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Traveling waves for a generalized Holling–Tanner predator–prey model ☆

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Abstract

We study traveling wave solutions for Holling–Tanner type predator–prey models, where the predator equation has a singularity at zero prey population. The traveling wave solutions here connect the prey only equilibrium $(1, 0)$ with the unique constant coexistence equilibrium (u^*, v^*). First, we give a sharp existence result on weak traveling wave solutions for a rather general class of predator–prey systems, with minimal speed explicitly determined. Such a weak traveling wave $(u(\xi), v(\xi))$ connects $(1, 0)$ at $\xi = -\infty$ but needs not connect (u^*, v^*) at $\xi = \infty$. Next we modify the Holling–Tanner model to remove its singularity and apply the general result to obtain a weak traveling wave solution for the modified model, and show that the prey component in this weak traveling wave solution has a positive lower bound, and thus is a weak traveling wave solution of the original model. These results for weak traveling wave solutions hold under rather general conditions. Then we use two methods, a squeeze method and a Lyapunov function method, to prove that, under additional conditions, the weak traveling wave solutions are actually traveling wave solutions, namely they converge to the coexistence equilibrium as $\xi \to \infty$.

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1. Introduction

The Holling–Tanner model is an important and well studied predator–prey model in the literature. Its simplified dimensionless form is

$$U_t = U(1-U) - \frac{\alpha U^m}{1+\beta U^m} V, \quad V_t = rV\left(1 - \frac{V}{U}\right), \tag{1.1}$$

where $U(t)$ and $V(t)$ are the population sizes of prey and predator respectively, and the parameters α, m and r are positive, with β nonnegative. Here the predation rate in the prey equation is governed by the so called Holling type functional response. Unlike in the conventional predator–prey models such as the Lotka–Volterra model, etc., the growth rate of the predator in (1.1) does not depend on predation rate explicitly, but rather obeys the logistic growth with the carrying capacity proportional to the prey population (with the proportional constant normalized to 1). Clearly, such an equation emphasizes the intra-specific competition among predators. The model was derived in Leslie [24], Leslie and Gower [25], May [31], and has been analyzed both for its mathematical properties (such as the local and global stabilities of the prey-only equilibrium $(1,0)$ and coexistence equilibrium (u^*, v^*), the existence and nonexistence of limit cycles, etc.) and its efficacy in describing real ecological systems such as mite/spider mite, hare/lynx, sparrow/sparrow hawk, etc. (see [12,14,15,17,18,20,37] and the references therein).

When investigating the spatial distributions of the predator and prey species, one is led to the study of the diffusive version of the Holling–Tanner model:

$$U_t = d_1\Delta U + U(1-U) - \frac{\alpha U^m}{1+\beta U^m} V,\ V_t = d_2 \Delta V + rV\left(1 - \frac{V}{U}\right),\ x \in \Omega,\ t > 0. \tag{1.2}$$

In the case that the underlying domain Ω is bounded and there is no population flux across the boundary $\partial\Omega$, the dynamics of the model has been extensively studied. The topics include the local and global stabilities of the constant equilibria $(1,0)$ and (u^*, v^*), the existence and non-existence of non-constant positive equilibria, Turing instability, Hopf bifurcation, etc. See, e.g., [6,8,14,17,18,34,35] and the references therein for details.

We are interested in the case that Ω is the whole space $\mathbb{R}^n$. In such a situation, an important topic is to understand the invasion of the predator into the prey habitat and the invasion speed. This leads to the study of traveling wave solutions of (1.2) that connect the equilibria $(1,0)$ and (u^*, v^*), and it is widely believed that the minimal wave speed of the waves gives the invasion speed (see [4,10,27] and the references therein).

Although there is extensive research on traveling wave solutions of various predator–prey models, very few cover the case of (1.2), where the singularity (at $U = 0$) in the predator equation causes extra difficulties. Some special cases of (1.2) (with $\Omega = \mathbb{R}^N$) have been investigated in [3,13]. In [13], the problem is studied when $m = 1$ and both diffusion coefficients d_1 and d_2 are sufficiently small, while in [3] the authors focus on the special case $m = 1$ and $\beta = 0$.

This research is partly motivated by [7], where the general case of (1.2) is considered, and the invasion speed is established without knowing the existence of traveling waves. It is shown in [7] that the invasion profile can be approximated by certain generalized transition waves, a notion introduced by Berestycki and Hamel [2] to describe traveling-wave-like transition phenomena. Such generalized transition waves include traveling waves as special cases but whether the generalized transition waves in [7] are actually traveling waves is an open problem.

In this paper we aim to obtain a better understanding of the traveling wave solutions of (1.2). As we will explain in detail below, some parts of our arguments can cover rather general systems of predator–prey type, while special properties of the nonlinear functions in (1.2) will be used in the other parts. In view of possible applications elsewhere, we will present results that cover more general cases when it is convenient to do so. One general system including (1.2) as a special case to be considered here is given by

$$U_t = d_1 \Delta U + B(U) - f(U)V, \qquad V_t = d_2 \Delta V + rV\left(1 - \frac{V}{U}\right), \tag{1.3}$$

which we call a generalized Holling–Tanner predator–prey model. Clearly both (1.2) and (1.3) are special cases of the following more general system

$$U_t = d_1 \Delta U + F(U,V), \quad V_t = d_2 \Delta V + G(U,V), \quad x \in \mathbb{R}^n,\ t > 0. \tag{1.4}$$

A traveling wave solution of (1.4) is a special solution $(U(x,t), V(x,t))$ taking the form

$$U(x,t) = u(x \cdot \nu + ct), \qquad V(x,t) = v(x \cdot \nu + ct),$$

where $\nu \in \mathbb{R}^n$ is a unit vector denoting the direction of wave propagation, $x \cdot \nu$ is the usual inner product in $\mathbb{R}^n$, $c > 0$ is the wave speed, and $(u(\xi), v(\xi))$ with $\xi = x \cdot \nu + ct$ satisfies, after the scalings

$$\xi/\sqrt{d_1} \to \xi, \quad c/\sqrt{d_1} \to c, \quad d_2/d_1 \to d, \tag{1.5}$$

the ODE system

$$cu' = u'' + F(u,v), \qquad cv' = dv'' + G(u,v), \qquad \xi \in \mathbb{R}, \tag{1.6}$$

and

$$\begin{cases} 0 < u(\xi) \le 1, \qquad 0 < v(\xi) \le v_0, \qquad \forall\, \xi \in \mathbb{R}, \\ (u,u',v,v')(-\infty) = E_0 := (1,0,0,0), \\ (u,u',v,v')(\infty) = E^* := (u^*,0,v^*,0), \end{cases} \tag{1.7}$$

where $v_0 > 0$ is a constant.

From the viewpoint of dynamical systems, a traveling wave solution corresponds to a heteroclinic orbit of (1.6) in the phase space $\mathbb{R}^4$ of (u,u',v,v') that connects the two equilibria E_0 and E^* (with the extra requirement that u and v are positive). The existence of such a heteroclinic orbit is equivalent to the nonempty intersection of the unstable manifold $W^u(E_0)$ and the stable manifold $W^s(E^*)$. In the well known paper [9], Dunbar proved the existence of traveling waves to a special predator–prey model in two steps. The first step is to show the existence of the so-called **weak traveling waves** that satisfy all the conditions in (1.7) except the boundary condition at $\xi = \infty$, and the second step is to show that these weak traveling waves converge to E^* as $\xi \to \infty$.

Dunbar's proof of the first step is complex, which involves detailed analysis of the dynamics of the orbits lying in the unstable manifold $W^u(E_0)$ and an application of a shooting argument that needs the homotopy theory in $\mathbb{R}^4$. Subsequently his method has been generalized and improved by many authors (see e.g., [1,5,16,19,21,22,29]). In particular, Huang [22] developed a new shooting argument which not only simplifies Dunbar's argument, but also produces a result on the existence of weak traveling waves for much more general predator–prey systems. Both Dunbar's and Huang's dynamical systems approaches depend on geometrical and topological properties in $\mathbb{R}^4$. (Other topological approaches, such as the use of the Conley index theory, can be found in [11,32].) On the other hand, there is an analytical approach to prove the existence of weak traveling waves in the literature that is based on the combination of lower and upper solutions and the Schauder fixed point theorem (see [8,23,28,30,38–40] and the references therein). Though this approach sometimes does not produce results as general as that in [22], it has its own strengths: the upper and lower solutions employed are simple, so the proofs involved are less complex; the approach can be generalized to nonlocal predator–prey systems (such as those with delays [30,39]), etc.

The second step in Dunbar's proof relies on Lyapunov functions and LaSalle's invariance principle. These techniques have been used as a general approach in the literature, and unlike in step one where rather general systems can be handled, here the detailed techniques vary greatly from model to model, and many general cases are still not covered.

In this paper, we will follow a similar two steps approach. Theorem 2.1 is our first main result in step one on weak traveling wave solutions, which covers very general predator–prey systems of the form (1.4). However, it does not apply directly to the Holling–Tanner type models (1.2) and (1.3), since the reaction function $rv(1 - \frac{v}{u})$ in the predator equation has a singularity at $u = 0$. To overcome this difficulty, we will replace this function by a smooth function $rv(1 - \frac{v}{\sigma_\varepsilon(u)})$ for $(u, v) \in [0, \infty) \times [0, \infty)$, where

$$\sigma_\varepsilon(u) = \begin{cases} u, & \text{if } u \geq \varepsilon, \\ u + \varepsilon e^{\frac{1}{u-\varepsilon}}, & \text{if } 0 \leq u < \varepsilon, \end{cases} \tag{1.8}$$

with $\varepsilon > 0$ sufficiently small, and apply our general theorem to obtain weak traveling waves $(u_\varepsilon, v_\varepsilon)$ for the modified system. We then prove that u_ε has a positive lower bound δ independent of small $\varepsilon > 0$, which ensures that $(u_\varepsilon, v_\varepsilon)$ is indeed a weak traveling wave solution of (1.3). We thus obtain our second main result Theorem 2.2, which in particular covers (1.2) with the parameters satisfying $m \geq 1$, $\alpha > 0$, $\beta \geq 0$ and $r > 0$. Our choice of $\sigma_\varepsilon(u)$ in (1.8) is purely based on the mathematical argument; it has no biological meaning, and the particular choice is not important.

In step two, we will focus on the generalized Holling–Tanner model (1.3). We show that under additional conditions, the weak traveling waves in Theorem 2.2 are actually traveling waves. To prove this, apart from a Lyapunov function method, we will further develop and use a squeeze method introduced in [7]. As a result, we obtain two main theorems, namely, Theorems 3.1 and 3.3, proved by these two different methods, on the existence of traveling waves for (1.3). Applied to the Holling–Tanner model (1.2) with $m = 1$ and $m = 2$ respectively, they yield different sets of conditions on α and β for the existence of traveling waves (see Theorems 3.2 and 3.4), suggesting that both methods have their own advantages.

The rest of the paper is organized as follows. In Section 2, we consider weak traveling waves for (1.4) and (1.3), with main results Theorems 2.1 and 2.2. In Section 3, we use a squeeze method and a Lyapunov function method, respectively, to prove that under additional conditions, the weak traveling waves of (1.3) established in Theorem 2.2 are actually traveling waves (see Theorems 3.1 and 3.3). The results are subsequently applied to the Holling–Tanner model (1.2) with $m = 1$ and $m = 2$ (see Theorems 3.2 and 3.4). In the Appendix, we give a detailed proof of Theorem 2.1 by an upper and lower solution approach.

In the rest of the paper, we are only concerned with traveling wave solutions $(u(\xi), v(\xi))$ with $\xi \in \mathbb{R}$. For convenience, we shall use x to replace the variable ξ.

2. Weak traveling waves

In this section, we first present an existence and nonexistence result for weak traveling waves for a very general predator–prey system, and then apply it to (1.3) through a perturbation argument.

2.1. Weak traveling waves for a general predator–prey system

In this subsection, we shall establish the existence of weak traveling wave solutions for a general predator–prey system of the form (1.6).

Theorem 2.1. *Let F and G be locally Lipschitz continuous on $[0,\infty) \times [0,\infty)$, and let $F(1,0) = 0$ and $G_v(1,0,)$ exist with $G_v(1,0) = r > 0$.*

(i) *Assume that there exists $v_0 > 0$ such that*

$$\begin{cases} G(u,0) = 0, \quad G(u,v_0) \le 0, \quad \forall u \in [0,1], \\ G(u,v) \le rv, \quad \forall (u,v) \in [0,1] \times [0,v_0], \\ G \in C^2([1-\delta,1] \times [0,\delta]) \text{ for some small } \delta > 0; \end{cases}$$

and

$$\begin{cases} F(u,0) \ge -M_0(1-u) \quad \text{for } u \in [0,1] \text{ and some } 0 \le M_0 < \min\{d,1\}r, \\ F(0,v) \ge 0, \ \ F(1,v) \le 0, \quad \forall v \in [0,v_0]. \end{cases}$$

Then for every $c \ge c^* := \sqrt{4dr}$, (1.6) admits a solution (u,v) satisfying

$$\begin{cases} 0 < u(x) \le 1, \quad 0 < v(x) \le v_0, \qquad \forall x \le 0, \\ 0 \le u(x) \le 1, \quad 0 \le v(x) \le v_0, \qquad \forall x > 0, \\ (u,u',v,v')(-\infty) = E_0. \end{cases} \tag{2.1}$$

Furthermore, u' and v' are bounded on $\mathbb{R}$, and

$$\begin{cases} u(x) > 0 \text{ in } \mathbb{R} \text{ if } F(0,v) = 0, \ \forall v \in [0,v_0], \\ u(x) < 1 \text{ in } \mathbb{R} \text{ if } F(1,v) < 0, \ \forall v \in (0,v_0], \\ v(x) < v_0 \text{ in } \mathbb{R} \text{ if } G(u,v_0) < 0, \ \forall u \in [0,1]. \end{cases}$$

(ii) *Assume that G is C^1 in a neighborhood of $(1,0)$ and $G(u,0)=0$ for $(u,0)$ in this neighborhood. Then for $0<c<\sqrt{4dr}$, (1.6) does not have a solution $(u(x),v(x))$ connecting $(1,0)$ as $x\to-\infty$ and satisfying $v(x)>0$ for sufficiently negative x.*

Remark 1.

(i) The speed $c^*=\sqrt{4dr}$ is referred to as the minimal wave speed. Using the scalings in (1.5), this minimal speed c^* becomes $\sqrt{4d_2r}$ for the system (1.4), which does not depend on the diffusion coefficient d_1 of the prey species.

