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Critical elliptic equations via a dynamical systems approach

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ABSTRACT

In this work we consider the existence of positive solutions to various equations of the form

$$\begin{cases} -\Delta u(x) &= (1+g(|x|,u))u(x)^p & \text{in } B_R, \\ u &= 0 & \text{on } \partial B_R, \end{cases}$$

where B_R is the open ball of radius R in $\mathbb{R}^N$ centered at the origin and $p = \frac{N+2}{N-2}$. We will generally assume g is nonnegative. Our approach will be to utilize some dynamical systems approaches.

1. Introduction

In this article we are interested in examining the existence of positive solutions of

$$\begin{cases} -\Delta u(x) = (1+g(x))u(x)^p & \text{in } B_R, \\ \qquad u = 0 & \text{on } \partial B_R, \end{cases} \tag{1}$$

where B_R is the open ball of radius R centered at the origin in $\mathbb{R}^N$ (where $N \geq 3$) and where $p = \frac{N+2}{N-2}$. In this work we will only consider the case of g radial and continuous. We first consider the subcritical case $1 < p < \frac{N+2}{N-2}$. In this case a standard variational approach easily yields a nonzero $H_0^1(B_R)$ solution and then one can apply elliptic regularity theory to show that the solution is in fact as smooth as g allows. In the case of $p = \frac{N+2}{N-2}$ the direct variational approach no longer works since one loses the compactness of the needed imbedding.

1.0.1. The Hénon equation

If one replaces $1 + g(x)$ with $|x|^\alpha$ in (1) then one obtains the well known and extensively studied Hénon equation given by

$$\begin{cases} -\Delta u = |x|^\alpha u^p & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

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A classical Pohozaev argument shows that there is no positive classical solution of (2) provided that Ω is a smooth bounded star shaped domain in $\mathbb{R}^N$ with $p \geq \frac{N+2+2\alpha}{N-2} =: p_\alpha(N)$. This suggests that one may hope to prove the existence of a positive classical solution of (2) in the case where $1 < p < p_\alpha(N)$, and indeed one has the following result:

Theorem A (*Ni [15]*)**.** *Suppose $N \geq 3$, $0 < \alpha$, $\Omega = B_1$ and $1 < p < p_\alpha(N)$. Then there exists a positive classical radial solution of (2).*

Proof. The idea of the proof is to show that $H^1_{0,rad}(B_1) := \{u \in H^1_0(B_1) : u \text{ is radial}\}$ is compactly imbedded in the weighted space $L^{p+1}(B_1, |x|^\alpha dx)$ for $1 < p < p_\alpha(N)$. One can then perform a standard minimization argument to obtain a positive solution of (2). $\square$

After the work of Ni [15] the Hénon equation did not receive much attention until [18], where they examined (2) in the case of $\Omega = B_1$. They showed, among many results, that for $1 < p < \frac{N+2}{N-2}$ the ground state solution is non radial provided that $\alpha > 0$ is sufficiently large. Since this work there have been many related works, see [2–4,19], which show various results regarding properties of solutions to (2) in the case where $\Omega = B_1$. Some of these works include certain ranges of $p > \frac{N+2}{N-2}$. We now mention the recent work [10] where they examine (2) for general bounded domains containing the origin. They show many interesting results, one of which is the existence of positive solutions provided $p = \frac{N+2+2\alpha}{N-2} - \varepsilon$ where $\varepsilon > 0$ is small. In addition they have another recent work [11] where they examine (2) on $\mathbb{R}^N$ and obtain many interesting results. We also mention the very interesting related works [12,13].

1.0.2. A generalized hénon equation

Consider replacing $1 + g(x)$ with $h(x)$ in (1) to get

$$\begin{cases} -\Delta u(x) = h(x)u(x)^p & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{3}$$

where $h \geq 0$ is radial. In the case of $h(0) = 0$ one can use approaches similar to those for the Hénon equation to obtain positive solutions of (3). We mention one result here is that if h is radial and continuous with $h(0) = 0$ and $p = \frac{N+2}{N-2}$ then there is a positive solution of (3); see [21].

In this work we consider (1) in the case that $g \geq 0$ and radial. Note importantly that if $h(x) = 1 + g(x)$ then $h(0) \geq 1$ and hence we need an alternate approach. The approach we will use is a dynamical systems approach developed by the first author in a prior work [1].

Theorem 1. *Let $N \geq 3$, $p = \frac{N+2}{N-2}$ and assume g is nonnegative, radial and Hölder continuous.*

1. *Suppose $\beta, b > 0$ and $g(r) = br^\beta$. Then for all $R > 0$ there is a positive solution of (1).*
2. *Suppose $g \geq 0$ is increasing. Then for sufficiently large R there is a positive solution of (1).*

Remark 1. We make a few remarks about Theorem 1. For the case of large R we can prove part 1 using a perturbation argument of the classical Hénon result. To prove the result for all $R > 0$ we use a dynamical systems approach. For part 2 we use a standard variational approach and again we need a large parameter R.

We now mention some previous related results. In [5,9] the following

$$\begin{cases} -\Delta u(x) = u^{\frac{N+2}{N-2}} + k(x)f(u) & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R, \end{cases} \tag{4}$$

was examined under various assumptions on k and f. In [5] the existence and nonexistence in the case $k(r) = r^\beta$ with $\beta > 0$ and $f(t) = t_+$ is completed. In [9] the case of $k(r) = r^\beta$ and $f(t) = t^q$ is considered for $1 < q < \frac{N+2+2\beta}{N-1}$.

In the next theorem we generalize the equation we are considering and here the proofs will fully utilize our dynamical systems approach.

Theorem 2. *Let $N \geq 3$, $k = \frac{1}{2}(N-2)$, and $\alpha \geq 0$ and $\beta > 0$ be constant that satisfy*

$$0 < \beta - \alpha k < 2k = N - 2. \tag{5}$$

Assume that $g = g(r,u) > 0$ is C^1 for $r > 0$ and $u > 0$ and satisfies

$$\begin{cases} g(rX, r^{-k}u) = r^{\beta-\alpha k} g(X,u) & \forall\, r > 0,\ 0 \leq X \leq 1,\ u > 0, \\ |g_u(r,u)| \leq M_1 u^{-\alpha_1} & \forall\, 0 < r \leq 1,\ 0 < u \leq 1, \end{cases} \tag{6}$$

where $M_1 > 0$ and $0 < \alpha_1 < p$ are constant, and $g_u = \frac{\partial g}{\partial u}$. Then for every $b > 0$ there is a positive solution of

$$\begin{cases} -\Delta u(x) = (1 + bg(|x|,u))\, u(x)^p & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases} \tag{7}$$

In particular, this result holds for $g(r,u) = r^{\beta-\alpha k} + dr^{\beta}u^{\alpha}$ where $d \geq 0$ is constant.

Theorem 3. *Let $N \geq 3$, $k = \frac{1}{2}(N-2)$, and $\alpha \geq 0$ and $\beta > 0$ be constant that satisfy (5). Let $N_0 := (\frac{1}{2}k^2(p+1))^{1/(p-1)}$ and assume that $g = g(r,u) > 0$ is C^1 for $r > 0$ and $u > 0$ and satisfies*

$$\begin{cases} \frac{d}{dr} g(r, r^{-k}u) > 0, & \forall\, r > 0,\ 0 < u \leq N_0, \\ g(r, r^{-k}u) \leq M_0 r^{\beta-k\alpha} & \forall\, r > 0,\ 0 < u \leq N_0, \\ |g_u(r,u)| \leq M_1 u^{-\alpha_1} & \forall\, 0 < r \leq 1,\quad 0 < u \leq 1, \\ g(r, r^{-k}u) \to \infty & \text{as } r \to \infty \text{ uniformly } \forall u \in [u_0, N_0] \text{ and } \forall\, u_0 \in (0, N_0), \end{cases} \tag{8}$$

where $M_0 > 0$, $M_1 > 0$ and $0 < \alpha_1 < p$ are constant. Fix any $0 < \gamma < p-1$ and $\rho > 0$ small such that $\beta - k\alpha < 2(k - 2\rho)$. Then for any sufficiently small $\delta > 0$, there is $\varepsilon_0 := \varepsilon_0(\delta) > 0$ such that for any $0 < \varepsilon < \varepsilon_0$ and any R satisfying

$$\varepsilon^{-1/2(k-2\rho)} < R < (\varepsilon\delta^{\gamma})^{-1/(\beta-\alpha k)}, \tag{9}$$

there exists a positive solution of

$$\begin{cases} -\Delta u(x) = (1 + \varepsilon g(|x|,u))\, u^p & \text{in } B_R, \\ u = 0 & \text{on } \partial B_R. \end{cases} \tag{10}$$

Remark 2.

