched on each.

This formula reflects the fact that there are three basic parameters in our model: the ideal monthly birth rate β_0, the initial number of rabbits $R(0)$, and the carrying capacity of the system κ. The first two we know or can figure out from basic biology. The last, κ, will have to be determined "from experiment". For example, we might return a year after releasing the original pair (at $t = 12$), count the number of rabbits on the ranch, $R(12)$, and then use this value along with the known values for R_0 and β_0 in formula (10.14) to create an equation for κ. Solving that equation will then give us κ. This, of course, assumes that the model is fairly accurate — an assumption that would require further experiment to verify or disprove. But, at least the model's prediction regarding the population growth seems a good deal more reasonable than that made by the simpler model in section 10.2.

10.5 Notes on the Art and Science of Modeling

Our current interest is in modeling situations in which the rate at which some quantity varies is fairly well understood. In these sorts of problems, it is often "relatively easy" to develop a first-order differential equation to serve as the basis for a mathematical model for the situation. We've seen several examples already, and will see more in the next few sections. But now, let us pause to discuss some of the steps and issues in developing and using such models.

First Steps in the Modeling Process

Naturally, one of your very first steps in modeling something should be to learn whatever you believe is needed for developing the model. Then identify and label the significant basic variables and decide on the units associated with these variables. In our rabbit ranch problems, those variables were R and t (with associated units *rabbits* and *months*, respectively); in the following, we'll use Q for the generic quantity of interest and assume it varies over time t.

Next, write out everything you know using these variables. This includes any initial values you may have for any of the variables. In our rabbit ranch problem, we did not know much at first, only the initial value for R, $R(0) = 2$. If you can draw an illuminating picture representing the situation, do so and label it for easy reference.

Then turn your attention to deriving a differential equation that accurately models the *rate* at which Q varies with t, dQ/dt. Do *not* attempt to directly derive a formula for $Q(t)$, at least, not with the sort of problems being considered here. We are now dealing with problems for which it is much easier to first find a formula $F(t, Q)$ for dQ/dt, and then find $Q(t)$ by solving the resulting differential equation.

Developing the Differential Equation for the Model

Coming up with a usable differential equation

$$\frac{dQ}{dt} = F(t, Q)$$

that accurately models how Q's rate of change depends on t and Q is the most important, and, for many, the hardest part of the of the modeling process. After all, as you now know, anyone can solve a first-order differential equation (or, at least, construct a slope field for one). Coming up with the right differential equation can be much more challenging.

Here are a few things you can do to make it less challenging:

Identify and Describe the Processes Driving the Model

Keep in mind that dQ/dt is the *rate* at which $Q(t)$ changes as t changes. This rate depends on the processes driving the situation, not on the particular value of Q at some particular time. In particular, the value of $Q(0)$ is irrelevant in setting up the differential equation.

Once you've determined your variables and drawn your illuminating pictures, write out[5]

$$\frac{dQ}{dt} = \text{the change in } Q \text{ per unit time}$$

and then identify the different processes that cause that Q to change. In our examples with rabbits, these processes were "birth" and "death", and we initially observed that

$$\begin{aligned} \frac{dR}{dt} &= \text{change in the number of rabbits per month} \\ &= \text{number of births per month} - \text{number of deaths per month} \quad . \end{aligned}$$

Then we worked out how many births and deaths should be expected each month given that we had R rabbits that month.

In general, you want to identify the different processes that cause $Q(t)$ to increase (e.g., births and immigration into the region) and to decrease (e.g., deaths and emigration out of the region). Each of these processes corresponds to a different term in $F(t, Q)$ (remember to *add* those terms corresponding to processes that increase Q, and *subtract* those terms corresponding to processes that decrease Q). For example, if $Q(t)$ is the number of, say, people in a certain region at time t, we may have

$$\frac{dQ}{dt} = F(t, Q)$$

---

[5] As noted in an earlier footnote, we are equating an "instantaneous rate of change", dQ/dt, to a "change in Q per unit time", which, in turn, will be based on the values of Q and t at a specific time t instead of over the unit interval of time. For a more detailed analysis justifying this, see appendix 10.8 on page 231.

where

$$F(t, Q) = F_{\text{birth}} + F_{\text{immig}} - F_{\text{death}} - F_{\text{emig}}$$

with

$$F_{\text{birth}} = \text{number of births per unit time} \quad ,$$

$$F_{\text{immig}} = \text{number of number of people immigrating into the region per unit time} \quad ,$$

$$F_{\text{death}} = \text{number of deaths per unit time} \quad ,$$

and

$$F_{\text{emig}} = \text{number of number of people emigrating out of the region per unit time} \quad .$$

Once you've identified the different terms making up $F(t, Q)$ (e.g., the above F_{birth}, F_{immig}, etc.); take each term, consider the process it is supposed to describe, and try to come up with a reasonable formula describing the change in Q during a unit time interval due to that process alone. Often, that formula will involve just Q, itself. For example, in our first "rabbit ranch" example (with $R = Q$),

$$\begin{aligned} F_{\text{birth}} &= \text{number of births per month} \\ &= \text{number of births per female rabbit per month} \\ &\qquad \times \text{number of female rabbits that month} \\ &= \cdots \\ &= \beta R \qquad \text{with} \quad \beta = \frac{5}{4} \quad . \end{aligned}$$

Often, you will make 'simplifications' and 'assumptions' to keep the model from becoming too complicated. In the above formula for F_{birth}, for example, we did not attempt to account for seasonal variations in birth rate, and we assumed that half the rabbit population were breeding females. We also assumed a constant monthly birthrate and death rate per rabbit, no matter how many rabbits we had.