(ii) The assumptions $G(u,0)=0$ for $u\in[0,1]$ and $G\in C^2([1-\delta,1]\times[0,\delta])$ for some small $\delta>0$ imply that

$$G(u,v)\ge rv-K[1-u+v]v, \qquad \forall\,(u,v)\in[1-\delta,1]\times[0,\delta], \tag{2.2}$$

where $K=\max\{|G_{uv}(u,v)|,|G_{vv}(u,v)|:(u,v)\in[1-\delta,1]\times[0,\delta]\}$. Indeed, using $G(u,0)=0$ for $u\in[0,1]$ and the fundamental theorem of calculus (FTC) we have, for $(u,v)\in[1-\delta,1]\times[0,\delta]$,

$$G(u,v)=vg(u,v)=:v\int_0^1 G_v(u,\theta v)\,d\theta, \qquad g(1,0)=G_v(1,0)=r.$$

Applying the FTC again yields, for $(u,v)\in[1-\delta,1]\times[0,\delta]$,

$$g(u,v)=r+(u-1)\int_0^1 g_u(\theta_1u+1-\theta_1,\theta_1v)\,d\theta_1+v\int_0^1 g_v(\theta_1u+1-\theta_1,\theta_1v)\,d\theta_1, \tag{2.3}$$

where $g_u(u,v)=\int_0^1 G_{uv}(u,\theta v)\,d\theta$ and $g_v(u,v)=\int_0^1 G_{vv}(u,\theta v)\theta\,d\theta$. Clearly this implies (2.2)

(iii) In part (i) of the above theorem, the assumption $G\in C^2([1-\delta,1]\times[0,\delta])$ can be replaced by

$$G(u,v)\ge rv-K[(1-u)^\sigma+v^\sigma]v, \quad \forall\,(u,v)\in[1-\delta,1]\times[0,\delta] \tag{2.4}$$

for some $\delta\in(0,1)$, $\sigma\in(0,1]$ and $K>0$.

(iv) The linear growth assumption $G(u,v)\le rv$ is the key to the minimal speed being equal to $\sqrt{4dr}$. If G is in $C^2([0,1]\times[0,v_0])$, then (2.3) holds for $(u,v)\in[0,1]\times[0,v_0]$, from which we deduce that a sufficient condition for this linear growth condition to hold is

$$G_{uv}(u,v)\ge 0, \qquad G_{vv}(u,v)\le 0, \qquad \forall\,(u,v)\in[0,1]\times[0,v_0].$$

(v) The condition on $F(u,0)$ (the rate of change of prey population without the presence of predator) in Theorem 2.1 (i) allows $F(u,0)$ to change signs over $[0,1]$, such as those in models with the strong Allee effect [33]. This condition weakens the condition $F(u,0)\ge 0$ for $u\in[0,1]$ required in [22].

Theorem 2.1 could perhaps be proved by modifications of the arguments in [22], though we assume less smoothness on F and G and weaker condition on $F(u,0)$ than [22]. In the Appendix, we give a complete proof using an upper and lower solution argument.

In the next subsection, we will apply Theorem 2.1 to the generalized Holling–Tanner model (1.3). Though we will not go into the details, we would like to point out that it also applies to many other models, such as the following ones:

Beddington–DeAngelis model ([22]).

$$\begin{cases} u'' - cu' + u(1-u) - \dfrac{\alpha uv}{a+mu+nv} = 0, \\ dv'' - cv' + \Big(-\mu_1 - \mu_2 v + \dfrac{\beta u}{a+mu+nv} \Big) v = 0, \end{cases}$$

where $\alpha > 0$, $\beta > 0$, $a \ge 0$, $m \ge 0$, $n \ge 0$, $\mu_1 > 0$, $\mu_2 \ge 0$ are constant and $\mu_2 + n > 0$.

Yodzis model ([22]).

$$\begin{cases} u'' - cu' + u(1-u) - \dfrac{\alpha u^2 v}{1+mu^2} = 0, \\ dv'' - cv' + v\Big(-\mu_1 - \mu_2 v + \dfrac{\beta u^2}{1+mu^2} \Big) = 0, \end{cases}$$

where $\alpha > 0$, $\beta > 0$, $m \ge 0$, $\mu_1 > 0$, $\mu_2 > 0$ are constant.

SIS model ([26]).

$$\begin{cases} S'' - cS' + 1 - S - \dfrac{\beta SI}{S+I} + \theta\gamma I = 0, \\ dI'' - cI' + \dfrac{\beta SI}{S+I} - \gamma I = 0, \end{cases}$$

where $\beta > 0$, $\gamma > 0$, and $\theta \in (0,1]$ are constant.

Leslie–Gower model with strong Allee effect ([33]).

$$\begin{cases} u'' - cu' + \dfrac{1}{b} u(1-u)(u-b) - \beta uv = 0, \\ dv'' - cv' + rv(1 - \dfrac{v}{u}) = 0, \end{cases}$$

where $\beta > 0$, $b \in (0,1)$, and $r > 0$ are constant.

2.2. *Weak traveling waves for the generalized Holling–Tanner model*

For our analysis to follow it is convenient to write the reduced traveling wave system for (1.3) in the following form:

$$u'' - cu' + g(u)[h_1(u) - h_2(u)v] = 0, \qquad dv'' - cv' + rv\left(1 - \frac{v}{u}\right) = 0, \tag{2.5}$$

where $r > 0$, and we assume that g, h_1, h_2 satisfy

$$(A_0):\begin{cases} g,\ h_1,\ h_2\in C[0,\infty);\ gh_1,\ gh_2 \text{ are locally Lipschitz continuous on } [0,\infty);\\ g(u)>0,\ h_1(u)\geq 0,\ \forall u\in(0,1];\\ g(0)=0,\ h_1(0)>0,\ h_1(1)=0,\ h_2(1)>0.\end{cases}$$

Theorem 2.2. *Assume (A_0). For $c<2\sqrt{dr}$, there is no weak traveling wave solution of (2.5). For any $c\geq 2\sqrt{dr}$, (2.5) has at least one weak traveling wave solution (u,v) satisfying (2.1) with $v_0=1$; moreover, $u(x)\geq\delta_0$ for $x\in\mathbb{R}$, and for any $0<\delta<\delta_0$, $v(x)>\delta$ for $x\geq x_0$ and some $x_0\in\mathbb{R}$, where $\delta_0>0$ is defined in (2.7).*

Proof. We first consider the modified system

$$u''-cu'+g(u)[h_1(u)-h_2(u)v]=0,\qquad dv''-cv'+rv\left(1-\frac{v}{\sigma_\varepsilon(u)}\right)=0,\tag{2.6}$$

where $c\geq 2\sqrt{dr}$ and $\sigma_\varepsilon(u)$ is defined in (1.8) with $\varepsilon>0$ small. Let $F(u,v)=g(u)[h_1(u)-h_2(u)v]$ and $G(u,v)=rv[1-v/\sigma_\varepsilon(u)]$. It is straightforward to verify that all the conditions in Theorem 2.1 with $v_0=1$ are satisfied, and hence there is a weak traveling wave solution $(u_\varepsilon,v_\varepsilon)$ of (2.6) satisfying (2.1) and $0<u_\varepsilon(x)<1$ and $0<v_\varepsilon(x)<1$ for all $x\in\mathbb{R}$. Applying Lemma 2.3 below we conclude that for any $0<\delta<\delta_0$ and sufficiently small $\varepsilon>0$, $u_\varepsilon(x)>\delta$ for all $x\in\mathbb{R}$ and $v_\varepsilon(x)>\delta$ for $x\geq x_0$ and some $x_0\in\mathbb{R}$. Consequently $\sigma_\varepsilon(u_\varepsilon)\equiv u_\varepsilon$, and $(u,v):=(u_\varepsilon,v_\varepsilon)$ is a weak traveling wave solution of (2.5) with the properties as stated in Theorem 2.2. □

Lemma 2.3. *Assume (A_0). Let*

$$\begin{cases} M_1=\sup\{g(u)(|h_2(u)|-h_1(u))/u: 0<u\leq 1\},\\ \rho_1^+:=\left(c+\sqrt{c^2+4M_1}\right)/2,\\ M_2=r/d+c\Big(|1/d-1|+1\Big)\rho_1^++2M_1.\end{cases}$$

Let

$$\delta_0:=\sup\left\{u\in(0,1):\sup_{0\leq s\leq u}\Big(h_1(s)-\frac{dM_2}{r}s|h_2(s)|\Big)>0\right\}.\tag{2.7}$$

Then for any $0<\delta<\delta_0$, there is $\varepsilon_0>0$ such that for every $0<\varepsilon<\varepsilon_0$, if $(u_\varepsilon,v_\varepsilon)$ is a weak traveling wave solution of (2.6), then $u_\varepsilon(x)>\delta$ for $x\in\mathbb{R}$, and $v_\varepsilon(x)>\delta$ for $x\geq x_0$ and some $x_0\in\mathbb{R}$.

Proof. Denote

$$\sigma(u):=\sigma_\varepsilon(u),\ F(u,v)=g(u)[h_1(u)-h_2(u)v],\ G(u,v)=rv(1-\frac{v}{\sigma(u)}).$$

Let $(u(x),v(x))$ be a weak traveling wave solution of (2.6). Note that $0<u(x)\leq 1$ and $0<v(x)\leq 1$ for all $x\in\mathbb{R}$. Since $h_1(0)>0$, we have $\delta_0>0$; since $h_2(1)>0$ and $h_1(1)=0$, we have $M_1>0$ and $-F(u(x),v(x))/u(x)\leq M_1$ for $x\in\mathbb{R}$. We complete the proof in 5 steps.

Step 1. *Show that* $\frac{|u'(x)|}{u(x)} \le \rho_1^+$ *for all* $x \in \mathbb{R}$.

Let $\rho_1 = u'/u$. Using the equation satisfied by u we have

$$\rho_1' = \frac{u''}{u} - \rho_1^2 = c\rho_1 - \frac{F(u,v)}{u} - \rho_1^2 \le c\rho_1 + M_1 - \rho_1^2. \tag{2.8}$$

We claim that $|\rho_1(x)| \le \rho_1^+$ for all x. Since $\rho_1(-\infty) = 0$ and ρ_1^+ is a positive constant solution of $\rho' = c\rho + M_1 - \rho^2$, it follows from the comparison theorem that $\rho_1(x) < \rho_1^+$ for all $x \in \mathbb{R}$. Similarly, if $\rho_1 < -\rho_1^+$ occurs at some x_0, then letting $\rho(x)$ be the solution of $\rho' = c\rho + M_1 - \rho^2$ with $\rho(x_0) = \rho_1(x_0)$, and using the comparison theorem gives $\rho_1(x) \le \rho(x)$ for $x \ge x_0$. Note that $c\rho(x_0) - M_1 - \rho^2(x_0) < c(-\rho_1^+) + M_1 - (-\rho_1^+)^2 < 0$ implies $\rho(x) \to -\infty$ as $x \to x_1$ for some finite value $x_1 > x_0$. It follows that $\rho_1(x) \to -\infty$ as $x \to x_2$ for some $x_2 \in (x_0, x_1]$, contradicting the fact that $\rho_1(x)$ is defined for all $x \in \mathbb{R}$. We thus conclude that $|u'|/u = |\rho_1| < \rho_1^+$ on $\mathbb{R}$.

Step 2. *Show that* $\frac{v'(x)}{v(x)} \le \left(c - \sqrt{c^2 - 4dr}\right)/(2d)$ *for* $x \in \mathbb{R}$.

To show this, we note that, in the (v, v') plane, the vector field determined by the v-equation in (2.6) on the line segment $v' = kv$, $0 < v < 1$, $k = \left(c - \sqrt{c^2 - 4dr}\right)/(2d)$ points upward across this line, because

$$\frac{v''}{v'} = \frac{cv' - G(u,v)}{dkv} > \frac{ckv - rv}{dkv} = \frac{ck - r}{dk} = k.$$

This implies that $\frac{v'(x)}{v(x)} < k$ for $x \in \mathbb{R}$, for otherwise there would exist x_0 such that $\frac{v'(x)}{v(x)} \ge k$, and so $\frac{v'(x)}{v(x)} > k$ for all $x > x_0$, yielding $v(x) > v(x_0)e^{k(x-x_0)} \to \infty$ as $x \to \infty$, contradicting the fact that $v(x) < 1$ for all x.

Step 3. *Show that* $\frac{v(x)}{\sigma(u(x))} \le \frac{dM_2}{r}$ *for* $x \in \mathbb{R}$.