1. We comment that in this paper we are considering various perturbations of $-\Delta u = u^{\frac{N+2}{N-2}}$ in $B_1 \subset \mathbb{R}^N$. The main thing we need to utilize in our approach is to perturb off a homoclinic orbit of this unperturbed equation. So we could have generalized our results in Theorems 2 and 3 to consider perturbations of

$$-\Delta u = |x|^{\alpha} u^{p_\alpha(N)} \quad \text{in } B_1 \subset \mathbb{R}^N, \qquad u = 0 \text{ on } \partial B_1.$$

2. We would like to point out that even though our results are new we hope that our dynamical systems approach might apply to other elliptic problems where a more classical approach is not available.

Remark 3.

1. An alternate proof of Theorem A, using a change of variables, is available. This approach is taken from [7] (and was also independently noticed in [11]) where it was used to analyze various numerically observed phenomena related to the extremal solution associated with equations of the form

$$\begin{cases} -\Delta u = \lambda(1 + \frac{\alpha}{2})^2 |x|^{\alpha} f(u) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1. \end{cases}$$

See the appendix for details regarding this change of variables.

2. Crucial in the proof of Theorem A is the fact that $|x|^\alpha$ is zero at the origin. If one considers $H^1_{0,rad}(B_1) \subset L^{p+1}(B_1, h(x)dx)$ where h is bounded away from zero and radial, one does not gain any improved imbeddings. It is precisely this case we consider in the current work.

Remark 4. After the completion of this work we were notified of the work of Naimen–Takahashi [14]. They examine similar equations to (1) but they use a purely variational approach to obtain positive solutions and they also consider nonexistence of positive solutions. Their results seem quite strong and generalize the results from Theorem 1. They do not consider as general equations as we consider in Theorems 2 and 3.

2. The classical perturbation and variational approaches for Theorem 1

Proof of Theorem 1 part 1; for large R. Here we want to find a solution of (1) for large R in the case of $g(r) = 1 + br^\beta$. For $R > 0$ define $u_R(r) = b^{\frac{1}{p-1}} R^{\frac{2+\beta}{p-1}} u(Rr)$ for $r < 1$. Then u is a positive solution of (1) exactly when u_R is a positive solution of

$$\begin{cases} -\Delta u_R(r) &= h_R(r) u_R(r)^p \quad \text{in } B_1, \\ \qquad u &= 0 \qquad\qquad\quad\;\; \text{on } \partial B_1, \end{cases} \tag{11}$$

where $h_R(r) = r^\beta + \frac{1}{bR^\beta}$. Note for large R that h_R is a small perturbation of r^β in $L^\infty(B_1)$. Hence provided the radial positive solution of the Hénon equation in the case of the above β, p is nondegenerate in the space of radial functions, then one can use a perturbation argument to obtain the desired result. One does in fact have this radial nondegenerate condition. To see this one notes that the positive radial solution of $-\Delta v = v^p$ in B_1 with $v = 0$ on ∂B_1 is nondegenerate; see [16]. One can then use the change of variables in Remark 3 to obtain the desired result; note one does not obtain directly the nondegeneracy of the solution in the full space $H^1_0(B_1)$ but rather just in $H^1_{0,rad}(B_1)$. One can use arguments developed in [8,17] to obtain the nondegeneracy on the full space for a reduced range of p; see [6] for details. The dynamical systems approach to prove the result for all $R > 0$ is given in the next section. $\square$

Proof of Theorem 1 part 2; for large R. For the proof we change notation slightly so as to agree with the more standard notation from the Concentration Compactness Lemma II, page 42 [20]. Set $h(r) = 1+g(r)$ and set $q := 2^*$ and so we are interested in finding positive solutions of

$$\begin{cases} -\Delta u(r) = h(r) u(r)^{q-1} & \text{in } B_R, \\ \qquad u = 0 & \text{on } \partial B_R, \end{cases} \tag{12}$$

for sufficiently large R. Consider the energy

$$E(u) = \frac{\int_{B_R} |\nabla u|^2 dx}{\|u\|^2_{L^q(B_R; hdx)}}$$

for $u \in H^1_{0,rad}(B_R) =: X$, here the L^q space in the denominator is using the measure $h(x)dx$. Set $T := \inf_{v \in X} E(v)$ and let $u_m \in X$ (which we can assume is nonnegative) such that

$$E(u_m) = T + \varepsilon_m$$

where $\varepsilon_m \searrow 0$. Let S_N denote the optimal constant in the critical Sobolev imbedding. We will show that if

$$T < \frac{S_N}{(h(0))^{\frac{2}{q}}}, \tag{13}$$

then there is a positive solution of (12). By normalizing we can assume $\|u_m\|_{L^q(B_R; hdx)} = 1$. Since h is bounded away from zero and bounded on B_R we see that u_m is bounded in $L^q(B_R)$ with the Euclidean measure. By passing to a subsequence we can assume $u_m \rightharpoonup u$ in $H^1_{0,rad}(B_R)$ and in $C^{0,\frac{1}{2}}_{loc}(\overline{B_R}\backslash\{0\})$. By the concentration compactness lemma there are constants $\nu^{(1)}, \mu^{(1)} \geq 0$ such that $\mu_m := |\nabla u_m|^2 dx \to \mu$ and $\nu_m := |u_m|^q dx \to \nu$ in the sense of measures where

$$\nu = |u|^q dx + \nu^{(1)}\delta_0, \qquad \mu \geq |\nabla u|^2 dx + \mu^{(1)}\delta_0$$

in the sense of measures and where δ_0 is the Dirac mass at the origin and

$$S_N \left(\nu^{(1)}\right)^{\frac{2}{q}} \leq \mu^{(1)}.$$

Now note that if we can show that $\nu^{(1)} = 0$ then we have $\int_{B_R} |u_m|^q dx \to \int_{B_R} |u|^q dx$ and hence we can prove $u_m \to u$ in $L^q(B_R)$. From this we can show that $\|u\|_{L^q(B_R;hdx)} = 1$ and from this one can see that u is a nonnegative nonzero minimizer of E over X and hence is a nonzero nonnegative solution of (12) and one can then argue that u is strictly positive.

So we now assume (13) holds and $\nu^{(1)} > 0$ and we hope to arrive at a contradiction. Then we have (after passing to limits)

$$T \geq \frac{\int_{B_R} |\nabla u|^2 dx + \mu^{(1)}}{\left(\int_{B_R} h|u|^q dx + \nu^{(1)} h(0)\right)^{\frac{2}{q}}} \geq \frac{\int_{B_R} |\nabla u|^2 dx + S_N(\nu^{(1)})^{\frac{2}{q}}}{\|u\|^2_{L^q(B_R;hdx)} + h(0)^{\frac{2}{q}}(\nu^{(1)})^{\frac{2}{q}}}$$

and this inequality is strict in the case of $u \neq 0$. In the case of $u = 0$ this contradicts (13) and so we can now assume $u \neq 0$. Let $\gamma > 1$ such that $T\gamma = \frac{S_N}{h(0)^{\frac{2}{q}}}$ and we write the above as

$$T > \frac{\int_{B_R} |\nabla u|^2 dx + S_N(\nu^{(1)})^{\frac{2}{q}}}{\|u\|^2_{L^q(B_R;hdx)} + h(0)^{\frac{2}{q}}(\nu^{(1)})^{\frac{2}{q}}} =: \frac{a+b}{c+d}$$

and note $\frac{a}{c} \geq T$ and $\frac{b}{d} = \gamma T$ and hence

$$\frac{a+b}{c+d} \geq \frac{Tc + T\gamma d}{c+d} > T,$$

which gives us a contradiction. So we have shown if (13) holds then we must have $\nu^{(1)} = 0$ and from our earlier arguments this implies we have the needed compactness of the minimizing sequence.