Balance the Units

As already noted, one of the first steps in modeling a situation is to decide on the main variables and to choose the units for measuring these variables. The subsequent computations and derivations are all in terms of these units, and we can often avoid embarrassing mistakes by just keeping track of our units and being sure that their use is consistent. In particular, the units implicit in any equation must remain balanced; that is, each term in any equation must have the same associated units as every other term in that equation.

For example, the basic quantities in our rabbit ranch models are R and t. Even though we treated these as numbers, we knew that

$$R = \text{number of } \textit{rabbits} \qquad \text{and} \qquad t = \text{time in } \textit{months} \quad .$$

So the units associated with R and t are, respectively, *rabbits* and *months*. Consequently, the units associated with

$$\frac{dR}{dt} = \lim_{\Delta t \to 0} \frac{R(t + \Delta t) - R(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\text{change in the number of } \textit{rabbits}}{\text{change in time as measured in } \textit{months}}$$

are ${}^{rabbits}/_{month}$ (i.e., rabbits per month), and every term in any formula for ${}^{dR}/_{dt}$ must also have ${}^{\text{rabbits}}/_{\text{month}}$ as the associated units. If someone suggested a term that was not "rabbits per month", then that term would be wrong and should be immediately rejected. Thus, for example,

$$\frac{dR}{dt} = Rt$$

is clearly wrong because the right side is describing *rabbits* $\times$*months*, not ${}^{rabbits}/_{month}$.

The constants used in our derivations also have associated units. The monthly birth rate per rabbit,

$$\begin{aligned}\beta &= \text{number of rabbits born per month per rabbit on the ranch}\\ &= \left(\frac{\text{number of } \textit{rabbits} \text{ born}}{\textit{month}}\right) \text{ per } \textit{rabbit} \text{ on the ranch}\\ &= \frac{\text{number of } \textit{rabbits} \text{ born}}{\textit{month} \times \textit{rabbit}} \quad ,\end{aligned}$$

has associated units ${}^{1}/_{\text{month}}$ (since the units of *rabbits* cancel out), and if we had wanted to be a bit more explicit, we would have written equation (10.4) on page 214 as

$$\beta = \frac{5}{4}\left(\frac{1}{\text{month}}\right)$$

instead of just

$$\beta = \frac{5}{4} \quad .$$

Often, you will not see the units being explicitly noted throughout the development and use of a model. There are several possible reasons for this:

1. If the formulas and equations are correctly developed, then the units in theses formulas and equations naturally remain balanced. The modeler knows this and trusts his or her skill in modeling.

2. The writer assumes the readers can keep track of the units themselves.

3. The writer is lazy or needs to save space.

There is much to be said in favor of explicitly giving the units associated with every element of every formula and equation. It helps prevent stupid mistakes and may help clarify the meaning of some of formulas for the reader (and for the model builder). We will do this, somewhat, in the next major example. Beginning modelers are strongly encouraged to keep track of the units in every step as they develop their own equations. At the very least, stop every so often and check that the units in the equations balance. If not, you did something wrong — go back, find your error, and correct it.

Oh yes, one more thing about "units": Be sure that anyone who is going to read your work or use any model you've developed knows the units you are using.[6]

[6] In 1999, the Mars Climate Orbiter crashed into Mars instead of orbiting the planet because the Orbiter's software gave instructions in terms of the imperial system (which measures force in pounds) while the hardware assumed the metric system (which measures force in newtons – with 1 pound $\approx$ 4.45 newtons). This failure to communicate the units being used caused an embarrassing end to a space project costing over 300 million dollars.

Testing and Using the Model

Once you've developed a differential equation modeling the way a quantity of interest $Q(t)$ varies, you will normally want to solve that differential equation or otherwise analyze it to see what it says about $Q(t)$ for various choices of t. If the situation being modeled is fairly simple and straightforward, and your modeling skills are adequate, then your model can probably be trusted to give fairly accurate predictions.

In practice, it is usually wise to check and see if predictions based on this model are reasonable before announcing your new model to the world. After all, it is quite possible that some of your 'simplifications' and 'assumptions' overly simplified your model and caused important issues to be ignored. That certainly happened with our first rabbit ranch model in which assuming constant the birth and death rates resulted in a model predicting far more rabbits in five years than possible.