Let $\rho = \frac{v}{\sigma(u)}$. We calculate

$$\rho' = \frac{v'}{\sigma(u)} - \frac{(\sigma(u))'v}{(\sigma(u))^2} = \left(\frac{v'}{v} - \frac{(\sigma(u))'}{\sigma(u)}\right)\rho, \tag{2.9}$$

and

$$\begin{aligned}
\rho'' &= \Big(\frac{v'}{v} - \frac{(\sigma(u))'}{\sigma(u)}\Big)\rho' + \Big[\frac{v''}{v} - \frac{\sigma'(u)u''}{\sigma(u)} - \frac{\sigma''(u)u'^2}{\sigma(u)} - \left(\frac{v'}{v}\right)^2 + \left(\frac{(\sigma(u))'}{\sigma(u)}\right)^2\Big]\rho \\
&= \left(\frac{v'}{v} - \frac{\sigma'(u)u'}{\sigma(u)}\right)\rho' + \Big[\frac{c}{d}\frac{v'}{v} - \frac{r}{d}(1-\rho) - c\frac{\sigma'(u)u'}{\sigma(u)} + \frac{\sigma'(u)F(u,v)}{\sigma(u)} \\
&\qquad - \frac{\sigma''(u)u'^2}{\sigma(u)} - \left(\frac{v'}{v}\right)^2 + \left(\frac{(\sigma(u))'}{\sigma(u)}\right)^2\Big]\rho \\
&= \Big(\frac{v'}{v} - \frac{(\sigma(u))'}{\sigma(u)}\Big)\rho' + \Big[\frac{c}{d}\left(\frac{\rho'}{\rho} + \frac{\sigma'(u)u'}{\sigma(u)}\right) - \frac{r}{d} + \frac{r}{d}\rho - c\frac{\sigma'(u)u'}{\sigma(u)}
\end{aligned}$$

$$+\frac{\sigma'(u)F(u,v)}{\sigma(u)}-\frac{\sigma''(u)u'^2}{\sigma(u)}\Big]\rho-\Big(\frac{v'}{v}+\frac{(\sigma(u))'}{\sigma(u)}\Big)\rho'$$
$$=\Big(\frac{c}{d}-\frac{2(\sigma(u))'}{\sigma(u)}\Big)\rho'+\Big[\frac{r}{d}\rho-\frac{r}{d}+c\left(\frac{1}{d}-1\right)\frac{\sigma'(u)u'}{\sigma(u)}$$
$$+\frac{\sigma'(u)F(u,v)}{\sigma(u)}-\frac{\sigma''(u)u'^2}{\sigma(u)}\Big]\rho. \quad (2.10)$$

To continue the proof we need the estimates for $\sigma(u)$ and its derivatives. Note that

$$\sigma'(u)=\begin{cases}1, & \text{if } u\ge\varepsilon,\\ 1-\frac{\varepsilon}{(u-\varepsilon)^2}e^{\frac{1}{u-\varepsilon}}, & \text{if } 0\le u<\varepsilon,\end{cases}$$
$$\sigma''(u)=\begin{cases}0, & \text{if } u\ge\varepsilon,\\ \frac{\varepsilon}{(u-\varepsilon)^4}[1+2(u-\varepsilon)]e^{\frac{1}{u-\varepsilon}}, & \text{if } 0\le u<\varepsilon,\end{cases}$$

and

$$\sigma'''(u)=\begin{cases}0, & \text{if } u\ge\varepsilon,\\ -\frac{\varepsilon}{(u-\varepsilon)^6}[1+6(u-\varepsilon)+6(u-\varepsilon)^2]e^{\frac{1}{u-\varepsilon}}, & \text{if } 0\le u<\varepsilon.\end{cases}$$

It follows that, for $u>0$ (taking $\varepsilon>0$ further smaller if necessary),

$$\max\{u,\varepsilon e^{-1/\varepsilon}\}\le\sigma(u)\le u+\varepsilon,\quad 0<\sigma'(u)\le 1,\quad 0\le\sigma''(u)\le\sigma''(0)<1, \quad (2.11)$$

so that

$$\frac{\sigma'(u)}{\sigma(u)}\le\frac{1}{u},\qquad \frac{\sigma''(u)}{\sigma(u)}\le\frac{1}{u}\le\frac{1}{u^2}.$$

Now using these estimates and

$$0<v(x)<1,\quad \left[\frac{u'(x)}{u(x)}\right]^2\le(\rho_1^+)^2=c\rho_1^++M_1,\quad \frac{F(u(x),v(x))}{u(x)}\ge -M_1\qquad \text{for all } x,$$

and the definitions of M_1 and M_2 defined in the lemma, we obtain

$$\rho''>\Big(\frac{c}{d}-\frac{2(\sigma(u))'}{\sigma(u)}\Big)\rho'+\Big[\frac{r}{d}\rho-\frac{r}{d}-c\Big|\frac{1}{d}-1\Big|\frac{|u'|}{u}-M_1-\frac{u'^2}{u^2}\Big]\rho$$
$$>\Big(\frac{c}{d}-\frac{2(\sigma(u))'}{\sigma(u)}\Big)\rho'+\Big(\frac{r}{d}\rho-M_2\Big)\rho. \quad (2.12)$$

Multiplying (2.12) by the integration factor $Q(x):=\sigma^2(u)e^{-cx/d}$ we get for $x\in\mathbb{R}$,

$$[Q(x)\rho'(x)]'>Q(x)\Big[\frac{r}{d}\rho(x)-M_2\Big]\rho(x). \quad (2.13)$$

We now show that $\rho(x) < dM_2/r$ for $x \in \mathbb{R}$. Assume that this is false. Since $\rho(-\infty) = 0$, there is a smallest x_0 such that $\rho(x_0) = dM_2/r$ and $\rho'(x_0) \geq 0$. It follows from (2.13) and $Q(x) > 0$ that $\rho'(x) > 0$ and $\rho(x) > dM_2/r$ for $x > x_0$ so that the right hand side of (2.13) is positive, and so $Q(x)\rho'(x) > Q(x_0)\rho'(x_0)$, which yields, for $x > x_0$,

$$\rho'(x) > \frac{Q(x_0)}{Q(x)}\rho'(x_0) = \frac{\sigma^2(u(x_0))}{\sigma^2(u(x))}e^{c(x-x_0)/d}\rho'(x_0) \geq \sigma^2(u(x_0))\rho'(x_0)e^{c(x-x_0)/d}.$$

This implies that $\rho'(x) > 0$ for $x > x_0$ and $\rho(x) \to \infty$ as $x \to \infty$.

On the other hand, $\rho'(x) > 0$ for $x > x_0$ implies $\frac{v'(x)}{v(x)} - \frac{\sigma'(u(x))u'(x)}{\sigma(u(x))} > 0$ from (2.9). Recall that

$$\frac{|\sigma'(u(x))u'(x)|}{\sigma(u(x))} \leq \frac{|u'(x)|}{u(x)} \leq \rho_1^+, \quad \forall x \in \mathbb{R}.$$

It follows that

$$\frac{v'(x)}{v(x)} > \frac{\sigma'(u(x))u'(x)}{\sigma(u(x))} > -\rho_1^+ \text{ for } x > x_0.$$

Combining with the estimate $\frac{v'(x)}{v(x)} < \frac{c}{2d}$ from Step 2 we obtain

$$\frac{|v'(x)|}{v(x)} < M_3 := \max\left\{\rho_1^+, \frac{c}{2d}\right\} \text{ for } x > x_0.$$

Using these estimates we obtain, for $x > x_0$,

$$\left|\left(\frac{c}{d} - \frac{2(\sigma(u))'}{\sigma(u)}\right)\rho'\right| = \left|\left(\frac{c}{d} - \frac{2\sigma'(u)u'}{\sigma(u)}\right)\left(\frac{v'}{v} - \frac{\sigma'(u)u'}{\sigma(u)}\right)\right|\rho \leq M_4\,\rho$$

with

$$M_4 := \left(\frac{c}{d} + 2\rho_1^+\right)(M_3 + \rho_1^+).$$

We may now use (2.12) to obtain

$$\rho'' > \left(\frac{r}{d}\rho - M_5\right)\rho, \qquad M_5 := M_4 + M_2.$$

Since $\lim_{x\to\infty}\rho(x) = \infty$, there is $x_1 > x_0$ such that $\frac{r}{d}\rho(x) > 2M_5$ for $x > x_1$, hence for $x > x_1$, $\rho''(x) > \frac{r}{2d}\rho^2(x)$, and so by multiplying by $\rho'(x) > 0$ on this inequality and then integrating we obtain

$$(\rho')^2(x) > (\rho'(x_0))^2 + \frac{r}{3d}[\rho^3(x) - \rho^3(x_1)].$$

Since $\rho(x) \to \infty$ as $x \to \infty$, we may take $x_2 > x_1$ such that for $x > x_2$, the right hand side of the above inequality is bigger than $\frac{r}{4d}\rho^3(x)$, yielding $\rho'(x) > \frac{1}{2}\sqrt{\frac{r}{d}}\rho^{3/2}(x)$ for $x > x_2$. This inequality implies that $\rho(x)$ blows up in finite time after x_2, which contradicts the definition of ρ. This contradiction shows $\rho(x) < dM_2/r$ for all $x \in \mathbb{R}$.

Step 4. *Show that $u(x) > \delta$ for $x \in \mathbb{R}$.*

Write the equation for u as $u'' - cu' + F(u, v) = 0$. Suppose for contradiction that $u(x) \le \delta$ for some $x \in \mathbb{R}$. Then since $u(-\infty) = 1$ there is a smallest x_0 such that $u(x_0) = \delta$ and $u'(x_0) \le 0$. Since $v(x) \le \frac{dM_2}{r}\sigma(u(x)) = \frac{dM_2}{r}u(x_0)$ from Step 3 (here we take $\epsilon < \delta$), we deduce $u''(x_0) = cu'(x_0) - F(\delta, v(x_0)) < 0$ from the choice of δ and (2.7), from which we conclude that $u(x) < \delta$, $u'(x) < 0$, and $u''(x) < 0$ for all $x > x_0$. This implies $u'(x) < u'(x_0 + 1) < 0$ for $x > x_0 + 1$ and so $u(x) \to -\infty$ as $x \to \infty$, a contradiction. This shows $u(x) > \delta$ for $x \in \mathbb{R}$.

Step 5. *Show that there exists $x_0 \in \mathbb{R}$ such that $v(x) > \delta$ for all $x \ge x_0$.*

This can be proved in the same way as in the proof of Claim 0 in the proof of Theorem 3.1, where γ_1 there is replaced by δ. Since it is more convenient and natural to present the proof there in view of the context, we are not giving the details here. □

If we take

$$g(u) = \frac{u}{1 + \beta u^m}, \quad h_1(u) = (1 - u)(1 + \beta u^m), \quad h_2(u) = \alpha u^{m-1}, \tag{2.14}$$

then (2.5) becomes

$$u'' - cu' + u(1 - u) - \frac{\alpha u^m}{1 + \beta u^m} v = 0, \qquad dv'' - cv' + rv(1 - \frac{v}{u}) = 0, \tag{2.15}$$

which is the reduced traveling wave system for (1.2). Clearly (A_0) is satisfied when $m \ge 1, \alpha > 0$ and $\beta \ge 0$. Moreover, $g(1)h_2(1) > 0$. We thus have

Corollary 2.4. *Suppose $m \ge 1$, $\alpha > 0$, $r > 0$ and $\beta \ge 0$. Then for $c < 2\sqrt{dr}$, (2.15) has no weak traveling wave solution; for every $c \ge 2\sqrt{dr}$, (2.15) has at least one weak traveling wave solution (u, v) satisfying (2.1) with $v_0 = 1$; moreover, $u(x) \ge \delta_0$ for $x \in \mathbb{R}$, and for any $0 < \delta < \delta_0$, $v(x) > \delta$ for $x \ge x_0$ and some $x_0 \in \mathbb{R}$, where $\delta_0 > 0$ is defined in (2.7) with g, h_1, h_2 given in (2.14).*

3. Traveling waves

In this section we focus on the generalized Holling–Tanner model (1.3). We show that under additional conditions, the weak traveling waves in Theorem 2.2 are actually traveling waves, that is, they converge to the coexistence equilibrium (u^*, v^*) as $x \to \infty$. We will use two different methods. In subsection 3.1 we use a squeeze method, while in subsection 3.2 a Lyapunov function method is used. The obtained results are applied to the Holling–Tanner model (1.2) with $m = 1$ and $m = 2$ respectively, and we will compare the results arising from the two different methods at the end of subsection 3.2.

3.1. Convergence of weak traveling waves by a squeeze method

In this subsection, we prove the following result for (2.5) by a squeeze method based on one introduced in [7], and then apply it to the Holling–Tanner model (2.15).

Theorem 3.1. *Assume* (A_0) *and that* $h_1(u) > 0$ *and* $h_2(u) > 0$ *for* $u \in (0, 1)$. *Let* $h(u) := \frac{h_1(u)}{h_2(u)}$ *for* $u \in (0, 1]$ *satisfy:*

- (A_1) *there is a unique* $u^* \in (0, 1)$ *such that* $h(u^*) = u^*$*;*
- (A_2) $\limsup_{u\to 0^+} h(u) > 1$*;*
- (A_3) *for some positive integer* k_0*, the* $2k_0$*-th iteration* $h^{2k_0}(u)$ *of* $h(u)$ *does not have any fixed point in the interval* $(0, 1)$ *other than* $u = u^*$.

Then for every $c \ge 2\sqrt{dr}$*, the system* (2.5) *has a traveling wave solution* (u, v) *satisfying* (1.7) *with* $v_0 = 1$ *and* $v^* = u^*$. *Furthermore,* $\max\{\delta_0, \gamma_1\} \le u(x) < 1$ *and* $0 < v(x) < 1$ *for* $x \in \mathbb{R}$ *with* $\gamma_1 := \min\{u \in (0, 1) : h(u) = 1\}$ *and* δ_0 *is given in* Theorem 2.2. *For* $c < 2\sqrt{dr}$*,* (2.5) *does not have a traveling wave solution.*

Proof. Let (u, v) be a weak traveling wave solution of (2.5) from Theorem 2.2. Then necessarily $c \ge 2\sqrt{dr}$. Since $u(x) > 0$, it follows that $(u(x), v(x))$ for $x \in \mathbb{R}$ satisfies

$$u'' - cu' + g_0(u)[h(u) - v] = 0, \quad dv'' - cv' + \frac{rv}{u}(u - v) = 0, \tag{3.1}$$

where $g_0(u) = g(u)h_2(u)$.

Using the assumption (A_2) and $h(u) > 0$ for $u \in (0, 1)$ and $h(1) = 0$, we define a sequence $\{\gamma_n\}_{n=-1}^{\infty}$ as follows: $\gamma_{-1} = 0$, $\gamma_0 = 1$, and, for $k = 0, 1, 2, \cdots$,

$$\begin{aligned}\gamma_{2k+1} &= \inf\{u > \gamma_{2k-1} : h(u) < \gamma_{2k}\} = \min\{u > \gamma_{2k-1} : h(u) = \gamma_{2k}\},\\ \gamma_{2k+2} &= \sup\{u < \gamma_{2k} : h(u) > \gamma_{2k+1}\} = \max\{u < \gamma_{2k} : h(u) = \gamma_{2k+1}\}.\end{aligned}$$

It follows that $\{\gamma_{2k}\}$ is strictly decreasing and $\{\gamma_{2k+1}\}$ is strictly increasing. Using (A_1) and (A_3) we have

$$\lim_{n\to\infty} \gamma_n = u^* \text{ exists, and } 0 = \gamma_{-1} < \gamma_1 < \gamma_3 < \cdots < u^* < \cdots < \gamma_2 < \gamma_0 = 1.$$

Note that in the case that h is strictly monotone decreasing on $[0, 1]$, we have

$$\gamma_{-1} = 0, \qquad \gamma_n = h(\gamma_{n+1}), \qquad \forall n = -1, 0, 1, \cdots.$$

We use the mathematical induction to prove the following squeezing lemma, which is the core of the squeeze method.