We now show that we do in fact have (13). To show the dependence on R we now write T_R for T. Fix $\varepsilon > 0$ sufficiently small such that

$$(1+\varepsilon)^{\frac{q}{2}} h(0) < h(1).$$

Then there is some $\delta > 0$ small and $0 \leq \phi \in H^1_{0,rad}(B_1)$ smooth, compactly supported in $B_1\backslash B_\delta$ such that

$$\frac{\int_{B_1} |\nabla\phi(x)|^2 dx}{\left(\int_{B_1} |\phi(x)|^q dx\right)^{\frac{2}{q}}} < (1+\varepsilon) S_N.$$

Set $\phi_R(x) = \phi(R^{-1}x)$ and then note we have

$$T_R \leq E(\phi_R) = \frac{\int_{\delta<|y|<1} |\nabla\phi(y)|^2 dy}{\left(\int_{\delta<|y|<1} h(Ry)|\phi(y)|^q dy\right)^{\frac{2}{q}}} \leq \frac{\int_{\delta<|y|<1} |\nabla\phi(y)|^2 dy}{(h(\delta R))^{\frac{2}{q}} \left(\int_{\delta<|y|<1} |\phi(y)|^q dy\right)^{\frac{2}{q}}} \leq \frac{(1+\varepsilon) S_N}{(h(\delta R))^{\frac{2}{q}}},$$

and note this quantity is strictly less than

$$\frac{S_N}{(h(0))^{\frac{2}{q}}} \left\{ \frac{h(1)}{h(\delta R)} \right\}^{\frac{2}{q}},$$

and this is strictly less than $\frac{S_N}{(h(0))^{\frac{2}{q}}}$ for $R > \frac{1}{\delta}$, which completes the proof. $\square$

3. Dynamical systems approach

We begin by looking for positive classical solutions of (10). A radial solution $u(r) = u(|x|)$ of (10) satisfies

$$u''(r) + \frac{n-1}{r}u'(r) + [1+\varepsilon g(r,u)]u(r)^p = 0, \quad 0 < r < R, \quad u(R) = 0, \tag{14}$$

note we are omitting the condition $u'(0) = 0$, which we get for free provided the solution is sufficiently regular. We make the standard change of variables

$$t = \ln r, \qquad v(t) := r^k u(r), \qquad k = \frac{2}{p-1} = \frac{1}{2}(N-2),$$

and yields that v is the solution of

$$v''(t) - k^2 v(t) + [1+\varepsilon g(e^t, e^{-kt}v(t))]v(t)^p = 0, \qquad -\infty < t \le T, \qquad v(T) = 0,$$

where $T = \ln R$. We shall prove the following result.

Theorem 4. *Let $N \ge 3$, $k = \frac{1}{2}(N-2)$, and $\alpha \ge 0$ and $\beta > 0$ be constant that satisfy (5). Let $N_0 := (\frac{1}{2}k^2(p+1))^{1/(p-1)}$ and assume that $g = g(r,u) > 0$ is C^1 for $r > 0$ and $u > 0$ and satisfies (8). The following hold.*

(i) For any sufficiently small $\delta > 0$, there is $\varepsilon_0 = \varepsilon_0(\delta)$ such that if $0 < \varepsilon < \varepsilon_0$, then for every $v_0 \in (0,\delta]$ the equation

$$v'' - k^2 v + [1+\varepsilon g(e^t, e^{-kt}v)]v^p = 0 \tag{15}$$

has a solution $v = v_{\varepsilon,v_0}$ defined on $(-\infty, T]$ for some $T := T_\varepsilon(v_0) \in (0,\infty)$ satisfying

$$\begin{cases} v(0) = v_0, \\ (v,v')(-\infty) = (0,0), \\ v(T) = 0, \quad v(t) > 0 \quad \forall\, t < T, \\ v'(T_0) = 0 \quad \text{for some } T_0 \in (0,T), \\ v' > 0 \text{ on } (-\infty, T_0) \text{ and } v' < 0 \text{ on } (T_0, T], \\ v(T_0) < N_0. \end{cases} \tag{16}$$

Furthermore, $h_\varepsilon(v_0) := v'_{\varepsilon,v_0}(0)$ and $T_\varepsilon(v_0)$ are continuous functions of $v_0 \in (0,\delta]$.

(ii) Let $0 < \gamma < p-1$ and let $\rho > 0$ be small such that $\beta - k\alpha < 2(k-2\rho)$. Let $\delta > 0$ be sufficiently small. If $\varepsilon > 0$ is sufficiently small, then the range of T_ε over $(0,\delta]$, namely, $T_\varepsilon((0,\delta]) := \{T_\varepsilon(v_0) : v_0 \in (0,\delta]\}$, satisfies

$$T_\varepsilon((0,\delta]) \supseteq \left[-\frac{1}{2(k-2\rho)}\ln\varepsilon, -\frac{1}{\beta - k\alpha}\ln(\varepsilon\delta^\gamma)\right]. \tag{17}$$

We need a series of lemmas to prove this theorem. In the proofs of these lemmas the energy function $E(t)$ of (15) plays key roles. Along any positive solution v of (15), $E(t)$ is defined as

$$E(t) := v'^2(t) - k^2v^2(t) + \frac{2}{p+1}v^{p+1}(t), \qquad E'(t) = -2\varepsilon g(e^t, e^{-kt}v(t))v^p(t)v'(t).$$

When $\varepsilon = 0$, Eq. (15) reduces to a Hamilton's equation $v'' - k^2 v + v^p = 0$ and $E(t)$ is constant along any positive solution of it; in particular, this equation has a homoclinic orbit Γ_0 (see Fig. 1) in the (v,v') phase plane that connects the origin (the trivial equilibrium point) and has also a continuum of closed orbit inside Γ_0 that surround the other equilibrium point $(k^{2/p-1}, 0)$; we also have that $E(t) < 0$ when $(v(t), v'(t))$ lying inside Γ_0 and $E(t) > 0$ when $(v(t), v'(t))$ lies outside Γ_0, and the maximum value of v along Γ_0 is N_0. When $\varepsilon > 0$ we have $E'(t) < 0$ whenever $v(t) > 0$ and $v'(t) > 0$ and $E'(t) > 0$ whenever $v(t) > 0$ and $v'(t) < 0$, and

$$v'(t) = \pm\sqrt{k^2v^2(t) - \frac{2}{p+1}v^{p+1}(t) + E(t)}.$$

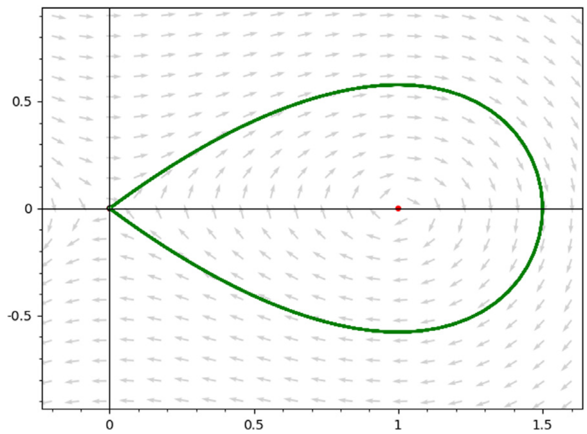

Fig. 1. The homoclinic orbit Γ_0 of $v' = w$, $w' = k^2 v - v^p$.

Lemma 1. *Let $\delta > 0$ such that $\delta^{p-1} < \frac{1}{2}(p+1)k^2$. Then for sufficiently small $\varepsilon > 0$, if*

$$0 < v(0) \leq \delta, \quad v'(0) > 0, \quad E(0) > 0,$$

then there is $T \in (-\infty, 0)$ such that $T > -\frac{v(0)}{\sqrt{E(0)}}$ and

$$\begin{cases} v(T) = 0, \quad v(t) > 0 \quad \forall t \in (T, 0], \\ v'(t) > 0 \quad \forall t \in [T, 0]. \end{cases}$$

Proof. Let $T_0 := -\frac{v(0)}{\sqrt{E(0)}}$ and $T = \inf\Big\{t \in (T_0, 0) : v(s) > 0, v'(s) > 0, \ \forall s \in [t, 0]\Big\}$. It follows that $T \geq T_0$. Since $v'(t) > 0$ for $t \in (T, 0]$, we have $E'(t) < 0$ and $E(t) > E(0)$ on $(T, 0)$, and

$$v'(t) = \sqrt{k^2 v^2(t) - \frac{1}{p+1} v^{p+1}(t) + E(t)} > \sqrt{E(t)} > \sqrt{E(0)} > 0,$$

where we used $0 < v(t) < v(0)$ and $\frac{1}{p+1} v^{p+1}(t) < \frac{1}{2} k^2 v^2(t)$ by the choice of δ, from which we obtain $v(T) < v(0) + \sqrt{E(0)} T$ and so $T > -v(0)/\sqrt{E(0)} = T_0$. By the definition of T and $v'(T) > 0$ we conclude $v(T) = 0$ as well as the rest of the assertions of the lemma. $\square$