If your predictions are not reasonable, go back, revisit your derivations, and see where a more careful modeling of the individual processes leads. In necessary, learn more about the processes themselves.[7] This should lead to a refined model for $Q(t)$ that, in turn, leads to more reasonable projections as to the behavior of $Q(t)$. The differential equation will probably be more complicated, but that is the price you pay for a better, more accurate model.

Of course, you should not automatically assume that 'apparently reasonable' predictions are accurate. If possible, compare results predicted by the model to "real world" data. You may need to do this anyway to determine the values of some of the constants in your model. Hopefully, the results predicted and the real world data will agree well enough that you can feel confident that your model is sufficiently accurate for the desired applications. If not, refine your model further.

By the way, in using your model, keep in mind the simplifications and assumptions made in deriving it so that you have some idea as to the limitations of this model.

10.6 Mixing Problems

In a "mixing problem", some substance is continually being added to some container in which the substance is mixed with some other material, and with the resulting mixture being constantly drained off at some rate. This container may be a large tank, a lake, or the system of veins and arteries in a body; and the substance being added may be some chemical, pollutant, or medicine being added to the liquid in the tank, the water in the lake, or the blood in the body. These problems are favorites of authors of differential equation texts because they can be modeled fairly easily using the basic observation that (usually)

$$\begin{aligned}&\text{the rate the amount of substance in the container changes}\\&\qquad = \text{the rate the substance is added} - \text{the rate the substance is drained off} \quad .\end{aligned}$$

We will do one simple mixing problem, and then briefly mention some possible variations.

[7] This author once read a paper describing a 'new' model for "laser interaction with a solid material". Using this model, you could then show that any solid can be chilled to absolute zero by suitably heating it with a laser — a rather dubious result. That paper's author should have better tested his model and learned more about thermodynamics.

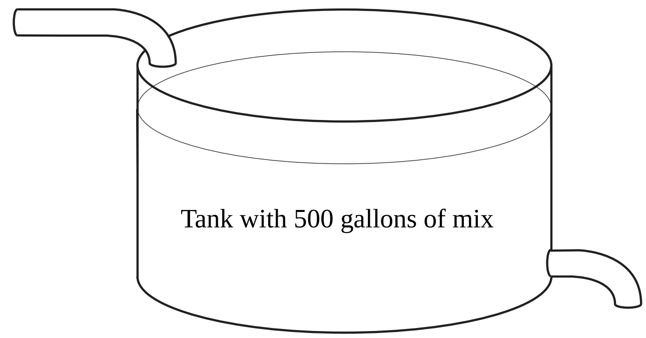

Figure 10.2: Figure illustrating a simple mixing problem.

A Simple Mixing Problem
The Situation to Be Modeled

We start out with a large tank containing 500 gallons of pure water. Each minute thereafter, two gallons of a alcohol-water mix is added, and two gallons of the mixture in the tank is drained. The alcohol-water mix being added is 90 percent alcohol. Throughout this entire process, we assume the mixture in the tank is thoroughly and uniformly mixed. The problem is to develop a formula describing the amount of alcohol in the tank at any given time. In particular, let's determine if and when the mixture in the tank is 50 percent alcohol.

Setting Up the Model

In this case, a simple, illustrative picture for the process is easily drawn. It is in figure 10.2. We will let

$$t = \text{number of minutes since we started adding the alcohol-water mix}$$

and

$$y = y(t) = \text{gallons of pure alcohol in the tank at time } t \quad .$$

Since we started with a tank containing pure water (no alcohol), the initial condition is

$$y(0) = 0 \quad .$$

Our derivation of the differential equation modeling the change in y starts with the observation that

$$\begin{aligned}\frac{dy}{dt} &= \text{change in the amount of alcohol in the tank per minute minute} \\ &= \text{rate alcohol is added to the tank} - \text{rate alcohol is drained from the tank} \quad .\end{aligned}$$

Since we are adding 2 gallons per minute of a 90 percent alcohol mix,

$$\begin{aligned}\text{rate alcohol is added to the tank} &= 2\left(\frac{\text{gallons of input mix}}{\text{minute}}\right) \times \frac{90}{100}\left(\frac{\text{gallons of alcohol}}{\text{gallons of input mix}}\right) \\ &= \frac{9}{5}\left(\frac{\text{gallons of alcohol}}{\text{minute}}\right) \quad .\end{aligned}$$

In determining how much is being drained away, we must determine the concentration of alcohol in the tank's mixture at any given time, which is simply the total amount of alcohol in the tank at that time (i.e., $y(t)$ gallons) divided by the total amount of the mixture in the tank (which, because we drain off as much as we add, remains constant at 500 gallons). So,

$$\begin{aligned}
\text{rate alcohol is drained from the tank} &= 2\left(\frac{\text{gallons of tank mix}}{\text{minute}}\right) \\
&\quad \times \text{amount of alcohol per gallon of tank mix} \\
&= 2\left(\frac{\text{gallons of tank mix}}{\text{minute}}\right) \times \frac{y(t)\text{ (gallons of alcohol)}}{500\text{ (gallons of tank mix)}} \\
&= \frac{y(t)}{250}\left(\frac{\text{gallons of alcohol}}{\text{minute}}\right) \quad .
\end{aligned}$$