Lemma A. *There exists an increasing sequence* $\{x_n\}_{n=0}^{\infty}$ *with* $x_n \to \infty$ *as* $n \to \infty$ *such that, for* $x \ge x_n$ *(*$n = 0, 1, 2, \cdots$*)*

$$\gamma_{2n+1} < u(x) < \gamma_{2n}, \qquad \gamma_{2n+1} < v(x) < \gamma_{2n}. \tag{3.2}$$

In the proof of Lemma A below, we will use the following fact several times: *If a function $f \in L^1[T,\infty)$ has a constant sign and a bounded derivative on $[T,\infty)$, $T \in \mathbb{R}$, then $f(\infty) = \lim_{x\to\infty} f(x) = 0$.* One could also use a PDE argument similar to that in [7] to prove Lemma A. The proof below uses an ODE approach, which might have independent interest.

Proof of (3.2) with $n = 0$. From Theorem 2.1 and $g(1)h_2(1) > 0$ we already have $0 < u(x) < 1$, $0 < v(x) < 1$ for $x \in \mathbb{R}$, and $u(-\infty) = 1$, $v(-\infty) = 0$.

We show that $u(x) > \gamma_1$ for all $x \in \mathbb{R}$ (recall that $h(\gamma_1) = 1$). Assume this is false. Then there is a smallest $T_{02} \in \mathbb{R}$ such that $u(T_{02}) = \gamma_1$ and $u'(T_{02}) \le 0$. Since $0 < v < 1$ we have $u'' \le g_0(u)[v - h(u)] = g_0(u)[v - 1] < 0$ at $x = T_{02}$. From this and the fact $h(s) > 1$ for $s \in (0, \gamma_1)$ we conclude using the first equation in (3.1) that $u''(x) < 0$, $u'(x) < 0$ and $u(x) < \gamma_1$ for all $x < T_{02}$, yielding $u(x) \to -\infty$ as $x \to \infty$, a contradiction. Hence, $u(x) > \gamma_1$ for all $x \in \mathbb{R}$.

We next show that $v(x) > \gamma_1$ for all large x, say $x \ge x_0$. As will become clear below, the key is to prove the following claim.

Claim 0. *There exists $x_0 \in \mathbb{R}$ such that $v(x_0) > \gamma_1$.*

Assume that Claim 0 is false. Then $v(x) \le \gamma_1$ for all $x \in \mathbb{R}$. We discuss two possibilities.

Case 1. Assume there exists $T_{03} \in \mathbb{R}$ such that $v'(T_{03}) \le 0$. Now using the v equation $dv'' = cv' + \frac{rv}{u}(v-u)$ and $v - u < v - \gamma_1 \le 0$ we derive $v''(x) < 0$, $v'(x) < 0$ for all $x > T_{03}$, concluding $v(x) \to -\infty$ as $x \to \infty$, a contradiction.

Case 2. Assume that $v'(x) > 0$ for all $x \in \mathbb{R}$. Then we would have $0 < v(\infty) \le \gamma_1$ and $v'(\infty) = 0$. We integrate the v equation over $[0,\infty)$ to get

$$-dv'(0) = c[v(\infty) - v(0)] + \int_0^\infty \frac{rv}{u}(v-u)\,dx,$$

so that $\int_0^\infty \frac{rv}{u}(v-u)\,dx < \infty$. Since $\frac{v(x)}{u(x)} > v(x) > v(0)$ for $x > 0$, and $v(x) - u(x) \le \gamma_1 - u(x) < 0$, it follows that $(v-u) \in L^1[0,\infty)$. Since $(v-u) < 0$ and its derivative is bounded on $[0,\infty)$, we have $\lim_{x\to\infty}(v-u)(x) = 0$ so that $u(\infty)$ exists with $u(\infty) = v(\infty)$. Since $u(x) > \gamma_1$ and $v(x) \le \gamma_1$ for $x \in \mathbb{R}$ (by the contradiction assumption), we necessarily have $u(\infty) = v(\infty) = \gamma_1$.

Now using the variation of constants formula for the u equation $(u')' - cu' = g_0(u)(v - h(u))$ and the L'Hopital's rule we derive, as $x \to \infty$,

$$cu'(x) \to g_0(u(\infty))[h(u(\infty)) - v(\infty)] = g_0(\gamma_1)(\gamma_0 - \gamma_1) > 0,$$

which implies that $u(\infty) = \infty$, a contradiction.

We have thus proved Claim 0. It is now easy to show $v(x) > \gamma_1$ for all $x > x_0$. Otherwise there is the smallest $T_{04} > x_0$ such that $v(T_{04}) = \gamma_1$ and $v'(T_{04}) \le 0$. Then the same argument used in the proof of Case 1 shows that $v(x) \to -\infty$ as $x \to \infty$, a contradiction. Thus we have showed that

$$\gamma_1 < u(x) < 1 \qquad \forall x \in \mathbb{R}, \qquad \gamma_1 < v(x) < 1 \qquad \forall x > x_0,$$

which is (3.2) with $n = 0$.

Proof of (3.2) for $n \ge 1$ by induction. Assume that (3.2) holds for $n = k$ with integer $k \ge 0$. We now show that it holds for $n = k + 1$. We divide the proof into four steps. Namely,

Step 1. Show that there exists $T_1 > x_k$ such that $u(x) < \gamma_{2k+2}$ for all $x \geq T_1$.
Step 2. Show that there exists $T_2 > T_1$ such that $v(x) < \gamma_{2k+2}$ for all $x \geq T_2$.
Step 3. Show that there exists $T_3 > T_2$ such that $u(x) > \gamma_{2k+3}$ for all $x \geq T_3$.
Step 4. Show that there exists $T_4 > T_3$ such that $v(x) > \gamma_{2k+3}$ for all $x \geq T_4$.

After these four steps are completed, we may take $x_{k+1} := T_4$ to see that (3.2) holds for $n = k+1$, as we wanted. We now carry out these steps.

Step 1. The key to complete this step is to prove the following claim.

Claim 1: *There exists $T_1 > x_k$ such that $u(T_1) < \gamma_{2k+2}$.*

Assume by contradiction that the claim is false. Then we have $u(x) \geq \gamma_{2k+2}$ and $h(u(x)) \leq h(\gamma_{2k+2}) = \gamma_{2k+1}$ (by the definition of γ_{2k+2}) for $x \geq x_k$. There are two possible cases.

Case 1. There exists a $T_{11} \geq x_k$ such that $u'(T_{11}) \geq 0$. Then we have, at $x = T_{11}$,

$$u'' = cu' + g_0(u)[v - h(u)] \geq g_0(u)(v - \gamma_{2k+1}) > 0,$$

from which we derive $u''(x) > 0$ and $u'(x) > 0$ for all $x > T_{11}$, yielding $u(x) \to \infty$ as $x \to \infty$, a contradiction.

Case 2. $u'(x) < 0$ for all $x \geq x_k$. Then $u(\infty)$ exists with $u(\infty) \geq \gamma_{2k+2}$. Since u'' is bounded and $u' \in L^1[0,\infty)$, it follows that $u'(\infty) = 0$. If $u(\infty) > \gamma_{2k+2}$, then since $v(x) > \gamma_{2k+1}$ for $x > x_k$, we have $u'' > cu' + g_0(u)[\gamma_{2k+1} - h(u(x))] \geq cu' + M_1$ for $x > x_k$, where

$$M_1 = \min_{u(\infty) \leq u \leq \gamma_{2k}} g_0(u)[\gamma_{2k+1} - h(u)] > 0,$$

and an integration over $[x_k, x]$ gives $u'(x) - cu(x) > u'(x_k) - cu(x_k) + M_1(x - x_k)$ and so $u'(x) > M_1(x - x_k)/2$ for all $x > x_k$, yielding $u(x) \to \infty$ as $x \to \infty$, a contradiction. Hence we must have $u(\infty) = \gamma_{2k+2}$.

Now we show that $v(\infty) = h(u(\infty)) = h(\gamma_{2k+2}) = \gamma_{2k+1}$. To see this, we integrate the u equation over $[x_k, \infty)$ to get

$$-u'(x_k) = c(\gamma_{2k+2} - u(x_k)) + \int_{x_k}^{\infty} g_0(u)(v - h(u))\,dx,$$

which implies $\int_{x_k}^{\infty} g_0(u)(v - h(u))\,dx < \infty$. Since $h(u) \leq \gamma_{2k+1}$ for $u \in [\gamma_{2k+2}, \gamma_{2k}]$ and $g_0(u) > 0$ in $(0,1]$, it follows that $(v - h(u)) \in L^1[x_k, \infty)$. As $(v - h(u))(x) \geq 0$ for $x > x_k$ and its derivative is bounded, we may conclude that $\lim_{x\to\infty}(v - h(u))(x) = 0$ so that $v(\infty) = h(u(\infty)) = \gamma_{2k+1}$.

Using the variation of constants formula for the v equation $(v')' - cv' = \frac{rv}{u}(v - u)$ and the L'Hopital's rule we derive

$$cv'(x) \to \frac{rv(\infty)}{u(\infty)}[u(\infty) - v(\infty)] = \frac{r\gamma_{2k+1}}{\gamma_{2k+2}}[\gamma_{2k+2} - \gamma_{2k+1}] > 0,$$

which implies $v(\infty) = \infty$, a contradiction. This completes our proof of Claim 1.

We are now ready to show that $u(x) < \gamma_{2k+2}$ for all $x \geq T_1$. If this is false, then Claim 1 implies there is $T_{12} > T_1$ such that $u'(T_{12}) \geq 0$ and $u(T_{12}) = \gamma_{2k+2}$, and the same argument used in the proof of Case 1 shows that $u''(x) > 0$, $u'(x) > 0$, and $u(x) > \gamma_{2k+2}$ for all $x > T_{12}$, yielding $u(x) \to \infty$ as $x \to \infty$, a contradiction. This completes Step 1.

Step 2. Similar to step 1, the key part in this step is to show

Claim 2: *There is $T_2 > T_1$ such that $v(T_2) < \gamma_{2k+2}$.*

Assume that the claim is false. Then $v(x) \geq \gamma_{2k+2}$ for all $x \geq T_1$. We discuss two possible cases.

Case 1. There is a $T_{21} \geq T_1$ such that $v'(T_{21}) \geq 0$. Then using $u(x) < \gamma_{2k+2}$ for $x \geq T_1$ we have, at $x = T_{21}$,

$$dv'' = cv' + \frac{rv}{u}(v-u) \geq \frac{rv}{u}(\gamma_{2k+2} - u) > 0,$$

from which we derive $v''(x) > 0$ and $v'(x) > 0$ for all $x > T_{21}$, yielding $v(x) \to \infty$ as $x \to \infty$, a contradiction.

Case 2. $v'(x) < 0$ for all $x \geq T_1$. Then we have $v(\infty) \geq \gamma_{2k+2}$ and $v'(\infty) = 0$. We integrate the v equation over $[T_1, \infty)$ to obtain

$$-dv'(T_1) = c[v(\infty) - v(T_1)] + \int_{T_1}^{\infty} \frac{rv}{u}(v-u)\,dx.$$

Thus, $\int_{T_1}^{\infty} \frac{rv}{u}(v-u)\,dx < \infty$ and consequently $\int_{T_1}^{\infty}(v-u)\,dx < \infty$. Since $(v-u) \geq 0$ and its derivative is bounded on $[T_1, \infty)$, it follows that $\lim_{x\to\infty}(v-u)(x) = 0$ so that $u(\infty)$ exists with $u(\infty) = v(\infty)$. Since $u(x) < \gamma_{2k+2}$ and $v(x) \geq \gamma_{2k+2}$ for $x > T_1$, it follows that $u(\infty) = v(\infty) = \gamma_{2k+2}$.

Then using the variation-constants formula for the u equation $(u')' - cu' = g_0(u)(v - h(u))$ and the L'Hopital's rule we derive

$$cu'(x) \to g_0(u(\infty))[h(u(\infty)) - v(\infty)] = g_0(\gamma_{2k+2})(\gamma_{2k+1} - \gamma_{2k+2}) < 0,$$

which implies that $u(\infty) = -\infty$, a contradiction. This proves Claim 2.

We now make use of Claim 2 to show that $v(x) < \gamma_{2k+2}$ for $x \geq T_2$. If this is false, then Claim 2 implies there is $T_{22} > T_2$ such that $v'(T_{22}) \geq 0$ and $v(T_{22}) = \gamma_{2k+2}$, and the same argument used in the proof of Case 1 shows that $v''(x) > 0$, $v'(x) > 0$, and $v(x) > \gamma_{2k+2}$ for all $x > T_{22}$, yielding $v(x) \to \infty$ as $x \to \infty$, a contradiction. This concludes Step 2.

Since Steps 3 and 4 can be carried out similarly, the details are left to the interested reader. This completes the proof of Lemma A.

We now continue the proof of Theorem 3.1. Since $\lim_{n\to\infty} \gamma_n = u^*$, it follows from Lemma A that $\lim_{x\to\infty}(u(x), v(x)) = (u^*, u^*)$. Then applying the variation of constants formulas for both equations in (3.1) and the L'Hopital's rule we obtain $\lim_{x\to\infty}(u'(x), v'(x)) = (0, 0)$. This shows $\lim_{x\to\infty}(u, u', v, v')(x) = E^*$. Consequently, (u, v) is a traveling wave solution of (2.5), and the proof of Theorem 3.1 is complete. □

Remark 2. The above squeeze method extends that in [7] to a more general situation. In [7], it is claimed that $\lim_{n\to\infty} \gamma_n = u^*$ under the assumptions (A_1) and (A_2) only. We explain below that even with a monotone decreasing function $h(u)$ as in [7], assumption (A_3) is still required for this claim to be true in general. Clearly, we have $\gamma_{2k+1} \nearrow \gamma_*$ and $\gamma_{2k} \searrow \gamma^*$ for some γ_* and γ^* in $[0,1]$, both being fixed points of the composition function $h(h(u))$. In the case that h is decreasing on $[0,1]$, we also have $\gamma_* \le u^* \le \gamma^*$. The following example shows that without (A_3), it is possible that $\gamma_* < u^* < \gamma^*$. Given $s \in (0,1)$ and $\varepsilon > 0$ small, let h be a smooth decreasing function on $[0,1]$ satisfying $h(u) = 2s - u$ for $s - \varepsilon \le u \le s + \varepsilon$, $h(u) > 2s - u$ for $0 \le u < s - \varepsilon$, and $h(u) < 2s - u$ for $u > s + \varepsilon$. It is not hard to verify that for this function h, $\gamma_* = s - \varepsilon$, $\gamma^* = s + \varepsilon$, and $u^* = s$.

We now apply Theorem 3.1 to the Holling–Tanner model (2.15). We only examine the cases $m = 1$ and $m = 2$ which correspond to the Holling type II and III functional responses respectively.