Lemma 2. *Let $\delta > 0$ be small and $M_2 := \sup_{0 < \xi \leq 1} g(1, \xi)$. Then for sufficiently small $\varepsilon > 0$, if*

$$0 < v(0) \leq \delta, \quad v'(0) > 0, \quad E(0) < -\frac{2M_2 \varepsilon}{p+1} v^{p+1}(0),$$

then there is $T \in (-\infty, 0)$ such that

$$\begin{cases} v'(T) = 0, \quad v'(t) > 0 \quad \forall t \in (T, 0], \\ v(t) > 0, \quad E(t) < -\frac{2M_2\varepsilon}{p+1} v^{p+1}(t), \quad \forall t \in [T, 0]. \end{cases}$$

Proof. Let

$$T = \inf\Big\{t < 0 : v(s) > 0, v'(s) > 0, E(s) < -\frac{2M_2 \varepsilon}{p+1} v^{p+1}(s), \quad \forall t \leq s < 0\Big\}.$$

W have $-\infty \leq T < 0$. We claim that $T > -\infty$. If this is not true, then we would have $T = -\infty$, $v' > 0$, and $v > 0$ on $(-\infty, 0]$, and so $0 \leq v(-\infty) < \delta$. Since $E'(t) < 0$ for $t < 0$, it follows that $E(-\infty)$ exists with

$E(0) \le E(-\infty) \le -\frac{M_2\varepsilon}{p+1} v^{p+1}(-\infty)$ and so $v'(-\infty)$ exists from the definition of E, which together with the finiteness of $v(-\infty)$ gives $v'(-\infty) = 0$. On the other hand, for $t \in (T, 0)$,

$$\begin{aligned} E(t) &= E(0) + 2\varepsilon \int_t^0 g(e^s, e^{-ks} v(s)) v^p(s) v'(s)\, ds \le E(0) + 2M_2\varepsilon \int_t^0 v^p(s) v'(s)\, ds \\ &= E(0) + \frac{2M_2\varepsilon}{p+1}[v^{p+1}(0) - v^{p+1}(t)] < E(0) + \frac{2M_2\varepsilon}{p+1} v^{p+1}(0), \end{aligned} \tag{18}$$

where we used the first assumption on g in (8) to get $g(e^s, e^{-ks}v(s)) \le g(1, v(s)) \le M_2$, then sending $t \to -\infty$ gives

$$E(-\infty) \le E(0) + \frac{2M_2\varepsilon}{p+1} v^{p+1}(0) < 0,$$

which together with the definition of $E(-\infty) < 0$ and $v'(-\infty) = 0$ yields $v(-\infty) > 0$. Now it follows from Eq. (15) and the smallness of δ that for $t \in (-\infty, 0)$, $v''(t) \ge v(t)[k^2 - (1 + \varepsilon M_2)v^{p-1}(t)] > v(-\infty)[k^2 - (1 + \varepsilon M_2)\delta^{p-1}] > 0$, implying $v'(-\infty) = -\infty$, a contradiction. Therefore we have $T > -\infty$.

Note that (18) still holds for $t \in (T, 0)$ and letting $t \to T^-$ in (18) yields

$$E(T) \le E(0) + \frac{2M_2\varepsilon}{p+1}[v^{p+1}(0) - v^{p+1}(T)] < -\frac{2M_2\varepsilon}{p+1} v^{p+1}(T).$$

Thus, by the definition of T we have either $v(T) = 0$ or $v'(T) = 0$. Since $E(T) < 0$, it follows from the definition of E again that $v'(T) = 0$ and $v(T) > 0$. The rest of the assertion of the lemma follows from the definition of T. □

Lemma 3. *Let $\delta > 0$ be sufficiently small. Then for sufficiently small $\varepsilon > 0$ and any $v_0 \in (0, \delta]$, there is a unique solution $v(t) := v_{\varepsilon, v_0}(t)$ of (15) satisfying*

$$\begin{cases} v(0) = v_0, \\ -\frac{2M_2\varepsilon}{p+1} v_0^{p+1} \le E(0) < 0, \\ (v, v')(-\infty) = (0, 0), \\ v' > 0 \ \textit{on}\ (-\infty, 0]. \end{cases} \tag{19}$$

Furthermore, $h_\varepsilon(v_0) := v'_{\varepsilon, v_0}(0)$ is a continuous functions of $v_0 \in (0, \delta]$.

Proof. Fix $v_0 \in (0, \delta]$. For each $v'_0 > 0$, let $v(t, v_0, v'_0)$ be the solution of (15) with $v(0, v_0, v'_0) = v_0$ and $v'(0, v_0, v'_0) = v'_0$ with the left maximal interval of existence $(t_{v'_0}, 0]$ where $v(t, v_0, v'_0) \ge 0$. Let

$$\mathcal{A}(v_0) = \left\{ v'_0 > 0 : \exists\, T \in (t_{v'_0}, 0) \text{ such that } \begin{cases} v'(t, v_0, v'_0) > 0 & \text{on } [T, 0], \\ v(t, v_0, v'_0) > 0 \text{ on } (T, 0], \\ v(T, v_0, v'_0) = 0, \end{cases} \right\}$$

and

$$\mathcal{B}(v_0) = \left\{ v'_0 > 0 : \exists\, T \in (t_{v'_0}, 0) \text{ such that } \begin{cases} v(t, v_0, v'_0) > 0 & \text{on } [T, 0], \\ v'(t, v_0, v'_0) > 0 \text{ on } (T, 0], \\ v'(T, v_0, v'_0) = 0. \end{cases} \right\}$$

It follows from Lemmas 1 and 2 that both sets $\mathcal{A}(v_0)$ and $\mathcal{B}(v_0)$ are not empty. Since any solution v of (15) with $v(t_0) = v'(t_0) = 0$ implies $v \equiv 0$, we see that $\mathcal{A}(v_0)$ and $\mathcal{B}(v_0)$ are disjoint. By the connectedness of $(0, \infty)$, it follows that

$$\mathcal{C}(v_0) := (0, \infty) \setminus (\mathcal{A}(v_0) \cup \mathcal{B}(v_0)) \ne \emptyset$$

and for any $v'_0 \in \mathcal{C}(v_0)$, the solution $v(t, v_0, v'_0)$ satisfies (19).

Next we show that $\mathcal{C}(v_0)$ is a singleton set and $h_\varepsilon(v_0)$ is continuous on $v_0 \in (0,\delta]$. Let $v_0' \in C(v_0)$ and $v(t) = v(t, v_0, v_0')$. Since v is bounded on $(-\infty, 0]$, it satisfies the integral equation for $t \le 0$:

$$v(t) = \left(v_0 - \frac{1}{2k}\int_{-\infty}^0 e^{ks} f(s,v(s))v^p(s)\,ds\right)e^{kt} + \frac{1}{2k}\int_t^0 e^{k(t-s)} f(s,v(s))v^p(s)\,ds \\ + \frac{1}{2k}\int_{-\infty}^t e^{-k(t-s)} f(s,v(s))v^p(s)\,ds, \tag{20}$$

where $f(t,v) := 1 + \varepsilon g(e^t, e^{-kt}v)$, and it holds

$$v'(0) = kv_0 - \int_{-\infty}^0 e^{ks} f(s,v(s))v^p(s)\,ds. \tag{21}$$

We show that

$$v(t) \le 3v_0 e^{kt} \qquad \forall\ t \le 0. \tag{22}$$

To do so, we let $w(\tilde t) := \sup\{v(t) : t \le \tilde t\}$ for $\tilde t \le 0$. Taking the supremum of (20) over $(-\infty, \tilde t]$ we have

$$w(\tilde t) \le v_0 e^{k\tilde t} + \frac{1}{2k}\sup_{-\infty<t\le\tilde t}\left(\int_t^{\tilde t} e^{k(t-s)} f(s,v(s))v^p(s)\,ds + \int_{\tilde t}^0 e^{k(t-s)} f(s,v(s))v^p(s)\,ds\right) \\ + \frac{1}{2k}\sup_{-\infty<t\le\tilde t}\int_{-\infty}^t e^{-k(t-s)} f(s,v(s))v^p(s)\,ds.$$