Combining the above gives us

$$\begin{aligned}
\frac{dy}{dt} &= \text{rate alcohol is added to the tank} - \text{rate alcohol is drained from the tank} \\
&= \frac{9}{5} - \frac{y(t)}{250} \quad \left(\frac{\text{gallons of alcohol}}{\text{minute}}\right) \quad .
\end{aligned}$$

Thus, the initial-value problem that $y = y(t)$ must satisfy is

$$\frac{dy}{dt} = \frac{9}{5} - \frac{y}{250} \qquad \text{with} \quad y(0) = 0 \quad . \tag{10.15}$$

Using the Model

Factoring out $1/250$ on the right side of our differential equation yields

$$\frac{dy}{dt} = \frac{1}{250}(450 - y) \quad .$$

From this we see that

$$y = 450$$

is the only constant solution. Moreover,

$$\frac{dy}{dt} = \frac{1}{250}(450 - y) > 0 \qquad \text{if} \quad y < 450 \quad ,$$

and

$$\frac{dy}{dt} = \frac{1}{250}(450 - y) < 0 \qquad \text{if} \quad y > 450 \quad .$$

So, we should expect the graphs of the possible solutions to this differential equation to be something like the curves in figure 10.3. In other words, no matter what the initial condition is, we should expect $y(t)$ to approach 450 as $t \to \infty$.

Fortunately, the differential equation at hand is fairly simple. It (the differential equation in initial-value problem (10.15)) is both separable and linear, and, using either the method we developed for separable equations or the method we developed for linear equations, you can easily show that

$$y(t) = 450 - Ae^{-t/250}$$

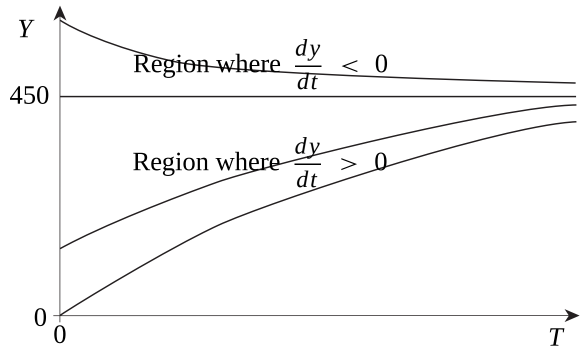

Figure 10.3: Crude graphs of solutions to the simple mixing problem from figure 10.2 based on the sign of dy/dt.

is the general solution. Note that, as $t \to \infty$,

$$y(t) = 450 - Ae^{-t/250} \to 450 - A \cdot 0 = 450 \quad ,$$

just as figure 10.3 suggests. Consequently, no matter how much alcohol is originally in the tank, eventually there will be nearly 450 gallons of alcohol in the tank. Since the tank holds 500 gallons of mix, the concentration of the alcohol in the mix will eventually be nearly $450/500 = 9/10$ (i.e., 90 percent of the liquid in the tank will be alcohol).

For our particular problem, $y(0) = 0$. So,

$$0 = y(0) = 450 - Ae^{-0/250} = 450 - A \quad .$$

Hence, $A = 450$ and

$$y(t) = 450 - 450e^{-t/250} \quad .$$

Finally, recall that we wanted to know when the mixture in the tank is 50 percent alcohol. This will be the time when half the liquid in the tank (i.e., 250 gallons) is alcohol. Letting τ denote this time, we must have

$$250 = y(\tau) = 450 - 450e^{-\tau/250}$$

$$\Longrightarrow \quad 450e^{-\tau/250} = 450 - 250$$

$$\Longrightarrow \quad e^{-\tau/250} = \frac{200}{450} = \frac{4}{9}$$

$$\Longrightarrow \quad -\frac{\tau}{250} = \ln\left(\frac{4}{9}\right) = -\ln\left(\frac{9}{4}\right) \quad .$$

So the mixture in the tank will be half alcohol at time

$$\tau = 250\ln\left(\frac{9}{4}\right) \approx 202.7 \text{ (minutes)} \quad .$$

Other Mixing Problems

All sorts of variations of the problem just discussed can be visualized:

1. Instead of adding an alcohol-water mix, we may be adding a mixture of so many ounces of some chemical (such as sugar or salt) dissolved in the water (or other solvent).

2. The flow rate into the tank may be different from the drainage flow rate. In this case, the volume of the mixture in the tank will be changing, and that will affect how the concentration in the tank is computed.

3. We may have the problem considered in our simple mixing problem, but with some of the drained flow being diverted to a machine that magically converts a certain fraction of the alcohol to water, and the flow from that machine being dumped back into the tank. (Think of that machine as the tank's 'liver'.)