Theorem 3.2. *For every $c \ge 2\sqrt{dr}$, (2.15) has a traveling wave solution (u, v) satisfying*

$$\delta_0 \le u < 1, \quad 0 < v < 1, \quad (u, u', v, v')(x) = \begin{cases} (1,0,0,0), & \text{if } x = -\infty, \\ (u^*, 0, u^*, 0), & \text{if } x = \infty, \end{cases} \tag{3.3}$$

in the following cases:

(i) *$m = 1$, $0 < \alpha < 1$ and $\beta \ge 0$;*
(ii) *$m = 2$, and*

$$\{\alpha > 0,\ 0 \le \beta < 3\} \quad \text{or} \quad \{0 < \alpha < \beta^{2/3}(3 - \beta^{1/3}),\ 0 < \beta < 27\}. \tag{3.4}$$

Moreover, in case (i)*, we have the following explicit expressions for u^* and γ_1:*

$$u^* = \frac{2}{\sqrt{(\beta - 1 - \alpha)^2 + 4\beta} + 1 + \alpha - \beta}, \qquad \gamma_1 = \frac{2(1-\alpha)}{\sqrt{(\beta - 1)^2 + 4\beta(1-\alpha)} + 1 - \beta}. \tag{3.5}$$

Proof. Note that (2.15) can be written in the forms of (2.5) and (3.1) with $g(u)$, $h_1(u)$ and $h_2(u)$ given in (2.14) and

$$g_0(u) = \frac{\alpha u^m}{1 + \beta u^m}, \qquad h(u) = \frac{1}{\alpha u^{m-1}}(1-u)(1+\beta u^m). \tag{3.6}$$

It is readily checked that (A_0), (A_1) and (A_2) hold, and $h_1(u) > 0$ and $h_2(u) > 0$ for $u \in (0,1)$ for any $m \ge 1$. Below we check that (A_3) holds with $k_0 = 1$ for cases (i) and (ii) separately. We only consider the case $\beta > 0$ since the verifications in both cases are trivial when $\beta = 0$.

Case (i). In this case we have $h(u) = \frac{1}{\alpha}[1 + (\beta - 1)u - \beta u^2]$ from (3.6). $\alpha < 1$ implies that $h(0) = \frac{1}{\alpha} > 1$. Since h is a quadratic function with $h'(u) = \frac{1}{\alpha}[(\beta - 1) - 2\beta u]$, we see that it has a maximum at $\hat{u} = \frac{\beta - 1}{2\beta} \in (-\infty, 1/2)$. So if $\beta \le 1$, then $h'(u) < 0$ for $u > 0$ and hence h is strictly decreasing in $[0,1]$; if $\beta > 1$, then h is increasing in $[0, \hat{u}]$ and decreasing in $[\hat{u}, 1]$. Since $h(0) > 1$, in either case we see that $h(u) = u$ and $h(u) = 1$ have unique solutions $u = u^* \in (0,1)$ and $u = \gamma_1 \in (0,1)$ respectively, with the formulas given in (3.5).

To show (A_3), we let $p(u) = h(h(u)) - u$. Since

$$h(h(u)) = \frac{1}{\alpha}\left\{1 + \frac{1}{\alpha}(\beta - 1)[1 + (\beta - 1)u - \beta u^2] - \frac{\beta}{\alpha^2}[1 + (\beta - 1)u - \beta u^2]^2\right\},$$

it follows that $p(u)$ is a polynomial of degree 4 with $p(\pm\infty) = -\infty$. Since $h(0) = \frac{1}{\alpha} > 1$, we have $p(0) = h(\frac{1}{\alpha}) < h(1) < 0$. Since $h(-\frac{1}{\beta}) = 0$, we have $p(-\frac{1}{\beta}) = h(0) + \frac{1}{\beta} = \frac{1}{\alpha} + \frac{1}{\beta} > 0$. Since $h(1) = 0$, we have $p(1) = h(0) - 1 = \frac{1}{\alpha} - 1 > 0$. We thus conclude that p has two negative roots lying in the intervals $(-\infty, -\frac{1}{\beta})$ and $(-\frac{1}{\beta}, 0)$ respectively, and two positive roots lying in the intervals $(0, 1)$ and $(1, \infty)$ respectively. Consequently, $h(h(u))$ has a unique fixed point in $(0, 1)$ which must be u^*. Thus (A_3) holds.

Case (ii). We have

$$h(u) = \frac{1}{\alpha u}\left(1 - u + \beta u^2 - \beta u^3\right) = \frac{1}{\alpha}\left(\frac{1}{u} - 1 + \beta u - \beta u^2\right),$$
$$h(h(u)) = \frac{1}{\alpha h(u)}\left[1 - h(u) + \beta h(u)^2 - \beta h(u)^3\right].$$

Thus, any fixed point $u \in (0, 1)$ of the function $h(h(u))$ satisfies

$$1 - h(u) + \beta h(u)^2 - \beta h(u)^3 = \alpha h(u)u = 1 - u + \beta u^2 - \beta u^3,$$

that is, $[u - h(u)] - \beta[u^2 - h(u)^2] + \beta[u^3 - h(u)^3] = 0$. If u is a fixed point of $h(h(u))$ other than u^*, by (A_2) we have $h(u) \neq u$ and hence we can divide the above identity by $u - h(u)$ to obtain

$$[1 - \beta u + \beta u^2] - \beta h(u)[1 - u - h(u)] = 0. \tag{3.7}$$

Assume the first set of conditions in (ii) holds, i.e., $\alpha > 0$ and $0 < \beta < 3$. Assume by contradiction that $h(h(u))$ has a fixed point $\bar{u} \in (0, 1)$ different from u^*. Since $\bar{u} \in (0, 1)$ we have $h(\bar{u})(1 - \bar{u} - h(\bar{u})) \leq (1 - \bar{u})^2/4$. It follows from (3.7) that

$$[1 - \beta\bar{u} + \beta\bar{u}^2] = \beta h(\bar{u})[1 - \bar{u} - h(\bar{u})] \leq \frac{\beta}{4}(1 - \bar{u})^2 = \frac{\beta}{4}(1 - 2\bar{u} + \bar{u}^2).$$

Simplifying gives $3\bar{u}^2 + 2\bar{u} + 4/\beta - 1 \leq 0$. However, the condition $\beta < 3$ implies that this inequality does not hold. Hence (A_3) holds.

Now we assume the second set of the conditions in (ii) holds. Noting that

$$h'(u) = \frac{1}{\alpha}\left(-\frac{1}{u^2} + \beta - 2\beta u\right), \quad h''(u) = \frac{2}{\alpha}\left(\frac{1}{u^3} - \beta\right),$$

we see that $h'(0) = -\infty$ and $h'(u)$ reaches its global maximum over the interval $(0, \infty)$ at $\tilde{u} = \beta^{-1/3}$ with $h'(\tilde{u}) = (\beta - 3\beta^{2/3})/\alpha$. The assumption in this case implies that $h'(\tilde{u}) < -1$. Thus, $h'(u) < -1$ for $u \in (0, \infty)$, so that $h(u)$ is strictly decreasing on $(0, \infty)$ with $h(u) > 0$ for $u \in (0, 1)$, and so $\frac{d}{du}h(h(u)) = h'(h(u))h'(u) > 1$ for $u \in (0, 1)$. This yields that $h(h(u))$ cannot have two different fixed points in $(0, 1)$, and thus (A3) holds. This completes the proof of Theorem 3.2. □

3.2. Convergence of weak traveling waves by Lyapunov functions

In this subsection, we use a Lyapunov function and LaSalle's invariance principle to prove the following result, and then apply it to the Holling–Tanner model (1.2).

Theorem 3.3. *Assume* (A_0) *and*

(A_1'): $h_1(u) - h_2(u)u = 0$ *has a unique solution* $u^* \in (0, 1)$, $h_2(u^*) > 0$*;*
(A_2'): $[h_1(u) - h_1(u^*)](u - u^*) < 0$, $[h_2(u) - h_2(u^*)](u - u^*) \geq 0$, $\forall u \in (0, 1] \setminus \{u^*\}$*;*
(A_3'): $u^* g(u) - u(u - u^*)g'(u) > 0$, $\quad \forall u \in (0, 1]$.

Then, the system (2.5) *has a traveling wave solution* (u, v) *satisfying* (1.7) *with* $v_0 = 1$ *and* $v^* = u^*$ *for every* $c \geq 2\sqrt{dr}$*. It has no such solutions when* $c < 2\sqrt{dr}$*.*

Proof. Let (u, v) be a weak traveling wave solution of (2.5). By Theorem 2.2, there are $\delta > 0$ and $x_0 > 0$ such that $u(x) > \delta$ for $x \in \mathbb{R}$ and $v(x) > \delta$ for $x > x_0$. Using a similar argument to the proof of Step 1 in the proof of Lemma 2.3 we can show that $|u'(x)|/u(x)$ and $|v'(x)|/v(x)$ are bounded for $x \in \mathbb{R}$, and hence there is $M > 0$ such that $|u'(x)| \leq M$ and $|v'(x)| \leq M$ for $x \in \mathbb{R}$. This implies that the orbit $(u, u', v, v')(x)$ lies in the set $\Omega_\delta =: [\delta, 1] \times [-M, M] \times [\delta, 1] \times [-M, M]$ for $x > x_0$. To show that $(u, u', v, v')(x) \to (u^*, 0, v^*, 0)$ as $x \to \infty$, we define a Lyapunov function L on $(0, 1] \times \mathbb{R} \times (0, 1] \times \mathbb{R}$ by

$$L(u, u', v, v') = cH(u, v) - \frac{\partial H}{\partial u}u' - d\frac{\partial H}{\partial v}v',$$

where

$$H(u, v) = \int_{u^*}^{u} \frac{u - u^*}{ug(u)}\,du + \frac{h_2(u^*)}{r}\int_{v^*}^{v} \frac{v - v^*}{v}\,dv.$$

Along the orbits of (2.5) with $x > x_0$ we have

$$\frac{d}{dx}L = \left[\frac{\partial H}{\partial u}F(u, v) + \frac{\partial H}{\partial v}G(u, v)\right] - \frac{\partial^2 H}{\partial u^2}(u')^2 - d\frac{\partial^2 H}{\partial v^2}(v')^2, \tag{3.8}$$

where $F(u, v) = g(u)[h_1(u) - h_2(u)v]$, $G(u, v) = rv(1 - \frac{v}{u})$. Below we show that under the assumptions of the theorem,

$$\left[\frac{\partial H}{\partial u}F(u, v) + \frac{\partial H}{\partial v}G(u, v)\right] < 0, \quad \forall (u, v) \in (0, 1] \times (0, 1] \setminus \{(u^*, v^*)\}, \tag{3.9}$$

and

$$\frac{\partial^2 H}{\partial u^2} > 0, \quad \frac{\partial^2 H}{\partial v^2} > 0, \quad \forall (u, v) \in (0, 1] \times (0, 1] \setminus \{(u^*, v^*)\}. \tag{3.10}$$

First we rewrite $F(u, v)$ and $G(u, v)$:

$$\begin{aligned}F(u,v) &= g(u)[h_1(u)-h_1(u^*)-h_2(u)v+h_2(u^*)v^*)]\\ &= g(u)[h_1(u)-h_1(u^*)-(h_2(u)-h_2(u^*))v-h_2(u^*)(v-v^*)],\end{aligned}$$

and

$$G(u,v)=\frac{rv}{u}[(u-u^*)-(v-v^*)].$$

Then, for $0<u\le 1$ and $0<v\le 1$,

$$\begin{aligned}&\frac{\partial H}{\partial u}F(u,v)+\frac{\partial H}{\partial v}G(u,v)=\frac{u-u^*}{ug(u)}F(u,v)+\frac{h_2(u^*)(v-v^*)}{rv}G(u,v)\\ &=\frac{u-u^*}{u}\Big[h_1(u)-h_1(u^*)-[h_2(u)-h_2(u^*)]v-h_2(u^*)(v-v^*)\Big]\\ &\quad+\frac{h_2(u^*)}{u}(v-v^*)\Big[(u-u^*)-(v-v^*)\Big]\\ &\le\frac{1}{u}\Big\{(h_1(u)-h_1(u^*))(u-u^*)-h_2(u^*)(v-v^*)^2\Big\}. \qquad \text{(by } (A_2')\text{)}\end{aligned} \tag{3.11}$$

By (A_2'), (3.9) follows.

For $0<u\le 1$ and $0<v\le 1$, we have $\frac{\partial^2 H}{\partial v^2}=\frac{h_2(u^*)v^*}{rv^2}>0$, and by (A_3')

$$\frac{\partial^2 H}{\partial u^2}=\frac{ug(u)-(u-u^*)[g(u)+ug'(u)]}{u^2g^2(u)}=\frac{u^*g(u)-u(u-u^*)g'(u)}{u^2g^2(u)}>0.$$

Thus, (3.10) follows.

It follows from (3.8), (3.9) and (3.10) that $\frac{d}{dx}L(u,u',v,v')\le 0$ for $(u,u',v,v')\in\Omega_\delta$, and equality holds only at E^*. Applying LaSalle's invariance principle gives $(u,u',v,v')(x)\to E^*$ as $x\to\infty$. This proves Theorem 3.3. □

Next we apply Theorem 3.3 to the Holling–Tanner model (2.15), and prove the following:

Theorem 3.4. *The conclusions of Theorem 3.2 hold in the following cases:*

(i) $m=1$, $\beta\ge 0$, *and*

$$0<\alpha<\bar\alpha:=\begin{cases}1+\frac{(3+2\beta)\beta}{2+\beta}, & \text{if } 0\le\beta\le\sqrt{2},\\ \frac{\beta}{\beta-1}, & \text{if } \beta>\sqrt{2}.\end{cases} \tag{3.12}$$

(ii) $m=2$, $\beta\ge 0$, *and*

$$0<\alpha<\bar\alpha:=\begin{cases}(1+\beta)\left[\frac{2}{(1-\beta)^2}+\frac{\beta}{2}\right], & \text{if } 0\le\beta\le\frac13,\\ 28\beta\left(1-\dfrac{1}{3\sqrt{3\beta}}\right), & \text{if } \frac13<\beta<3,\\ \dfrac{h_1(\hat u)}{\hat u^2}, & \text{if } \beta>3,\end{cases}$$

where $h_1(u) = (1-u)(1+\beta u^2)$ *and*

$$\hat{u} = \frac{1}{3}\left(1 + 2\sqrt{1 - \frac{3}{\beta}}\right). \tag{3.13}$$

Proof. We show that all the assumptions in Theorem 3.3 are satisfied. The assumptions (A_0), (A_1') and the second inequality in (A_2') are easily verified for $m \geq 1$ with g, h_1 and h_2 given in (2.14). It remains to verify (A_3') and the first inequality in (A_2'). We do so only for $\beta > 0$ since the verifications in both cases are trivial for $\beta = 0$.