Since $f(s,v) \le 1 + M_2\varepsilon \le 2$ for $s \le 0$ and small $\varepsilon > 0$, $w(\tilde t)$ is non-decreasing, $v(t) \le w(\tilde t)$ for $t \le \tilde t$,

$$\int_t^{\tilde t} e^{k(t-s)} f(s,v(s))v^p(s)\,ds \le \frac{2}{k}\delta^{p-1}w(\tilde t), \quad \int_{-\infty}^t e^{-k(t-s)} f(s,v(s))v^p(s)\,ds \le \frac{2}{k}\delta^{p-1}w(\tilde t),$$

and

$$\int_{\tilde t}^0 e^{k(t-s)} f(s,v(s))v^p(s)\,ds \le 2\delta^{p-1}\int_{\tilde t}^0 e^{k(\tilde t-s)}w(s)\,ds,$$

it follows that for $\tilde t \le 0$,

$$w(\tilde t) \le \left(1 - \frac{2}{k^2}\delta^{p-1}\right)^{-1} v_0 e^{k\tilde t} + \left(1 - \frac{2}{k^2}\delta^{p-1}\right)^{-1}\frac{1}{k}\delta^{p-1}\int_{\tilde t}^0 e^{k(\tilde t-s)}w(s)\,ds \\ \le 2v_0 e^{k\tilde t} + \frac{2}{k}\delta^{p-1}\int_{\tilde t}^0 e^{k(\tilde t-s)}w(s)\,ds \quad \text{(by taking } \delta \text{ small)},$$

yielding that $w(\tilde t)e^{-k\tilde t} \le 2v_0 + \frac{2}{k}\delta^{p-1}\int_{\tilde t}^0 e^{-ks}w(s)\,ds$, and applying the Gronwall's inequality gives that $w(\tilde t)e^{-k\tilde t} \le 2v_0 e^{-\frac{2}{k}\delta^{p-1}\tilde t}$, hence $v(t) \le w(t) \le 2v_0 e^{(k-\frac{2}{k}\delta^{p-1})t}$ for $t \le 0$. Using this estimate and (20) we get, for $t < 0$,

$$v(t) \le v_0 e^{kt} + \frac{1}{k}\int_t^0 e^{k(t-s)}v^p(s)\,ds + \frac{1}{k}\int_{-\infty}^t v^p(s)\,ds \\ \le v_0 e^{kt} + \frac{(2v_0)^p}{k}e^{kt}\int_t^0 e^{[p(k-\frac{2}{k}\delta^{p-1})-k]s}\,ds + \frac{(2v_0)^p}{k}\int_{-\infty}^t e^{p(k-\frac{2}{k}\delta^{p-1})s}\,ds \\ \le v_0 e^{kt} + \frac{(2v_0)^p}{k[p(k-\frac{2}{k}\delta^{p-1})-k]}e^{kt} + \frac{(2v_0)^p}{kp(k-\frac{2}{k}\delta^{p-1})}e^{p(k-\frac{2}{k}\delta^{p-1})t} \\ \le 3v_0 e^{kt} \qquad \text{(by taking } \delta > 0 \text{ further smaller if needed)},$$

which shows (22).

Next we first prove the following: If v_1 and v_2 are solutions of (15) with $v_1(0) = v_0^1$ and $v_2(0) = v_0^2$ and $v_1'(0) \in \mathcal{C}(v_0^1)$ and $v_2'(0) \in \mathcal{C}(v_0^2)$, then

$$|v_1 - v_2|_0 := \sup_{t\le 0} |v_1(t) - v_2(t)| \le 2|v_0^1 - v_0^2|. \tag{23}$$

To this end, subtracting the Eqs. (20) for v_1 and v_2 we have

$$\begin{aligned} v_1(t) - v_2(t) = v_0^1 - v_0^2 &- \frac{1}{2k}\int_{-\infty}^0 e^{ks}[f(s,v_1(s))v_1^p(s) - f(s,v_2(s))v_2^p(s)]\,ds\,e^{kt} \\ &+ \frac{1}{2k}\int_t^0 e^{k(t-s)}[f(s,v_1(s))v_1^p(s) - f(s,v_2(s))v_2^p(s)]\,ds \\ &+ \frac{1}{2k}\int_{-\infty}^t e^{-k(t-s)}[f(s,v_1(s))v_1^p(s) - f(s,v_2(s))v_2^p(s)]\,ds. \end{aligned} \tag{24}$$

Note that, for $s \le 0$,

$$\begin{aligned} &f(s,v_1(s))v_1^p(s) - f(s,v_2(s))v_2^p(s) \\ &= [f(s,v_1(s)) - f(s,v_2(s))]v_1^p(s) + f(s,v_2(s))[v_1^p(s) - v_2^p(s)] \\ &= \varepsilon[g(e^s, e^{-ks}v_1(s)) - g(e^s, e^{-ks}v_2(s))]v_1^p(s) + f(s,v_2(s))[v_1^p(s) - v_2^p(s)]; \end{aligned}$$

use the mean value theorem, $v_i^0 \le \delta$, $v_i(t) \le 3\delta e^{kt}$ $(i = 1,2)$, and the third condition in (8) with $p - \alpha_1 > 0$ to get

$$|v_1^p(s) - v_2^p(s)| \le p(3\delta)^{p-1}e^{(p-1)ks}|v_1(s) - v_2(s)| \le p(3\delta)^{p-1}e^{(p-1)ks}|v_1 - v_2|_0,$$

and

$$\begin{aligned} &|g(e^s, e^{-ks}v_1(s)) - g(e^s, e^{-ks}v_2(s))|v_1^p(s) \le M_1[e^{-ks}v_1(s)]^{-\alpha_1}e^{-ks}|v_1(s) - v_2(s)|v_1^p(s) \\ &\le M_1 e^{(\alpha_1-1)ks}[v_1(s)]^{p-\alpha_1}|v_1(s) - v_2(s)| \le M_1(3\delta)^{p-\alpha_1}e^{(p-1)ks}|v_1 - v_2|_0. \end{aligned}$$

Hence, for $s \le 0$,

$$\begin{aligned} &|f(s,v_1(s))v_1^p(s) - f(s,v_2(s))v_2^p(s)| \\ &\le M_1(3\delta)^{p-\alpha_1}\varepsilon e^{(p-1)ks}|v_1 - v_2|_0 + 2p(3\delta)^{p-1}e^{(p-1)ks}|v_1 - v_2|_0 \\ &\le p3^p\delta^{p-1}e^{(p-1)ks}|v_1 - v_2|_0 \qquad \text{(by taking } \varepsilon \text{ small)}, \end{aligned}$$

and hence,

$$\begin{aligned} &\int_{-\infty}^0 e^{ks}|f(s,v_1(s))v_1^p(s) - f(s,v_2(s))v_2^p(s)|\,ds \\ &\qquad \le p3^p\delta^{p-1}|v_1 - v_2|_0\int_{-\infty}^0 e^{pks}\,ds \le \frac{3^p}{k}\delta^{p-1}|v_1 - v_2|_0, \end{aligned} \tag{25}$$

$$\begin{aligned} &\int_t^0 e^{k(t-s)}|f(s,v_1(s))v_1^p(s) - f(s,v_2(s))v_2^p(s)|\,ds \\ &\qquad \le p3^p\delta^{p-1}|v_1 - v_2|_0\int_t^0 e^{k(t-s)}e^{(p-1)ks}\,ds \le \frac{p3^p}{k}\delta^{p-1}|v_1 - v_2|_0, \end{aligned}$$

and

$$\int_{-\infty}^{t} e^{-k(t-s)}|f(s,v_1(s))v_1^p(s)-f(s,v_2(s))v_2^p(s)|\,ds$$
$$\leq p3^p\delta^{p-1}|v_1-v_2|_0\int_{-\infty}^{t} e^{-k(t-s)}e^{(p-1)ks}\,ds \leq \frac{p3^p}{k}\delta^{p-1}|v_1-v_2|_0.$$

Therefore, from (24)

$$|v_1-v_2|_0 \leq |v_0^1-v_0^2|+\frac{p3^{p+1}}{2k^2}\delta^{p-1}|v_1-v_2|_0 \leq |v_0^1-v_0^2|+\frac{1}{2}|v_1-v_2|_0$$

by taking δ small which yields (23).

Now it readily follows from (23) that for any $v_0 \in (0,\delta]$, if $v_0^1=v_0^2=v_0$ then $|v_1-v_2|_0=0$, showing that $\mathcal{C}(v_0)$ is a singleton set. Hence we denote the solution of (15) with $v(0)=v_0$ and $v'(0)\in\mathcal{C}(v_0)$ by $v_{\varepsilon,v_0}(t)$, and $h_\varepsilon(v_0):=v'_{\varepsilon,v_0}(0)$ is a well defined function of $v_0\in(0,\delta]$.