4. Instead of adding an alcohol-water mix, we may be adding a certain quantity of some microorganism (yeast, e-coli bacteria, etc.) in a nutrient solution. Then we would have to consider a mixture/population dynamics model to also account for the growth of the microorganism in the tank, as well as the in-flow and drainage.

5. And so on … .

10.7 Simple Thermodynamics

Bring a hot cup of coffee into a cool room, and, in time, the coffee cools down to room temperature. Put a similar hot cup of coffee into a refrigerator, and you will discover that the coffee cools down faster. Let's try to describe this cooling process a little more precisely.

To be a little more general, let us simply assume we have some object (such as a hot cup of coffee or a cold glass of water) that we place in a room in which the air is at temperature T_{room} . To keep matters simple, assume T_{room} remains constant. Let $T = T(t)$ be the temperature at time t of the object we placed in the room. As time t goes on, we expect T to approach T_{room} . Now consider

$$\frac{dT}{dt} = \text{rate at which } T \text{ approaches } T_{\text{room}} \text{ as time } t \text{ increases} \quad .$$

It should seem reasonable that this rate at any instant of time t depends just on the difference between the temperature of the object and the temperature of the room, $T - T_{\text{room}}$; that is

$$\frac{dT}{dt} = F(T - T_{\text{room}}) \quad . \tag{10.16}$$

for some function F . Moreover,

1. If $T - T_{\text{room}} = 0$, then the object is the same temperature as the room. In this case, we do not expect the object's temperature to change. Hence, we should have ${}^{dT}/_{dt} = 0$ when $T = T_{\text{room}}$.

2. If $T - T_{\text{room}}$ is a large positive value, then the object is much warmer than the room. We then expect the object to be rapidly cooling; that is, T should be a rapidly decreasing function of t. Hence $^{dT}/_{dt}$ should be large and negative.

3. If $T - T_{\text{room}}$ is a large negative value, then the object is much cooler than the room. We then expect the object to be rapidly warming; that is, T should be a rapidly increasing function of t. Hence $^{dT}/_{dt}$ should be large and positive.

In terms of the function F on the right side of equation (10.16), these three observations mean

$$T - T_{\text{room}} = 0 \quad \Longrightarrow \quad F(T - T_{\text{room}}) = 0 \quad ,$$

$$T - T_{\text{room}} \text{ is a large positive value} \quad \Longrightarrow \quad F(T - T_{\text{room}}) \text{ is a large negative value}$$

and

$$T - T_{\text{room}} \text{ is a large negative value} \quad \Longrightarrow \quad F(T - T_{\text{room}}) \text{ is a large positive value} \quad .$$

The simplest choice of F satisfying these three conditions is

$$F(T - T_{\text{room}}) = -\kappa(T - T_{\text{room}})$$

where κ is some positive constant. Plugging this into equation (10.16) yields

$$\frac{dT}{dt} = -\kappa(T - T_{\text{room}}) \quad . \tag{10.17}$$

This equation is often known as *Newton's law of heating and cooling*. The positive constant κ describes how easily heat flows between the object and the air, and must be determined by experiment.

Equation (10.17) states that the change in the temperature of the object is proportional to the difference in the temperatures of the object and the room. It's not exactly the same as equation (10.8) on page 216 (unless $T_{\text{room}} = 0$), but it is quite similar in spirit. We'll leave its solution and further discussion as exercises for the reader.

10.8 Appendix: Approximations That Are Not Approximations

In our first rabbit ranch model, (after assuming a death rate of zero), our derivation of the model can, essentially, be described by

$$\frac{dR}{dt} = \text{number of births per month} = \beta R(t)$$

where

$$\beta = \text{monthly birth rate per rabbit} \quad .$$

Those who are comfortable with calculations involving rates should be comfortable with this. Others, however, may be concerned that we have two approximations here: The first is in approximating the derivative $^{dR}/_{dt}$ (an ''instantaneous rate of change at time t ") by the monthly rate of change. The second is in describing this monthly rate of change in terms of $R(t)$, the number of rabbits at the instant of time t, even though the number of rabbits clearly changes over a month.

Let us reassure those concerned readers by looking at this derivation a little more carefully: We start by recalling the definition of the derivative of R at time t:

$$\frac{dR}{dt} = \lim_{\Delta t \to 0} \frac{\Delta R}{\Delta t}$$

where

$$\Delta R = R(t + \Delta t) - R(t) = \textit{change} \text{ in } R \text{ as time changes from } t \text{ to } t + \Delta t \quad .$$

Of course, the ‘ $R(t + \Delta t) - R(t)$ ’ formula for ΔR is pretty useless since we don’t have the formula for R. However, we can approximate ΔR via

$$\begin{aligned} \Delta R &= \text{bunnies born as time changes from } t \text{ to } t + \Delta t \\ &\leq \text{monthly birth rate per rabbit} \\ &\qquad \times \text{maximum number of rabbits at any one time between } t \text{ and } t + \Delta t \\ &\qquad\quad \times \text{length of time (in months) between } t \text{ and } t + \Delta t \\ &= \beta R_{\max} \Delta t \end{aligned}$$

where

$$R_{\max} = \text{maximum number of rabbits at any one time between } t \text{ and } t + \Delta t \quad .$$