Case (i). In this case we have $h_1(u) = (1-u)(1+\beta u)$, which is a quadratic function with its maximum achieved at $\bar{u} = \frac{1}{2} - \frac{1}{2\beta}$. Thus, if $\beta \leq 1$, then h_1 is strictly decreasing in $(0, \infty)$ so that $(h_1(u) - h_1(u^*))(u - u^*) < 0$ for $u \in (0, 1) \setminus \{u^*\}$. Assume now $\beta > 1$. Note that $(h_1(u) - h_1(u^*))(u - u^*) < 0$ for $u \in (0, 1) \setminus \{u^*\}$ is equivalent to

$$h_1(u) > h_1(u^*), \quad \forall u \in (0, u^*), \qquad h_1(u) < h_1(u^*), \quad \forall u \in (u^*, 1).$$

From the graph of h_1 we see that these inequalities are equivalent to $h_1(u^*) < h_1(0) = 1$. This reduces to, by the definition of h_1, $(\beta - 1)u^* - \beta(u^*)^2 < 0$, that is $\beta u^* > \beta - 1$. Now the formula for u^* gives $\sqrt{(\beta - \alpha - 1)^2 + 4\beta} > \beta + \alpha - 1$, and simplifying this inequality yields $\alpha < \frac{\beta}{\beta - 1}$. Thus we have showed that the first inequality in (A_2') holds if either $\alpha > 0$ and $0 \leq \beta \leq 1$ or $0 < \alpha < \frac{\beta}{\beta - 1}$ and $\beta > 1$.

To verify (A_3'), we note that

$$g(u) = \frac{u}{1 + \beta u}, \quad g'(u) = \frac{1}{(1 + \beta u)^2},$$
$$u^* g(u) - u(u - u^*)g'(u) = \frac{u}{(1 + \beta u)^2}[2u^* + (\beta u^* - 1)u].$$

So for (A_3') to hold, it suffices to have $(2 + \beta)u^* > 1$, which combined with the formula of u^* reduces to

$$\frac{2(2 + \beta)}{\sqrt{(\beta - \alpha - 1)^2 + 4\beta} + (\alpha + 1 - \beta)} > 1,$$

that is, $3(1 + \beta) - \alpha > \sqrt{(\beta - \alpha - 1)^2 + 4\beta}$. After simplifying this inequality we find that it is equivalent to

$$\alpha < 1 + \frac{(3 + 2\beta)\beta}{2 + \beta} = 1 + 2\beta - \frac{\beta}{2 + \beta}.$$

Finally, using the fact that for $\beta \geq 1$,

$$\frac{\beta}{\beta - 1} \leq 1 + \frac{(3 + 2\beta)\beta}{2 + \beta} \quad \Longleftrightarrow \quad \beta \geq \sqrt{2},$$

we conclude that the condition (3.12) implies (A_3') and the first inequality in (A_2'). This completes the proof for case (i).

Case (ii). In this case we have

$$h_1(u) = (1-u)(1+\beta u^2) = 1 - u + \beta u^2 - \beta u^3,$$
$$h_1'(u) = -1 + 2\beta u - 3\beta u^2 = -\beta(3u^2 - 2u + \frac{1}{\beta}), \quad h_1''(u) = 2\beta - 6\beta u = 2\beta(1-3u).$$

Hence $h_1'(u)$ is increasing on $(0, 1/3)$ and decreasing on $(1/3, 1)$. Note that $h_1'(0) = -1$, $h_1'(1) = -\beta - 1$, and $h_1'(\frac{1}{3}) = -1 + \frac{1}{3}\beta$. Hence, if $\beta \le 3$, then $h_1'(u) < 0$ and so h_1 is strictly decreasing on $(0, 1]$, yielding the first inequality in (A_2') (for any $\alpha > 0$).

Next we show this inequality holds as well if $\beta > 3$ and $\alpha \in (0, \bar{\alpha})$. So we assume $\beta > 3$. Then $h_1'(u) = 0$ at $u = \bar{u}$ and $u = \tilde{u}$ with $0 < \bar{u} < \tilde{u} < 1$ given by

$$\bar{u} = \frac{1}{3}\left(1 - \sqrt{1 - \frac{3}{\beta}}\right), \qquad \tilde{u} = \frac{1}{3}\left(1 + \sqrt{1 - \frac{3}{\beta}}\right).$$

Moreover, $h_1(\bar{u})$ is a local minimum and $h_1(\tilde{u})$ is a local maximum, so that h_1 is decreasing in $(0, \bar{u}) \cup (\tilde{u}, 1)$ and is increasing in $(\bar{u}, \tilde{u})$. Let $\hat{u} \in (\tilde{u}, 1)$ be such that $h_1(\hat{u}) = h_1(\bar{u})$. We derive now that $\hat{u}$ has the formula given in (3.13). To see this, using $h_1(\bar{u}) = h_1(\hat{u})$ gives $(\hat{u} - \bar{u}) - \beta(\hat{u}^2 - \bar{u}^2) + \beta(\hat{u}^3 - \bar{u}^3) = 0$, yielding $1 - \beta(\hat{u} + \bar{u}) + \beta(\hat{u}^2 + \hat{u}\bar{u} + \bar{u}^2) = 0$. Since $h'(\bar{u}) = 0$ we have $1 = 2\beta\bar{u} - 3\beta\bar{u}^2$. Inserting this into the above equation gives $2\beta\bar{u} - 3\beta\bar{u}^2 - \beta(\hat{u} + \bar{u}) + \beta(\hat{u}^2 + \hat{u}\bar{u} + \bar{u}^2) = 0$, and simplifying gives $(\bar{u} - \hat{u}) + (\hat{u}^2 + \hat{u}\bar{u} - 2\bar{u}^2) = 0$. Removing the common factor $\bar{u} - \hat{u}$ gives $1 - (\hat{u} + 2\bar{u}) = 0$, which together with the formula of $\bar{u}$ above leads to (3.13).

Now to guarantee that $(h_1(u) - h_1(u^*))(u - u^*) < 0$ for $u \in (0, u^*) \cup (u^*, 1)$, it suffices to require $u^* \in (\hat{u}, 1)$. This is equivalent to $u^* > \hat{u}$ and $h_1(u^*) < h_1(\hat{u})$. Since $h_1(u^*) = \alpha(u^*)^2$, this reduces to $\alpha(u^*)^2 < h_1(\hat{u})$. On the other hand, note that u^* is the horizontal coordinate of the intersection point of the graphs of $h_1(u)$ and αu^2. Using these graphs we see that u^* is decreasing and $\alpha(u^*)^2$ is increasing as α increases in the interval $(0, \bar{\alpha})$ with $u^* \nearrow 1$ as $\alpha \searrow 0$ and $u^* \searrow u^*(\bar{\alpha}) = \hat{u}$ as $\alpha \nearrow \bar{\alpha}$. Thus we have $\bar{\alpha}(\hat{u})^2 = h_1(\hat{u})$, and so

$$\bar{\alpha} = \frac{h_1(\hat{u})}{\hat{u}^2}, \qquad \hat{u} < u^*(\alpha) < 1, \qquad \forall\, \alpha \in (0, \bar{\alpha}).$$

This shows that the first inequality in (A_2') holds if $\beta > 3$ and $\alpha \in (0, \bar{\alpha})$, as we wanted.

It remains to verify (A_3'). We have $g(u) = \frac{u}{1+\beta u^2}$ and $g'(u) = \frac{1-\beta u^2}{(1+\beta u^2)^2}$, and so

$$u^* g(u) - u(u - u^*)g'(u) = u\frac{2u^* + u(\beta u^2 - 1)}{(1+\beta u^2)^2}. \tag{3.14}$$

Note that the function $u(\beta u^2 - 1)$ reaches its minimum on $(0, \infty)$ at $u = \sqrt{1/(3\beta)}$ with the value $-\frac{2}{3}\sqrt{1/(3\beta)}$. Hence,

$$u^*g(u) - u(u-u^*)g'(u) \geq \frac{2u}{(1+\beta u^2)^2}\left(u^* - \frac{1}{3}\sqrt{\frac{1}{3\beta}}\right). \tag{3.15}$$

Note that $0 < \frac{1}{\sqrt{3\beta}} < 1$ when $\beta > \frac{1}{3}$. We consider three cases.

Case 1. Assume that $\beta > 3$ and $0 < \alpha < \bar{\alpha} = h(\hat{u})/(\hat{u})^2$. Then from the above proof we know that $u^* > \hat{u}$ for $\alpha \in (0, \bar{\alpha})$. Using this and (3.15) we get, for $u \in (0, 1]$,

$$\begin{aligned} u^*g(u) - u(u-u^*)g'(u) &\geq \frac{2u}{(1+\beta u^2)^2}\left(\hat{u} - \frac{1}{3}\sqrt{\frac{1}{3\beta}}\right) \\ &\geq \frac{2u}{3(1+\beta u^2)^2}\left(1 + 2\sqrt{1-\frac{3}{\beta}} - \sqrt{\frac{1}{3\beta}}\right) > 0. \end{aligned}$$

So (A_3') holds in this case.

Case 2. Assume that $\frac{1}{3} < \beta \leq 3$ and $0 < \alpha < \bar{\alpha} = 28\beta\left(1 - \frac{1}{3\sqrt{3\beta}}\right)$. Recall when $\beta \leq 3$, $h_1(u)$ is decreasing on $(0, 1]$ so that u^* is decreasing as α increases on $(0, \infty)$ with $u^* \nearrow 1$ as $\alpha \searrow 0$ and $u^* \searrow 0$ as $\alpha \nearrow \infty$. So to have $u^* > \frac{1}{3}\sqrt{\frac{1}{3\beta}}$ for $0 < \alpha < \bar{\alpha}$, it is equivalent to having $u^* = \frac{1}{3}\sqrt{\frac{1}{3\beta}}$ for $\alpha = \bar{\alpha}$, which is equivalent to having

$$\bar{\alpha} = \frac{h_1(u^*)}{(u^*)^2} = 27\beta h_1\Big(\frac{1}{3\sqrt{3\beta}}\Big) = 28\beta\left(1 - \frac{1}{3\sqrt{3\beta}}\right).$$

Hence, it follows from (3.15) that (A_3') holds in the current case.

Case 3. Assume that $0 < \beta \leq \frac{1}{3}$ and $0 < \alpha < \bar{\alpha} = (1+\beta)\left[\frac{2}{(1-\beta)^2} + \frac{\beta}{2}\right]$. In this case we have $\frac{1}{\sqrt{3\beta}} \geq 1$ and $u(\beta u^2 - 1)$ is decreasing in $[0, 1]$, and so $u(\beta u^2 - 1) \geq \beta - 1$ for $u \in [0, 1]$; thus from (3.14) we have

$$u^*g(u) - u(u-u^*)g'(u) \geq \frac{u}{(1+\beta u^2)^2}\left[2u^* - (1-\beta)\right]).$$

To have $2u^* - (1-\beta) > 0$ for $0 < \alpha < \bar{\alpha}$, it is equivalent to having $u^* = \frac{1}{2}(1-\beta)$ for $\alpha = \bar{\alpha}$, yielding

$$\bar{\alpha} = \frac{h_1(u^*)}{(u^*)^2} = \frac{4}{(1-\beta)^2}h_1\Big(\frac{1}{2}(1-\beta)\Big) = (1+\beta)\left[\frac{2}{(1-\beta)^2} + \frac{\beta}{2}\right].$$

So (A_3') holds in this case as well. This completes the proof of Theorem 3.4. $\square$

We conclude this section by giving some comparisons of the results in Theorems 3.2 and 3.4, which are consequences of Theorems 3.1 and 3.3, respectively. Since Theorems 3.1 and 3.3 are obtained by completely different methods, we hope the comparison might reveal the strengths of these different methods.

For the case $m = 1$, Theorem 3.4 clearly covers a bigger range of the parameter α, thus giving a better result than Theorem 3.2. For $m = 2$, when $\beta \in [0, 3)$, Theorem 3.2 gives a better result (no restriction on $\alpha > 0$) than Theorem 3.4; on the other hand, Theorem 3.4 covers ranges for $\beta \geq 27$ which is absent in Theorem 3.2.

In conclusion, the above comparison seems to suggest that both the squeeze method and the Lyapunov function method have their own advantages.

4. Appendix

Here we give the proof of Theorem 2.1 by an upper and lower solution approach. We believe that this method should have many applications elsewhere.

We start with the definition of upper and lower solutions of (1.6), which generalizes those in the literature when F and G are required to satisfy monotone or quasi-monotone conditions (see [3,30] and references therein).

Definition. The continuous functions $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ on $\mathbb{R}$ are called a pair of lower and upper solutions of the system (1.6) if they satisfy:
(i)

$$0 \leq \underline{u}(x) \leq \bar{u}(x) \leq U_0, \qquad 0 \leq \underline{v}(x) \leq \bar{v}(x) \leq V_0, \qquad \forall\ x \in \mathbb{R},$$

for some positive constants U_0 and V_0.
(ii) There exists a set $\mathbb{D}$ consisting of at most finitely many real numbers such that
(a) $\bar{u},\ \underline{u},\ \bar{v},\ \underline{v}$ are in $C^2(\mathbb{R} \setminus \mathbb{D})$,
(b) The right and left limits of $\underline{u}',\ \bar{u}',\ \underline{v}',\ \bar{v}'$ all exist at each $x \in \mathbb{D}$ and satisfy

$$\bar{u}'(x-) \geq \bar{u}'(x+), \quad \underline{u}'(x-) \leq \underline{u}'(x+), \quad \bar{v}'(x-) \geq \bar{v}'(x+), \quad \underline{v}'(x-) \leq \underline{v}'(x+).$$

(iii) At $\pm\infty$, the first and second derivatives of $\bar{u},\ \bar{v},\ \underline{u},\ \underline{v}$ have at most exponential growth.
(iv) For every pair of continuous functions (u, v) with $\underline{u} \leq u \leq \bar{u}$ and $\underline{v} \leq v \leq \bar{v}$,

$$\begin{cases} \bar{u}'' - c\bar{u}'(x) + F(\bar{u}(x), v(x)) \leq 0, \\ d\bar{v}''(x) - c\bar{v}'(x) + G(u(x), \bar{v}(x)) \leq 0, \\ \underline{u}''(x) - c\underline{u}'(x) + F(\underline{u}(x), v(x)) \geq 0, \\ d\underline{v}''(x) - c\underline{v}'(x) + G(u(x), \underline{v}(x)) \geq 0, \end{cases} \qquad \forall\ x \in \mathbb{R} \setminus \mathbb{D}.$$

Lemma 4.1. *Assume that $(\underline{u}, \underline{v})$ and $(\bar{u}, \bar{v})$ are a pair of lower and upper solutions of (1.6), where we only assume that F and G are Lipschitz continuous on $[0, U_0] \times [0, V_0]$. Then there is a solution (u, v) of (1.6) satisfying*

$$\underline{u}(x) \leq u(x) \leq \bar{u}(x), \qquad \underline{v}(x) \leq v(x) \leq \bar{v}(x), \qquad \forall x \in \mathbb{R},$$

and u', u'', v' and v'' are bounded on $\mathbb{R}$.