Next we show that $h_\varepsilon(v_0)$ is Lipschitz continuous on v_0. To see this, let $v_0^1, v_0^2 \in (0,\delta]$, then use (21) for v_1 and v_2 to get the equations for $v_1'(0)$ and $v_2'(0)$, then subtract these equations to give

$$v_1'(0)-v_2'(0)=k(v_0^1-v_0^2)-\int_{-\infty}^{0} e^{ks}[f(s,v_1(s))v_1^p(s)-f(s,v_2(s))v_2^p(s)]\,ds,$$

and then use (25) and (23) to get

$$|h_\varepsilon(v_0^1)-h_\varepsilon(v_0^2)|=|v_1'(0)-v_2'(0)|\leq\left(k+\frac{2\cdot 3^p}{k}\delta^{p-1}\right)|v_0^1-v_0^2|,$$

which shows that $h_\varepsilon(v_0)$ is Lipschitz continuous on v_0. This completes the proof of the lemma. □

Lemma 4. *Let $\delta>0$ be sufficiently small. If $\varepsilon>0$ is sufficiently small, then for every $v_0\in(0,\delta]$, the solution v_{ε,v_0} of (15) given in Lemma 3 has the following properties:*

(i) There is $t_1^{\varepsilon,v_0}\in(0,\infty)$ such that $v'_{\varepsilon,v_0}(t_1^{\varepsilon,v_0})=0$, $v''_{\varepsilon,v_0}(t_1^{\varepsilon,v_0})<0$, and $v'_{\varepsilon,v_0}(t)>0$ for $t\in[0,t_1^{\varepsilon,v_0})$;

(ii) There is $t_2^{\varepsilon,v_0}\in(t_1^{\varepsilon,v_0},\infty)$ such that $v_{\varepsilon,v_0}(t_2^{\varepsilon,v_0})=0$ and $v'_{\varepsilon,v_0}(t)<0$ for $t\in(t_1^{\varepsilon,v_0},t_2^{\varepsilon,v_0}]$. Furthermore, t_2^{ε,v_0} is continuous on v_0.

(iii) For all $t\in(-\infty,t_2^{\varepsilon,v_0}]$, $v_{\varepsilon,v_0}(t)\leq v_{\varepsilon,v_0}(t_1^{\varepsilon,v_0})<N_0$.

Proof. Let $v(t):=v_{\varepsilon,v_0}(t)$ with the maximal interval of existence $(-\infty,\omega^+)$. Let $t_1=\sup\{t\in(0,\omega^+): v'>0\}$. For $t\in(0,t_1)$, since $E'(t)=-2\varepsilon g v^p(t)v'(t)<0$, we have $E(t)<E(0)<0$, so $(v(t),v'(t))$ lies inside the homoclinic orbit Γ_0, so $v(t)$ is bounded. We claim that $t_1<\infty$. Suppose this is not true. We have $v(t)\nearrow v_\infty$ as $t\nearrow\infty$; Now taking $T>0$ sufficiently large such that for $t>T$, using $v(t)>v(0)>0$ for $t>T$, the first and fourth assumptions in (8) give

$$\varepsilon g(e^t,r^{-kt}v(t))v^p(t)\geq\varepsilon g(e^T,r^{-kT}v(t))v^p(0)\geq 2k^2v_\infty.$$

Hence, for $t>T$,

$$v''(t)=k^2v(t)-v^p(t)-\varepsilon g(e^t,e^{-kt}v(t))v^p(t)<-k^2v_\infty,$$

yielding $v'(t)<v'(T)-k^2v_\infty(t-T)<0$ for sufficiently large t, a contradiction. Hence we have $t_1<\infty$, and from the definition of t_1, $v'(t_1)=0$, and furthermore, $v''(t_1)<0$, for if it is false, then we have $v''(t_1)=0$ and so

$$v'''(t_1)=\left[-\varepsilon\frac{d}{dr}g(r,r^{-k}v(t_1))|_{r=e^{t_1}}+\varepsilon g_u(e^{t_1},e^{-kt_1}v(t_1))e^{-kt_1}v'(t_1)\right]v^p(t_1)$$
$$=-\varepsilon\frac{d}{dr}g(r,r^{-k}v(t_1))|_{r=e^{t_1}}v^p(t_1)<0,$$

yielding that $v'(t_1) = 0$ is a local maximum of v' and $v'(t) < 0$ for $t < t_1$, which is a contradiction. This shows $v''(t_1) < 0$. Letting $t_1^{\varepsilon,v_0} := t_1$ completes the proof of (i).

Next we show (ii). Let $t_2 := \sup\{t > t_1 : v'(t) < 0, v(t) > 0\}$. We claim that $t_2 < \infty$. If not, then we have $v(t) \searrow v_\infty$ as $t \nearrow \infty$ for some $v_\infty \geq 0$. If $v_\infty > 0$, then taking $T > t_1$ sufficiently large and using the similar reasoning as above with $v(t) > v_\infty > 0$ for $t > T$ we have for $t > T$,

$$\varepsilon g(e^t, r^{-kt}v(t))v^p(t) \geq \varepsilon g(e^T, r^{-kT}v(t))v_\infty^p \geq 2k^2 v(t_1),$$

so $v''(t) < k^2 v(t_1) - 2k^2 v(t_1) = -k^2 v(t_1)$, so $v'(t) < v'(T)$, and so $v(t) < v(T) + V'(T)(t - T) \to -\infty$ as $t \to \infty$, a contradiction. Hence, $t_2 < \infty$.

By the definition of t_2, we have either $v'(t_2) = 0$ or $v(t_2) = 0$. Assume that $v'(t_2) = 0$. Let $t_0 < t_1$ be the time where $v(t_0) = v(t_2)$. Then we have

$$\begin{aligned}
E(t_2) - E(t_0) &= -2\varepsilon \int_{t_0}^{t_2} g(e^t, e^{-kt}v(t))v^p(t)v'(t)\,dt \\
&= -2\varepsilon \int_{t_0}^{t_1} g(e^t, e^{-kt}v(t))v^p(t)v'(t)\,dt - 2\varepsilon \int_{t_1}^{t_2} g(e^t, e^{-kt}v(t))v^p(t)v'(t)\,dt \\
&= -2\varepsilon \int_{v(t_0)}^{v(t_1)} g(e^{t_-(v)}, e^{-kt_-(v)}v)v^p\,dv + 2\varepsilon \int_{v(t_2)}^{v(t_1)} g(e^{t_+(v)}, e^{-kt_+(v)}v)v^p\,dv \\
&= 2\varepsilon \int_{v(t_0)}^{v(t_1)} \left[g(e^{t_+(v)}, e^{-kt_+(v)}v) - g(e^{t_-(v)}, e^{-kt_-(v)}v)\right]v^p\,dv > 0,
\end{aligned}$$

which contradict to the fact that $E(t_2) - E(t_0) = -v'(t_0)^2 < 0$. Hence $v'(t_2) < 0$ and $v(t_2) = 0$. Let $t_2^{\varepsilon,v_0} := t_2$. Since $v'_{\varepsilon,v_0}(0)$ is continuous on v_0 from Lemma 3 and $v'_{\varepsilon,v_0}(t_2^{\varepsilon,v_0}) \neq 0$, it follows from the continuous dependence of solution on initial data that t_2^{ε,v_0} is continuous on $v_0 \in (0, \delta]$. This shows (ii).

Finally, since $E(t_1) < E(-\infty) = 0$ and $v'(t_1) = 0$, it follows from the definition of E that $v(t_1) < N_0$. Since $v(t_1)$ is the unique maximum of $v(t)$ for $t < t_2$, (iii) follows. □

In the next two important lemmas we estimate t_2^{ε,v_0} and $t_2^{\varepsilon,\delta}$ where v_0 is defined in Lemma 5.

Lemma 5. *Let $\delta > 0$ be sufficiently small, $0 < \gamma < p - 1$, and $v_0 := \delta(\varepsilon\delta^\gamma)^{\frac{k}{\beta - \alpha k}}$. If $\varepsilon > 0$ is sufficiently small, then t_2^{ε,v_0} for the solution v_{ε,v_0} of (15) given in Lemma 3 satisfies*

$$t_2^{\varepsilon,v_0} > -\frac{1}{\beta - k\alpha}\ln(\varepsilon\delta^\gamma). \tag{26}$$