Note that

$$\begin{aligned} \lim_{\Delta t \to 0} R_{\max} &= \text{maximum number of rabbits at any one time between } t \text{ and } t + 0 \\ &= \text{number of rabbits at time } t \\ &= R(t) \quad . \end{aligned}$$

Consequently,

$$\frac{dR}{dt} = \lim_{\Delta t \to 0} \frac{\Delta R}{\Delta t} \leq \lim_{\Delta t \to 0} \beta R_{\max} = \beta R(t) \quad .$$

Similar arguments with

$$R_{\min} = \text{minimum number of rabbits at any one time between } t \text{ and } t + \Delta t$$

yields

$$\frac{dR}{dt} \geq \lim_{\Delta t \to 0} \beta R_{\min} = \beta R(t) \quad .$$

Together the two above inequalities involving ${}^{dR}/_{dt}$ tells us that

$$\beta R(t) \leq \frac{dR}{dt} \leq \beta R(t)$$

which, of course, means that

$$\frac{dR}{dt} = \beta R(t) \quad ,$$

just as we originally derived.

More generally, this sort of analysis can be used to justify letting

$$\frac{dQ}{dt} = \frac{\Delta Q}{\Delta t}$$

where Δt is the unit time interval in whatever units we are using, and then deriving a formula for ${}^{\Delta Q}/_{\Delta t}$ in terms of t and $Q(t)$, just as we do in our examples, and just as you should do in the exercises.

Additional Exercises

10.2. *Do the following using formula (10.7) on page 215 from the simple model for the rabbit population on our rabbit ranch:*

a. *Find the approximate number of rabbits on the ranch after one year.*

b. *How long does it take for the number of rabbits to increase*

i. *from* 2 *to* 4 *?* **ii.** *from* 4 *to* 8 *?* **iii.** *from* 8 *to* 16 *?*

c. *How long does it take for the number of rabbits to increase*

i. *from* 2 *to* 20 *?* **ii.** *from* 5 *to* 50 *?* **iii.** *from* 10 *to* 100 *?*

d. *Approximately how long does it take for the mass of the rabbits on the ranch to equal the mass of the Earth?*

10.3. (Epidemiology) *Imagine the following situation:*

> *A stranger infected with a particularly contagious strain of the sniffles enters a city. Let* $I(t)$ *be the number of people in the city infected with the sniffles* t *days after the stranger entered the city. Assume that only the stranger has the sniffles on day* 0 *, and that the number of people with the sniffles increases exponentially thereafter (as derived in the simple population growth model in section 10.3). Assume further that 50 people have the sniffles on the tenth day after the stranger entered the city, .*

Let $I(t)$ *be the number of people in the city with sniffles on day* t .

a. *What is the formula for* $I(t)$ *?*

b. *How many people have the sniffles on day 20?*

c. *Approximately how long until 250,000 people in the city have the sniffles?*

10.4. *Assume that* $A(t) = A_0 e^{-\delta t}$ *is the amount of some radioactive substance at time* t *having a half-life* $\tau_{1/2}$.

a. *Verify that, for each value of* t *(not just* $t = 0$ *),*

$$A(t + \tau_{1/2}) = \frac{1}{2} A(t) \quad .$$

b. *Verify that the formula* $A(t) = A_0 e^{-\delta t}$ *can be rewritten as*

$$A(t) \;=\; A_0 \left(\frac{1}{2}\right)^{t/\tau_{1/2}} \quad .$$

10.5. *Cesium-137 is a radioactive isotope of cesium with a half-life of about 30 years.*

a. *Find the corresponding decay constant* δ *for cesium-137.*

b. *Suppose we have a bottle (which we never open) containing 20 grams of cesium-137. Approximately how many grams of cesium-137 will still be in the bottle*

i. *after 10 years?* **ii.** *after 25 years?* **iii.** *after 100 years?*

10.6. (Carbon-14 dating) *A little background:*

> *Most of the carbon in living tissue comes, directly or indirectly, from the carbon dioxide in the air. A tiny fraction (about one part per trillion) of this carbon is the radioactive isotope carbon-14 (which has a half-life of approximately 5,730 years). The rest of the carbon is not radioactive. As a result, about one trillionth of the carbon in the tissues of a living plant or animal is that radioactive form of carbon. This ratio of carbon-14 to nonradioactive carbon in the air and living tissue has remained fairly constant*[8] *because the rate at which carbon-14 is created (through an interaction of cosmic radiation with the nitrogen in the upper atmosphere) matches the rate at which it decays.*
>
> *At death, however, the plant or animal stops absorbing carbon, and the tiny amount of carbon-14 in its tissues begins to decrease due to radioactive decay. By measuring the current ratio of carbon-14 to the nonradioactive carbon in a tissue sample (say, a piece of old bone or wood), and then comparing this ratio to the ratio in comparable living tissue, a good estimate of fraction of the carbon-14 that has decayed can be made. Using that and our model for radioactive decay, the age of the bone or wood can then be approximated.*