Proof. Since F and G satisfy the Lipschitz condition on $[0, U_0] \times [0, V_0]$, there is $\Lambda > 0$ such that, for $(u_i, v_i) \in [0, U_0] \times [0, V_0]$, $i = 1, 2$,

$$|F(u_1, v_1) - F(u_2, v_2)| + |G(u_1, v_1) - G(u_2, v_2)| \leq \Lambda\Big(|u_1 - u_2| + |v_1 - v_2|\Big). \tag{4.1}$$

Define the functions $\hat{F}(u,v) := F(u,v) + \Lambda u$ and $\hat{G}(u,v) = G(u,v) + \Lambda v$. It follows from (4.1) that $\hat{F}(u,v)$ is nondecreasing in $u \in [0, U_0]$ for each fixed $v \in [0, V_0]$, $\hat{G}(u,v)$ is nondecreasing in $v \in [0, V_0]$ for each fixed $u \in [0, U_0]$, and (1.6) can be written as

$$u'' - cu' - \Lambda u + \hat{F}(u,v) = 0, \quad dv'' - cv' - \Lambda v + \hat{G}(u,v) = 0.$$

Now let

$$X = \{(u,v) \in [C(\mathbb{R})]^2 : \underline{u}(x) \le u(x) \le \bar{u}(x), \underline{v}(x) \le v(x) \le \bar{v}(x), \forall x \in \mathbb{R}\},$$

and define the map $T = (T_1, T_2) : X \to [C(\mathbb{R})]^2$ by

$$T_1(u,v)(x) = \frac{1}{\lambda_1^+ - \lambda_1^-} \left(\int_{-\infty}^{x} e^{\lambda_1^-(x-y)} + \int_{x}^{\infty} e^{\lambda_1^+(x-y)} \right) \hat{F}(u,v)(y)\,dy,$$

$$T_2(u,v)(x) = \frac{1}{d(\lambda_2^+ - \lambda_2^-)} \left(\int_{-\infty}^{x} e^{\lambda_2^-(x-y)} + \int_{x}^{\infty} e^{\lambda_2^+(x-y)} \right) \hat{G}(u,v)(y)\,dy,$$

where

$$\lambda_1^{\pm} = \frac{1}{2}(c \pm \sqrt{c^2 + 4\Lambda}), \quad \lambda_2^{\pm} = \frac{1}{2d}(c \pm \sqrt{c^2 + 4d\Lambda}).$$

It is easy to check that for each $(u,v) \in X$, $(U,V) = T(u,v)$ is the unique bounded solution of the linear equations

$$U'' - cU' - \Lambda U + \hat{F}(u,v) = 0, \quad dV'' - cV' - \Lambda V + \hat{G}(u,v) = 0,$$

and any fixed point of T in X gives a solution of (1.6). Therefore, it suffices to show by the Schauder fixed point theorem that T has a fixed point in X. To do so, we define the Banach space

$$C_\rho(\mathbb{R}, \mathbb{R}^2) = \{(u,v) \in [C(\mathbb{R})]^2 : \|(u,v)\|_\rho < \infty\},$$

with the exponentially weighted norm

$$\|(u,v)\|_\rho = \sup_{x\in\mathbb{R}} |(u(x), v(x))| e^{-\rho|x|} := \sup_{x\in\mathbb{R}} [|u(x)| + |v(x)|] e^{-\rho|x|}, \quad 0 < \rho < \min\{|\lambda_1^-|, |\lambda_2^-|\},$$

and it follows that X is a bounded, closed, and convex subset of $C_\rho(\mathbb{R}, \mathbb{R}^2)$.

It is easily checked that T maps X into itself. Moreover, by rather standard arguments similarly to those in the references [28,30,36,38,40], we can show that T is completely continuous on X. Thus we can apply the Schauder fixed point theorem to conclude that T has a fixed point (u,v) in X, which gives a solution of (1.6).

Note that for $x \in \mathbb{R}$,

$$u'(x) = \frac{1}{\lambda_1^+ - \lambda_1^-} \left(\lambda_1^- \int_{-\infty}^{x} e^{\lambda_1^-(x-y)} + \lambda_1^+ \int_{x}^{\infty} e^{\lambda_1^+(x-y)} \right) \hat{F}(u,v)(y)\,dy.$$

$$v'(x) = \frac{1}{d(\lambda_2^+ - \lambda_2^-)} \left(\lambda_2^- \int_{-\infty}^{x} e^{\lambda_2^-(x-y)} + \lambda_2^+ \int_{x}^{\infty} e^{\lambda_2^+(x-y)} \right) \hat{G}(u,v)(y)\,dy.$$

It follows that $|u'(x)| \le M_0/(\lambda_1^+ - \lambda_1^-)$ and $|v'(x)| \le M_0/[d(\lambda_2^+ - \lambda_2^-)]$ for $x \in \mathbb{R}$, where $M_0 = \max\{|\hat{F}(u,v)|, |\hat{G}(u,v)| : 0 \le u \le U_0, 0 \le v \le V_0\}$. This shows that u' and v' are bounded on $\mathbb{R}$, and then using the equations in (1.6) yields the boundedness of u'' and v'' as well. This completes the proof of Lemma 4.1. $\square$

In the following two lemmas, we construct upper and lower solutions for $c > c^* = \sqrt{4dr}$ and for $c = c^*$ respectively. Since the lower solution $\underline{v}$ to be constructed for $c > c^*$ goes to zero as $c \searrow \sqrt{4dr}$, it is unclear whether the limit of the weak traveling wave solutions as $c \searrow \sqrt{4dr}$ is the trivial equilibrium $(1, 0)$. Therefore we cannot use the usual approach of taking the limit of these weak traveling wave solutions (or a sequence of these solutions) to obtain a weak traveling wave solution for the case $c = \sqrt{4dr}$. Instead, we construct the upper and lower solutions for $c = \sqrt{4dr}$ separately.

Our construction of the upper and lower solutions is inspired by those appearing in the literature, such as [3,8,40], where various special cases are considered. Here we give a unified approach that cover a very general class of predator–prey systems.

In view of Remark 1 (ii), under the assumptions of Theorem 2.1 part (i), the weaker requirement (2.4) always holds. In the lemmas below, we will only assume this less restrictive condition and the constants δ, σ and K from (2.4) will be used. Furthermore, by the local Lipschitz continuity of F and G, there is $M > 0$ such that, for $(u_i, v_i) \in [0, 1] \times [0, v_0]$, $i = 1, 2$,

$$|F(u_1, v_1) - F(u_2, v_2)| + |G(u_1, v_1) - G(u_2, v_2)| \le M\Big[|u_1 - u_2| + |v_1 - v_2|\Big]. \tag{4.2}$$

Lemma 4.2. *Suppose all the assumptions in part* (i) *of Theorem 2.1 are satisfied. Let $c > c^*$, $\lambda = (c - \sqrt{c^2 - 4dr})/(2d)$, and M be the constant in (4.2). Choose auxiliary constants γ, β, η, A one by one in that order such that*

$$\frac{1}{2}\left(c - \sqrt{c^2 - 4M_0}\right) < \gamma < \min\left\{\lambda, \frac{1}{2}\left(c + \sqrt{c^2 - 4M_0}\right)\right\},$$

$$\beta > \max\left\{\left(\frac{M}{c\gamma - \gamma^2 - M_0}\right)^{\gamma/\lambda}, \left(\frac{1}{v_0}\right)^{\gamma/\lambda}, 1\right\},$$

$$0 < \eta < \sigma\gamma, \quad -d(\lambda+\eta)^2 + c(\lambda+\eta) - r > 0,$$

$$A > \max\left\{\beta^{\gamma/\eta}, \left(\frac{1}{\delta}\right)^{\eta/\lambda}, \left(\frac{\beta}{\delta}\right)^{\eta/\gamma}, \frac{K(1+\beta)}{-d(\lambda+\eta)^2 + c(\lambda+\eta) - r}\right\}.$$

Then define $\bar{u}(x)$, $\underline{u}(x)$, $\bar{v}(x)$ *and* $\underline{v}(x)$ *on* $\mathbb{R}$ *by*

$$\bar{u}(x)=1, \quad \underline{u}(x)=\begin{cases} 1-\beta e^{\gamma x}, & \forall\ x \le a_1, \\ 0, & \forall x > a_1, \end{cases}$$

$$\bar{v}(x)=\begin{cases} e^{\lambda x}, & \forall x \le a_2, \\ v_0, & \forall x > a_2, \end{cases} \quad \underline{v}(x)=\begin{cases} e^{\lambda x}(1-Ae^{\eta x}), & \forall x \le a_0, \\ 0, & \forall x > a_0, \end{cases}$$

where

$$a_0=-\frac{1}{\eta}\ln A, \qquad a_1=-\frac{1}{\gamma}\ln \beta, \qquad a_2=\frac{1}{\lambda}\ln v_0.$$

Then $(\bar{u},\bar{v})$ *and* $(\underline{u},\underline{v})$ *are a pair of upper and lower solutions of* (1.6).

Proof. We first point out that since $M_0<\min\{d,1\}r$, we have $\frac{1}{2}\left(c-\sqrt{c^2-4M_0}\right)<\lambda=\frac{1}{2d}(c-\sqrt{c^2-4dr}\,)$ so that γ is well defined. If $d<1$, then we have $M_0<dr$, so this inequality is clearly true. If $d\ge 1$, then $M_0<r\le dr$, which implies that an equivalent inequality

$$\frac{M_0}{c+\sqrt{c^2-4M_0}}<\frac{r}{c+\sqrt{c^2-4dr}}$$

holds. The choice of γ yields that $c\gamma-\gamma^2-M_0>0$ so that β is well defined.

We also point out that by the definitions of a_1, a_2 and a_0, $\underline{u}$, $\bar{v}$ and $\underline{v}$ are continuous at a_1, a_2 and a_0 respectively, and $a_0<a_1<\min\{0,a_2\}$ by the assumptions on γ, β, η, A and the definitions of $\underline{u}$, $\bar{v}$, $\underline{v}$. It is clear that $\underline{u}(x)<\bar{u}(x)$ and $\underline{v}(x)<\bar{v}(x)$ for all $x\in\mathbb{R}$, and

$$\begin{aligned}
\underline{u}'(a_1-)&=-\gamma<0=\underline{u}'(a_1+),\\
\underline{v}'(a_0-)&=-\eta e^{\lambda a_0}<0=\underline{v}'(a_0+),\\
\bar{v}'(a_2-)&=\lambda v_0>0=\bar{v}'(a_2+).
\end{aligned}$$

Let (u,v) be a pair of continuous functions with $\underline{u}\le u\le\bar{u}$ and $\underline{v}\le v\le\bar{v}$.

Since $\bar{u}\equiv 1$ and $F(1,v(x))\le 0$ by the assumptions on F, it follows that

$$\bar{u}''(x)-c\bar{u}'(x)+F(\bar{u}(x),v(x))=F(1,v(x))\le 0 \text{ for all } x\in\mathbb{R}.$$

For $x<a_1$, we have $\underline{u}(x)=1-\beta e^{\gamma x}$, $\underline{u}''(x)-c\underline{u}'(x)=\beta\gamma(c-\gamma)e^{\gamma x}$, and by (4.2),

$$F(\underline{u}(x),v(x))\ge F(\underline{u}(x),0)-Mv(x)\ge -M_0[1-\underline{u}(x)]-M\bar{v}(x).$$

Thus, noting $a_1<a_2$ and $\bar{v}(x)=e^{\lambda x}$ for $x<a_1$, we obtain

$$\begin{aligned}
\underline{u}''(x)-c\underline{u}'(x)+F(\underline{u}(x),v(x))&\ge\beta\gamma(c-\gamma)e^{\gamma x}-M_0\beta e^{\gamma x}-Me^{\lambda x}\\
&=\beta e^{\gamma x}[\gamma(c-\gamma)-M_0-\frac{1}{\beta}Me^{(\lambda-\gamma)x}]\\
&\ge\beta e^{\gamma x}[\gamma(c-\gamma)-M_0-Me^{(\lambda-\gamma)a_1}]
\end{aligned}$$

$$
\begin{aligned}
&= \beta e^{\gamma x}[\gamma(c-\gamma) - M_0 - M\beta^{-\lambda/\gamma}] \\
&\geq 0, \ \forall x < a_1,
\end{aligned}
$$

where the last inequality is guaranteed by the assumptions on γ and β. For $x > a_1$, since $\underline{u}(x) = 0$ and $F(0, v(x)) \geq 0$, we also have

$$\underline{u}''(x) - c\underline{u}'(x) + F(\underline{u}(x), v(x)) \geq 0.$$

For $x < a_2$, we have $\bar{v}(x) = e^{\lambda x}$, and using $G(u(x), \bar{v}(x)) \leq r\bar{v}(x)$ we get

$$d\bar{v}''(x) - c\bar{v}'(x) + G(u(x), \bar{v}(x)) \leq d\bar{v}''(x) - c\bar{v}'(x) + r\bar{v}(x) = 0.$$

For $x > a_2$, we have $\bar{v}(x) = v_0$ so that $G(u(x), \bar{v}(x)) \leq 0$, and thus

$$d\bar{v}''(x) - c\bar{v}'(x) + G(u(x), \bar{v}(x)) \leq 0.$$

For $x < a_0$, since $a_0 < 0$ and $a_0 < a_1 < a_2$, we deduce

$$\underline{v}(x) = e^{\lambda x} - Ae^{(\lambda+\eta)x}, \ \bar{v}(x) = e^{\lambda x} \text{ and } 1 - \underline{u}(x) = \beta e^{\gamma x}.$$

By the choice of A, for $x < a_0$ we have $\underline{v}(x) \leq \bar{v}(x) \leq e^{\lambda a_0} < \delta$, and $1 - u(x) \leq 1 - \underline{u}(x) \leq \beta e^{\gamma a_0} < \delta$. Hence, for $x < a_0$,

$$
\begin{aligned}
G(u(x), \underline{v}(x)) &\geq r\underline{v}(x) - K[(1-u(x))^\sigma + \underline{v}(x)^\sigma]\underline{v}(x) \\
&\geq r\underline{v}(x) - K[(1-\underline{u}(x))^\sigma + \bar{v}(x)^\sigma]\bar{v}(x) \\
&\geq r\underline{v}(x) - K(\beta^\sigma e^{\sigma\gamma x} + e^{\sigma\lambda x})\bar{v}(x) \\
&\geq r\underline{v}(x) - K(\beta+1)e^{\sigma\gamma x}e^{\lambda x},
\end{aligned}
$$

which together with the formula of $\underline{v}(x)$, the choice of A, $a_0 < 0$, and $0 < \eta < \sigma\gamma$ leads to

$$
\begin{aligned}
&d\underline{v}''(x) - c\underline{v}'(x) + G(u(x), \underline{v}(x)) \\
&\quad \geq d\underline{v}''(x) - c\underline{v}'(x) + r\underline{v}(x) - K(\beta+1)e^{\sigma\gamma x}e^{\lambda x} \\
&\quad = e^{(\lambda+\eta)x}\{A[-d(\lambda+\eta)^2 + c(\lambda+\eta) - r] - K(\beta+1)e^{(\sigma\gamma-\eta)x}\} \\
&\quad \geq 0.
\end{aligned}
$$

For $x > a_0$, we have $\underline{v}(x) = 0$, and so $d\underline{v}''(x) - c\underline{v}'(x) + G(u(x), \underline{v}(x)) = G(u(x), 0) = 0$ by the assumptions on G.