Proof. Let $v(t) := v_{\varepsilon,v_0}(t)$ and $t_0 := \frac{1}{k}\ln\frac{\delta}{v_0} = -\frac{1}{\beta - k\alpha}\ln(\varepsilon\delta^\gamma)$, and $T = \sup\{t \in (0, t_0) : v' > 0 \text{ on } [0, t]\}$. We show that $T = t_0$. First for $t \in [0, T]$, $v'(t) = \sqrt{k^2 v^2(t) - \frac{2}{p+1}v^{p+1}(t) + E(t)}$, and since $E(t) < E(0) < 0$, we have $v'(t) < kv(t)$ and so $v(t) < v_0 e^{kt} \leq v_0 e^{kt_0} = \delta$. Using the second assumption in (8) we have $\varepsilon g(e^t, e^{-kt}v(t)) \leq M_0\varepsilon e^{(\beta - k\alpha)t_0} = M_0\delta^{-\gamma}$ for $t \in [0, T]$, which together with the fact that $-\frac{2M_2\varepsilon}{p+1}v_0^{p+1} \leq E(0) < 0$ gives

$$\begin{aligned}
0 > E(T) &= E(0) - 2\varepsilon \int_0^T g(e^s, e^{-ks}v(s))v^p(s)v'(s)\,ds \\
&\geq -\frac{2M_2\varepsilon}{p+1}v_0^{p+1} - \frac{M_0\delta^{-\gamma}}{p+1}(v^{p+1}(T) - v_0^{p+1}) \\
&\geq -\frac{M_0\delta^{-\gamma}}{p+1}v^{p+1}(T) = -\frac{M_0\delta^{-\gamma}}{p+1}v^{p-1}(T)v^2(T) \geq -M_0\delta^{p+1-\gamma}v^2(T),
\end{aligned}$$

hence,

$$
\begin{aligned}
v'(T) &= \sqrt{k^2 v^2(T) - \frac{2}{p+1} v^{p+1}(T) + O(\delta^{p-1-\gamma}) v^2(T)} \\
&= v(T)\sqrt{k^2 + O(\delta^{p-1}) + O(\delta^{p-1-\gamma})} > 0,
\end{aligned}
$$

and hence by the definition of T we have $T = t_0$ and $v(t_0) < \delta$. By the definition of t_1^{ε,v_0}, we have $t_1^{\varepsilon,v_0} > t_0 = -\frac{1}{\beta-k\alpha}\ln(\varepsilon\delta^\gamma)$, which implies (26). This completes the proof of the lemma. □

In the following lemma we estimate $t_2^{\varepsilon,\delta}$. To this end, we need to study the properties of the solution $v_{\varepsilon,\delta}$. We show that for sufficiently small $\varepsilon > 0$, in the (v, v') phase plane, $(v_{\varepsilon,\delta}, v'_{\varepsilon,\delta})$ lies in an ε neighborhood of the homoclinic solution $(V(t), V'(t))$ of (15) when $\varepsilon = 0$ with $V(0) = \delta$ and $V'(0) > 0$. Note that the following properties of $V(t)$ are useful in the proof of Lemma 6: $V(t)$ is defined for all $t \in (-\infty, \infty)$, $V(t)$ has a unique maximum value reached at some $T_1 > 0$ with $V'(t) > 0$ for $t < T_1$, $V'(t) < 0$ for $t > T_1$, $V''(T_1) < 0$), and $V(T_1) = N_0$; the graph of $V(t)$ is symmetric about $t = T_1$ and the graph of $V'(t)$ is anti-symmetric about T_1; $V(2T_1) = V(0) = \delta$; $E(t) \equiv 0$ along $V(t)$, and

$$
V'(t) = \begin{cases} \sqrt{k^2V^2(t) - \frac{2}{p+1}V^{p+1}(t)} & \text{if } t \le T_1, \\ -\sqrt{k^2V^2(t) - \frac{2}{p+1}V^{p+1}(t)} & \text{if } t > T_1. \end{cases}
$$

Also note that $T_1 \to \infty$ as $\delta \to 0$.

Lemma 6. *Let $\rho > 0$ be small such that $\beta - k\alpha < 2(k - 2\rho)$ and let $\delta > 0$ be sufficiently small. If $\varepsilon > 0$ is sufficiently small, then*

$$
t_2^{\varepsilon,\delta} \le -\frac{1}{2(k-2\rho)} \ln \varepsilon.
$$

Proof. Let $(v(t), v'(t)) := (v_{\varepsilon,\delta}(t), v'_{\varepsilon,\delta}(t))$, $t_1 := t_1^{\varepsilon,\delta}$, and $t_2 := t_2^{\varepsilon,\delta}$, and let $E(t)$ be evaluated along $(v(t), v'(t))$ for $t \in (-\infty, t_2)$. We proceed the proof in two steps.

Step 1. We first show: If $\varepsilon > 0$ is sufficiently small, then there is a constant $M > 0$ independent of ε such that

$$
|v(t) - V(t)| + |v'(t) - V'(t)| \le M\varepsilon \qquad \forall\, t \in [0, 2T_1 + 1], \tag{27}
$$

and furthermore, letting $t_3 \in (t_1, t_2)$ such that $v(t_3) = v(0) = \delta$ we have

$$
t_1 = T_1 + O(\varepsilon), \qquad t_3 = 2T_1 + O(\varepsilon).
$$

Now we start to prove the above claim. Since $-\frac{2M_2\varepsilon}{p+1}\delta^{p+1} \le E(0) < 0$ from Lemma 3 and $V'(0) = \sqrt{k^2\delta^2 - \frac{2}{p+1}\delta^{p+1}}$, we have

$$
\begin{aligned}
v'(0) &= \sqrt{k^2\delta^2 - \frac{2}{p+1}\delta^{p+1} + E(0)} = V'(0)\sqrt{1 + \frac{E(0)}{k^2\delta^2 - \frac{2}{p+1}\delta^{p+1}}} \\
&= V'(0)\sqrt{1 + O(\delta^{p-1})\varepsilon} = V'(0)\left[1 + O(\delta^{p-1})\varepsilon\right] = V'(0) + O(\delta^p)\varepsilon.
\end{aligned}
$$

It follows from the continuous dependence of solutions with respect to the initial data and parameters that, for sufficiently small $\varepsilon > 0$,

$$
|v(t) - V(t)| + |v'(t) - V'(t)| \le 1 \qquad \forall\, t \in [0, 2T_1 + 1].
$$

Integrating Eq. (15) for both v and V over $[0,t] \subset [0, 2T_1+1]$ and then subtracting the resulting integral equations gives

$$\begin{aligned}
&|v(t) - V(t)| + |v'(t) - V'(t)| \\
&\le |v'(0) - V'(0)| + \int_0^t \Big[|v'(s) - V'(s)| + k^2 |v(s) - V(s)| + |v^p(s) - V^p(s)| \Big]\, ds \\
&\quad + \varepsilon \int_0^t |g(e^s, e^{-ks} v(s))| |v^p(s)|\, ds \\
&\le M_3 \varepsilon + M_4 \int_0^t \Big[|v'(s) - V'(s)| + |v(s) - V(s)| \Big]\, ds,
\end{aligned}$$

where $M_3 = (2T_1+1) \max_{(s,v)\in[0,2T_1+1]\times[0,N_0]} g(e^s, e^{-ks}v) N_0^p + O(\delta^p)$ where we used $v(t) < N_0$ for $t \in [0, 2T_1+1]$, and $M_4 = k^2 + pN_0^{p-1}$. Applying the Gronwall's inequality gives (27) with $M = M_3 e^{M_4(2T_1+1)}$.

Next we use $t_1 = T_1 + o(1)$ to show that $t_1 = T_1 + O(\varepsilon)$. Since $v'(t_1) = 0$, it follows from (27) that $M\varepsilon \ge |v'(t_1) - V'(t_1)| = |V'(t_1)| = |V'(t_1) - V'(T_1)| = |V''(T_1) + o(1)||t_1 - T_1|$. This shows that $t_1 = T_1 + O(\varepsilon)$.

Similarly we use $t_3 = 2T_1 + o(1)$ to show that $t_3 = 2T_1 + O(\varepsilon)$. Since $v(t_3) = \delta$, it follows from (27) that $M\varepsilon \ge |v(t_3) - V(t_3)| = |\delta - V(t_3)| = |V(2T_1) - V(t_3)| = |V'(2T_1) + o(1)||t_3 - 2T_1|$ for some $\theta \in (0,1)$. This shows that $t_3 = 2T_1 + O(\varepsilon)$. This shows the Step 1.