Using the above information:

a. *Find the (approximate) decay constant* δ *for carbon-14.*

b. *Suppose a piece of wood came from a tree that died* t *years ago. Approximately what percentage of the carbon-14 that was in piece of wood when the tree died still remains undecayed if*

i. $t = 10$ *years?* **ii.** $t = 100$ *years?* **iii.** $t = 1000$ *years?*

iv. $t = 5000$*years?* **v.** $t = 10000$*years?* **vi.** $t = 50000$*years?*

c. *Suppose a skeleton of a person found in an ancient grave contains 30 percent of the carbon-14 normally found in (equally sized) skeletons of living people. Approximately how long ago did this person die?*

---

[8] but not perfectly constant — see a good article on carbon-14 dating.

d. *The wood in the ornate funeral mask of the Egyptian pharaoh Rootietootiekoomin*[9] *is found to contain 60 percent of the carbon-14 originally in the wood. Approximately how long ago did Rootietootiekoomin die?*

e. *Let* A *be the amount of carbon-14 measured in a tissue sample (e.g., an old bone or piece of wood), and let* A_0 *be the amount of carbon-14 in the tissue when the plant or creature died. Derive a formula for the approximate length of time since that plant's or creature's demise in terms of the ratio* A/A_0 *.*

10.7. *Consider the "better model" for the rabbit population in section 10.4.*

a. *Solve the logistic equation derived there (equation (10.13) on page 221), and verify that the solution can be written as given in formula (10.14) on page 221.*

b. *Assume the same values for the initial number of rabbits and ideal birth rate as assumed in section 10.4,*

$$R(0) = 2 \qquad \text{and} \qquad \beta_0 = \frac{5}{4} \quad .$$

Also assume that our rabbit ranch has a carrying capacity κ *of 10,000,000 rabbits (it's a big ranch). How many rabbits (approximately) does our "better model" predict will be on our ranch*

i. *at the end of the first 6 months?*

ii. *at the end of the first year? (Compare this to the number predicted by the simple model in exercise 10.2 a, and to the carrying capacity.)*

iii. *at the end of the second year? (Compare this to the carrying capacity.)*

c. *Solve formula (10.14) on page 221 for the carrying capacity* κ *in terms of* R_0 *,* $R(t)$ *,* β *and* t *.*

d. *Using the formula for the carrying capacity just derived (and assuming the ideal birth rate* $\beta_0 = 5/4$ *, as before), determine the approximate carrying capacity of a rabbit ranch under each of the following conditions:*

i. *You have 1,000 rabbits 6 months after starting with a single breeding pair.*

ii. *You have 2,000 rabbits 6 months after starting with a single breeding pair.*

10.8. *Suppose we have a rabbit ranch and have begun harvesting rabbits. Let*

$$R(t) = \text{number of rabbits on the ranch } t \text{ months after beginning harvesting}$$

and assume the following:

1. *The monthly birth rate per rabbit,* β *, is* $5/4$ *(as we derived).*

2. *We have no problems with overpopulation (i.e., for all practical purposes, we can assume the natural death rate is* 0 *.)*

3. *Each month we harvest* 500 *rabbits. (Assume this is done "over the month", so the rabbits are still reproducing as we are harvesting.)*

[9] from a fictional dynasty

a. *Derive the differential equation for $R(t)$ based on the above assumptions.*

b. *Find any equilibrium solutions to your differential equation (this may surprise you), and, using crude slope fields as we did in class, analyze how the rabbit population varies over time, based on how many we had when we first began harvesting.*

c. *Solve the differential equation. Get your final answer in terms of t and $R_0 = R(0)$.*

10.9. *Repeat the previous problem, only, instead of harvesting 500 rabbits a month, harvest 25 percent of the rabbits on the ranch each month.*

10.10. *Again, assume we have a rabbit ranch, and let*

$$R(t) = \text{number of rabbits on the ranch after } t \text{ months.}$$

Taking into account the problems that arise when the population is too large, we obtained the differential equation

$$\frac{dR}{dt} = \beta R - \gamma R^2$$

where β is the monthly birth rate per rabbit (which we figured was $^5/_4$) and γ was some positive constant that would have to be determined later.