Step 2. Since $v(t)$ is strictly increasing on $(0, t_1)$ and decreasing on (t_1, t_3), let $t =: t_-(v)$ be the inverse function of $v = v(t)$ for $t \in [0, t_1]$ and $t =: t_+(v)$ be the inverse function of $v = v(t)$ for $t \in [t_1, t_3]$. It follows that $t_-(v)$ is strictly increasing for $v \in [\delta, \bar{v}]$ and $t_+(v)$ is strictly decreasing for $v \in [\delta, v(t_1)]$, and we can write

$$E(t_3) - E(0) = 2\varepsilon \int_\delta^{v(t_1)} \Big[g(e^{t_+(v)}, e^{-kt_+(v)} v) - g(e^{t_-(v)}, e^{-kt_-(v)} v) \Big] v^p \, dv. \tag{28}$$

We fix a number $\nu \in (0, V(T_1))$ such that $V(T_1) - \nu$ is very small. Since $v(t_1) < V(T_1)$, for sufficiently small ε we may assume that $v(t_1) > \nu$ from (27). By the mean value theorem, for given $v \in [\delta, \nu]$,

$$g(e^{t_+(v)}, e^{-kt_+(v)} v) - g(e^{t_-(v)}, e^{-kt_-(v)} v) = \Big(\frac{d}{dr} g(r, r^{-k} v)\Big|_{r=\tilde{r}} \Big) e^{\tilde{t}} \Big(t_+(v) - t_-(v) \Big),$$

where for some $\theta_1, \theta_2 \in (0,1)$,

$$\tilde{r} := \theta_1 e^{t_+(v)} + (1-\theta_1) e^{t_-(v)} \in [1, e^{2T_1+1}], \qquad \tilde{t} := \theta_2 t_-(v) + (1-\theta_2) t_+(v) \in [0, 2T_1+1],$$

there is also $m_0 > 0$ such that for sufficiently small $\varepsilon > 0$, $t_+(v) - t_-(v) \ge m_0$ for $v \in [\delta, \nu]$, and hence (with $e^{\tilde{t}} > 1$)

$$g(e^{t_+(v)}, e^{-kt_+(v)} v) - g(e^{t_-(v)}, e^{-kt_-(v)} v) \ge m_1,$$

where $m_1 := \Big(\max_{(r,v)\in[1,e^{2T_1+1}]\times\in[\delta,\nu]} \frac{d}{dr} g(r, r^{-k} v) \Big) m_0$. Hence from (28) we have

$$E(t_3) - E(0) \ge 2m_1 \varepsilon \int_\delta^\nu v^p \, dv = \frac{2m_1\varepsilon}{p+1} \Big[\nu^{p+1} - \delta^{p+1} \Big].$$

Using $E(0) = O(\delta^{p+1})\varepsilon$ we have, by taking $\delta > 0$ sufficiently small if necessary,

$$E(t_3) \ge \varepsilon \left\{ O(\delta^{p+1}) + \frac{2m_1}{p+1} \Big[\nu^{p+1} - \delta^{p+1} \Big] \right\} > m\varepsilon, \qquad \text{where } m := \tfrac{m_1}{p+1}.$$

Finally, for $t \in (t_3, t_2)$, since $v'(t) < 0$, $E(t)$ is increasing in this interval and so $E(t) \ge E(t_3) > m\varepsilon$ and so by choosing $\delta > 0$ small enough and using $v(t) < v(t_3) = \delta$ gives

$$v'(t) = -\sqrt{k^2 v^2(t) - \frac{2}{p+1} v^{p+1}(t) + E(t)} \le -\sqrt{(k-\rho)^2 v^2(t) + m\varepsilon},$$

and so

$$t_2 - t_3 = \int_{t_3}^{t_2} \frac{-v'(t)}{-v'(t)}\, dt \le \int_{t_3}^{t_2} \frac{-v'(t)}{\sqrt{(k-\rho)^2 v^2(t) + m\varepsilon}}\, dt = \int_0^{\delta} \frac{dv}{\sqrt{(k-\rho)^2 v^2 + m\varepsilon}}$$
$$= \frac{1}{k-\rho} \ln \frac{(k-\rho)\delta + \sqrt{(k-\rho)^2\delta^2 + m\varepsilon}}{\sqrt{m\varepsilon}} \le \frac{1}{k-\rho} \ln \frac{3k\delta}{\sqrt{m\varepsilon}},$$

and so by taking $\varepsilon > 0$ sufficiently small

$$t_2 \le t_3 + \frac{1}{k-\rho} \ln \frac{3k\delta}{\sqrt{m\varepsilon}} \le 2T_1 + 1 + \frac{1}{k-\rho} \ln \frac{3k\delta}{\sqrt{m\varepsilon}} \le -\frac{1}{2(k-2\rho)} \ln \varepsilon.$$

This completes the proof of Lemma 6. □

Proof of Theorem 4. Let $T_\varepsilon(v_0) = t_2^{\varepsilon, v_0}$. The first part of Theorem 4 follows from Lemmas 3 and 4. We now show (ii).

Since t_2^{ε,v_0} is continuous on $v_0 \in (0, \delta]$ from Lemma 4, it follows that $T_\varepsilon((0,\delta])$ is a connected set in $\mathbb{R}$ and hence an interval. Since from Lemma 5 we have $T_\varepsilon(v_0) \ge -\frac{1}{\beta - k\alpha} \ln(\varepsilon\delta^\gamma)$ for $v_0 = (\varepsilon\delta^\gamma)^{\frac{k}{\beta-k\alpha}}\delta$ and from Lemma 6 we have $T_\varepsilon(\delta) \le -\frac{1}{2(k-2\rho)} \ln \varepsilon$, it follows that (17) holds. This shows (ii) and whence Theorem 4. □

Proof of Theorem 3. It follows from Theorem 4 that for sufficiently small ε and $v_0 \in (0,\delta]$, the solution of v_{ε,v_0} of (15) satisfying (16) and (17). In particular, from (17) we obtain

$$e^{T_\varepsilon((0,\delta])} \supseteq \left[\varepsilon^{-1/2(k-2\rho)}, (\varepsilon\delta^\gamma)^{-1/(\beta-\alpha k)}\right].$$

Hence, for any given any R satisfying (9), there is $v_0 \in (0,\delta]$ such that $e^{T_\varepsilon(v_0)} = R$. Then $u(x) := |x|^{-k} v_{\varepsilon,v_0}(\ln|x|)$ solves the problem

$$-\Delta u(x) = (1 + \varepsilon g(|x|, u)) u(x)^p, \quad |x| < R, \qquad u(x) = 0 \quad \text{on} \quad |x| = R.$$

This completes the proof of Theorem 3. □

Proof of Theorem 2. Note that g satisfies (6) implies that g satisfies (8). Let u be a solution of (10) given in Theorem 3. Then $w(x) := R^k u(Rx)$ solves the problem

$$\begin{aligned} -\Delta w(x) &= \Big(1 + \varepsilon g(R|x|, R^{-k} w(x))\Big) w(x)^p = \Big(1 + \varepsilon R^{\beta - k\alpha} g(|x|, w(x))\Big) w(x)^p, \qquad |x| < 1, \\ w(x) &= 0 \quad \text{on} \quad |x| = 1. \end{aligned}$$

Since from (9) the range of R is $[\varepsilon^{-1/2(k-2\rho)}, (\varepsilon\delta^\gamma)^{-1/(\beta-\alpha k)}]$, it follows that the range of $\varepsilon R^{\beta-k\alpha}$ is the interval $[\varepsilon^{1-(\beta-k\alpha)/2(k-2\rho)}, \delta^{-\gamma}]$. Note by (5) that $\varepsilon^{1-(\beta-k\alpha)/2(k-2\rho)} \to 0$ and $\delta^{-\gamma} \to \infty$ as $\varepsilon \to 0$ and $\delta \to 0$. Hence, for any given $b > 0$, we can take ε and δ sufficiently small such that $\varepsilon R^{\beta-k\alpha} = b$ and $w(x) = R^k u(Rx)$ is the solution of (7). This shows Theorem 2. □

Appendix

Given a radial function we define the m dimensional Laplacian by

$$\Delta_m v(r) = v''(r) + \frac{m-1}{r} v'(r).$$

Note this is well defined for fractional dimensions. The following theorem gives the precise change of variables result, which has been modified for our particular nonlinearity. We remark this change of variables was independently noticed in [11].

Theorem B. ([7]). *For any $\alpha > -2$, the change of variable $u(r) = (1+\frac{\alpha}{2})^{\frac{2}{p-1}}\tilde{u}(r^{1+\frac{\alpha}{2}})$ gives a correspondence between the radially symmetric solutions of the equation*

$$\begin{cases} -\Delta_N u = |x|^\alpha u^p & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{29}$$

in dimension N and those of the equation

$$\begin{cases} -\Delta_{N(\alpha)} \tilde{u} = \tilde{u}^p & \text{in } \tilde{B}, \\ \tilde{u} = 0 & \text{on } \partial \tilde{B}, \end{cases} \tag{30}$$

in – the potentially fractional – dimension $N(\alpha) = \frac{2(N+\alpha)}{2+\alpha}$.

Proof. A computation shows that

$$\Delta_N u(r) + r^\alpha u(r)^p = (1+\frac{\alpha}{2})^{\frac{2p}{p-1}} r^\alpha \left(\Delta_{N(\alpha)}\tilde{u}(s)\big|_{s=r^{\frac{\alpha}{2}+1}} + \tilde{u}(r^{\frac{\alpha}{2}+1})^p \right),$$

and the desired result easily follows. $\square$

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