This differential equation was obtained assuming we were not harvesting rabbits. Assume, instead, that we are harvesting h rabbits each month. How do we change the above differential equation to reflect this if

a. *we harvest a constant number h_0 of rabbits each month?*

b. *we harvest one fourth of all the rabbits on the ranch each month?*

10.11. *Consider the following situation:*

> *Mullock the Barbarian begins a campaign of self-enrichment with a horde of 200 vicious warriors. Each week he loses 5 percent of his horde to the unavoidable accidents that occur while sacking and pillaging. Fortunately, the horde's lifestyle of wanton violence and mindless destruction attracts 50 new warriors to the horde each week.*

Let $y(t)$ be the number of warriors in Mullock's horde t weeks after starting the campaign.

a. *Derive the differential equation describing how $y(t)$ changes each week. Is there also an initial value given?*

b. *To what size does the horde eventually grow? (Use equilibrium solutions and graphical methods to answer this. Don't actually solve the initial-value problem.)*

c. *Solve the initial-value problem from the first part.*

d. *How long does it take Mullock's horde to reach 90 percent of its final size?*

10.12. (mixing) *Consider the following mixing problem:*

> *We have a large tank initially containing 1,000 gallons of pure water. We begin adding a alcohol-water mix at a rate of 3 gallons per minute. This alcohol-water mix being added is 75 percent alcohol. At the same time, the mixture in the tank is drained at a rate of 3 gallons per minute. Throughout this entire process, the mixture in the tank is thoroughly and uniformly mixed.*

Let $y(t)$ *be the number of gallons of pure alcohol in the tank* t *minutes after we started adding the alcohol-water mix.*

a. *Find the differential equation for* $y(t)$ *.*

b. *Sketch a crude slope field for the differential equation just obtained, and find any equilibrium solutions.*

c. *Using the differential equation just obtained, find the formula for* $y(t)$ *.*

d. *Approximately how many gallons of alcohol are in the tank at*

i. $t = 10$ *?* ***ii.*** $t = 60$ *?* ***iii.*** $t = 1000$ *?*

e. *Approximately when will the mixture in the tank be half alcohol?*

10.13. *Redo exercise 10.12, but assuming the tank initially contains 900 gallons of pure water and 100 gallons of alcohol.*

10.14. *Consider the following mixing problem:*

> *We have a tank initially containing 5000 gallons of pure water, and start adding saltwater (containing 2 ounces of salt per gallon of water) at the rate of 2 gallons per minute. At the same time, the resulting mixture in the tank is drained at the rate 2 gallons per minute. As usual, the mixture in the tank is thoroughly and uniformly mixed at all times.*

Let $y(t)$ *be the number of ounces of salt in the tank at* t *minutes after we started adding the saltwater.*

a i. *Find the differential equation for* $y(t)$ *.*

ii. *Sketch a crude slope field for the differential equation just obtained, and find any equilibrium solutions.*

iii. *Using the differential equation just obtained along with any given initial values, find the formula for* $y(t)$ *.*

b. *Approximately how many ounces of salt are in the tank at*

i. $t = 10$ *?* ***ii.*** $t = 50$ *?* ***iii.*** $t = 100$ *?*

c. *Approximately when will the concentration of the salt in the tank be 1 ounce of salt per gallon of water?*

10.15. *Redo exercise 10.14, but assuming that a device has been attached to the tank that, each minute, filters out half the salt in a single gallon from the mixture in the tank.*

10.16. *Consider the following variation of the mixing problem in exercise 10.12:*

> *We have a large tank initially containing 500 gallons of pure water, and start adding saltwater (containing 2 ounces of salt per gallon of water) at the rate of* 2 *gallons per minute. At the same time, the resulting mixture in the tank is drained at the rate* 3 *gallons per minute. As usual, assume the mixture in the tank is thoroughly and uniformly mixed at all times.*

Note that the tank is being drained faster that it is being filled.

Let $y(t)$ *the number of ounces of salt in the tank at* t *minutes after we started adding the saltwater.*

a. *What is the formula for the volume of the liquid in the tank* t *minutes after we started adding the saltwater.*

b i. *Find the differential equation for* $y(t)$ *. (Keep in mind that the concentration of salt in the outflow at time* t *will depend both on both the amount of salt* and *the volume of the liquid in the tank at that time.)*

ii. *Using the differential equation just obtained along with any given initial values, find the formula for* $y(t)$ *.*

c. *Approximately how many ounces of salt are in the tank at*

i. $t = 10$ *?* **ii.** $t = 60$ *?* **iii.** $t = 100$ *?*

d i. *When will there be exactly 1 gallon of saltwater in the tank?*

ii. *Approximately how much salt will be in that gallon of saltwater?*

10.17. (heating/cooling) *Consider the following situation:*

> *At 2 o'clock in the afternoon, the butler reported discovering the dead body of his master, Lord Hakky d'Sack, in the Lord's personal wine cellar. The Lord had apparently been bludgeoned to death with a bottle of* Rip'le 04*. At 4 o'clock, the forensics expert arrived and measured the temperature of the body. It was 90 degrees at that time. One hour later, the body had cooled down to 80 degrees. It was also noted that the wine cellar was maintained at a constant temperature of 50 degrees.*

Should the butler be arrested for murder? (Base your answer on the time of death as determined from the above information, Newton's law of heating and cooling and the fact that a reasonably healthy person's body temperature is about 98.2 degrees.)