Combining the above we have proved the assertions of Lemma 4.2. □

Lemma 4.3. *Let the assumptions in part* (i) *of Theorem 2.1 hold. Suppose* $c := \sqrt{4dr}$. *Let* $\lambda = c/(2d)$, $M_1 = \lambda e v_0$, *and* M *be the constant in* (4.2). *Choose auxiliary constants* γ *and* β *such that*

$$\frac{1}{2}\left(c-\sqrt{c^2-4M_0}\right)<\gamma<\min\left\{\lambda,\frac{1}{2}\left(c+\sqrt{c^2-4M_0}\right)\right\},$$
$$\beta>\max\left\{e^{\frac{\gamma}{\lambda-\gamma}},\ \frac{MM_1}{(c\gamma-\gamma^2-M_0)(\lambda-\gamma)e}\right\}.$$

There exists $N_0>0$ large so that for all $N\geq N_0$, if we define

$$a_2=-\frac{1}{\lambda},\qquad a_1=\frac{1}{\gamma}\ln\frac{1}{\beta},\qquad a_0=-\frac{N^2}{M_1^2},$$

and

$$\bar{u}\equiv 1,\qquad \underline{u}(x)=\begin{cases}1-\beta e^{\gamma x}, & x\leq a_1,\\ 0, & x>a_1,\end{cases}$$
$$\bar{v}(x)=\begin{cases}M_1|x|e^{\lambda x}, & x\leq a_2,\\ v_0, & x>a_2,\end{cases}\qquad \underline{v}(x)=\begin{cases}\left(M_1|x|-N\sqrt{|x|}\right)e^{\lambda x}, & x\leq a_0,\\ 0, & x>a_0,\end{cases}$$

then $(\bar{u},\bar{v})$ and $(\underline{u},\underline{v})$ are a pair of upper and lower solutions of (1.6).

Proof. We first point out that γ and β are well defined. By the choice of M_1, γ, β, and the definitions of a_0, a_1 and a_2, we have $\underline{u}(a_1)=\underline{v}(a_0)=0$, $\bar{v}(a_2)=v_0$ and $a_0<a_1<a_2<0$ provided that N is sufficiently large.

Let u and v be continuous functions on $\mathbb{R}$ such that $\underline{u}\leq u\leq\bar{u}$ and $\underline{v}\leq v\leq\bar{v}$. Combined with the proof of Lemma 4.2, it is sufficient to do the following three steps to confirm that $\bar{u}$, $\underline{u}$, $\underline{v}$ and $\bar{v}$ are a pair of upper and lower solutions of (1.6).

Step 1. Checking the inequalities for $\underline{u}$. For $x<a_1$ we have

$$\underline{u}'(x)=-\beta\gamma e^{\gamma x},\ \underline{u}''(x)=-\beta\gamma^2e^{\gamma x},\ F(\underline{u}(x),v(x))\geq -M_0[1-\underline{u}(x)]-M\bar{v}(x),$$

and so

$$\begin{aligned}\underline{u}''(x)-c\underline{u}'(x)+F(\underline{u}(x),v(x))&\geq\beta(c\gamma-\gamma^2)e^{\gamma x}-M_0\beta e^{\gamma x}-MM_1|x|e^{\lambda x}\\&=\beta e^{\gamma x}\left[(c\gamma-\gamma^2)-M_0-\frac{1}{\beta}MM_1|x|e^{(\lambda-\gamma)x}\right].\end{aligned}$$

Since $|x|e^{(\lambda-\gamma)x}$ is monotone increasing over $(-\infty,-\frac{1}{\lambda-\gamma})$ and $a_1<-\frac{1}{\lambda-\gamma}$, it follows that $|x|e^{(\lambda-\gamma)x}\leq\frac{1}{(\lambda-\gamma)e}$ for $x<a_1$. By the choice of β, we have, for $x<a_1$,

$$\underline{u}''(x)-c\underline{u}'(x)+F(\underline{u},v)\geq\beta e^{\gamma x}\left[(c\gamma-\gamma^2)-M_0-\frac{MM_1}{\beta(\lambda-\gamma)e}\right]\geq 0.$$

Furthermore, $\underline{u}'(a_1-)=-\beta e^{\gamma a_1}\leq 0=\underline{u}'(a_1+)$.

Step 2. Checking the inequalities for $\underline{v}$. For $x<a_0$, we have

$$(\underline{v}+M_1xe^{\lambda x})' = \left[\frac{N}{2\sqrt{-x}} + \lambda(-N\sqrt{-x})\right]e^{\lambda x} = N\left[\frac{1}{2\sqrt{-x}} - \sqrt{-x}\lambda\right]e^{\lambda x},$$

and

$$(\underline{v}+M_1xe^{\lambda x})'' = N\left[-\frac{1}{4x\sqrt{-x}} + \frac{1}{\sqrt{-x}}\lambda - \sqrt{-x}\lambda^2\right]e^{\lambda x},$$

and so

$$\begin{aligned} d\underline{v}'' - c\underline{v}' + r\underline{v} &= N\left[\frac{-d}{4x\sqrt{-x}} + \frac{d}{\sqrt{-x}}\lambda - d\sqrt{-x}\lambda^2 - \frac{c}{2\sqrt{-x}} + c\sqrt{-x}\lambda\right]e^{\lambda x} - rN\sqrt{-x}e^{\lambda x} \\ &= \frac{-dN}{4x\sqrt{-x}}e^{\lambda x}. \end{aligned}$$

Note that for $x < a_0$, since $a_0 < 0$ and $a_0 < a_1 < a_2$, we have $\underline{v}(x) = (M_1|x| - N\sqrt{|x|})e^{\lambda x}$, $\bar{v}(x) = M_1|x|e^{\lambda x}$ and $1 - \underline{u}(x) = \beta e^{\gamma x}$. For all large N and $x < a_0$ we have $\underline{v}(x) \le \bar{v}(x) < \delta$, and $1 - u(x) \le 1 - \underline{u}(x) \le \beta e^{\gamma a_0} < \delta$. Hence, for such N and x,

$$\begin{aligned} G(u,\underline{v}) &\ge r\underline{v} - K[(1-u)^\sigma + \underline{v}^\sigma]\underline{v} \\ &\ge r\underline{v} - K[(1-\underline{u})^\sigma + \bar{v}^\sigma]\bar{v} \\ &\ge r\underline{v} - K\left[\beta e^{\sigma\gamma x} + M_1^\sigma |x|^\sigma e^{\sigma\lambda x}\right]\bar{v} \\ &\ge r\underline{v} - M_1K[\beta + M_1|x|]|x|e^{\sigma\gamma x}e^{\lambda x} \quad (\text{since } x < a_0 < -1,\ 0 < \sigma \le 1) \end{aligned}$$

and

$$\begin{aligned} d\underline{v}'' - c\underline{v}' + G(u,\underline{v}) &\ge \left(\frac{-dN}{4x\sqrt{-x}} - M_1K\left[\beta + M_1|x|\right]|x|e^{\sigma\gamma x}\right)e^{\lambda x} \\ &= \frac{1}{4|x|\sqrt{|x|}}\left(dN - M_1K\left[\beta + M_1|x|\right]4x^2\sqrt{|x|}e^{\sigma\gamma x}\right)e^{\lambda x} \\ &\ge \frac{1}{4|x|\sqrt{|x|}}\left[dN - 4M_1K(\beta + M_1)x^4e^{\sigma\gamma x}\right]e^{\lambda x} \\ &\ge 0. \end{aligned}$$

Furthermore, using $\underline{v}'(x) = [-M_1 + \frac{N}{2\sqrt{-x}} + \lambda(-M_1x - N\sqrt{-x})]e^{\lambda x}$, we have

$$\underline{v}'(a_0-) = \left[-M_1 + \frac{N}{2\sqrt{|a_0|}}\right]e^{\lambda a_0} = -\frac{M_1}{2}e^{\lambda a_0} < 0 = \underline{v}'(a_0+).$$

Step 3. Checking the inequalities for $\bar{v}$. For $x < a_2 = -\frac{1}{\lambda}$, we have $\bar{v}(x) = -M_1xe^{\lambda x}$, and $\bar{v}'(x) = -M_1[1+\lambda x]e^{\lambda x}$, so that $\bar{v}(a_2) = M_1/(\lambda e) = v_0$ and $\bar{v}'(a_2-) = 0 = \bar{v}'(a_2+)$. Since $d\bar{v}'' - c\bar{v}' + r\bar{v} = 0$ and $G(u,\bar{v}) \le r\bar{v}$, it follows that $d\bar{v}'' - c\bar{v}' + G(u,\bar{v}) \le 0$.

We thus conclude the proof of Lemma 4.3. □

Proof of Theorem 2.1. We first show (i). Applying Lemmas 4.2 and 4.3 and Lemma 4.1 with $U_0 = 1$ and $V_0 = v_0$ yields the existence of a solution (u, v) of (1.6), satisfying $\underline{u} \le u \le \bar{u}$ and $\underline{v} \le v \le \bar{v}$. The definitions of $\underline{u}$, $\bar{u}$, $\underline{v}$, and $\bar{v}$ imply that $(u, v)(x) \to (1, 0)$ as $x \to -\infty$, and that, after a translation in x, $0 < u(x) \le 1$ and $0 < v(x) < v_0$ for $x \le 0$, and that $0 \le u(x) \le 1$ and $0 \le v(x) \le v_0$ for $x > 0$. Using the expressions

$$u'(x) = e^{cx}u'(0) + \int_x^0 e^{c(x-y)}F(u(y), v(y))\,dy,$$

$$v'(x) = e^{cx/d}u'(0) + \frac{1}{d}\int_x^0 e^{c(x-y)/d}G(u(y), v(y))\,dy,$$

and L'Hospital's rule we get $(u'(x), v'(x)) \to 0$ as $x \to -\infty$. Therefore, (u, v) is a weak traveling wave of (1.6).

We next prove the remaining assertions in (i). We first show that $v(x) > 0$ for $x > 0$. If not, then there exists $x_0 > 0$ such that $v(x_0) = 0$ and $v'(x_0) \le 0$. If $v'(x_0) < 0$, then $v(x) < 0$ for small $x - x_0 > 0$, contradicting the non-negativeness of v. So $v'(x_0) = 0$, and by the uniqueness of the IVP: $dV'' - cV' + G(u(x), V) = 0$, $V(x_0) = V'(x_0) = 0$ we conclude that $v \equiv 0$, a contradiction again. Thus we have $v > 0$ on $\mathbb{R}$. Using the same argument and the assumption that $F(0, v) = 0$ for $v \in (0, v_0)$ we can show $u > 0$ on $\mathbb{R}$.

Assuming $G(u, v_0) < 0$ for $u \in [0, 1]$, we show that $v(x) < v_0$ for $x > 0$. If not, then there is $x_1 > 0$ such that $v(x_1) = v_0$ and $v'(x_1) \ge 0$. If $v'(x_1) > 0$, then $v(x) > v_0$ for all $x - x_1 > 0$ small, contradicting that $v(x) \le v_0$ for all $x \in \mathbb{R}$. So $v'(x_1) = 0$, and then the v equation gives $v''(x_1) = -G(u(x_1), v_0)) > 0$, yielding that $v(x_1) = v_0$ is a strict local minimum of v. This again contradicts the fact that $v \le v_0$, showing that $v(x) < v_0$ for all $x \in \mathbb{R}$. Applying the same argument with the assumption that $F(1, v) < 0$ for $v \in (0, v_0]$ leads to the assertion that $u(x) < 1$ for all $x \in \mathbb{R}$. This shows (i).

We now show (ii). Under the assumptions, we can write in a neighborhood of $(1, 0)$ the v equation in (1.6) as

$$dv'' - cv' + rv + [g(u, v) - r]v = 0$$

where $g(1, 0) = G_v(1, 0) = r$ and $g(u, v) - r \to 0$ as $(u, v) \to (1, 0)$. Note that the characteristic equation $d\lambda^2 - c\lambda + r = 0$ has a pair of complex roots $\lambda = \alpha \pm i\beta := c/(2d) \pm i\sqrt{4dr - c^2}/(2d)$. Assume by contradiction there is a solution (u, v) of (1.6) satisfying $(u(x), v(x)) \to (1, 0)$ as $x \to -\infty$ and $v(x) > 0$ for sufficiently negative x. Then using the variation of constants formula one can show that, for sufficiently negative x_0 and x,

$$v(x) = e^{\alpha(x-x_0)}\left\{v(x_0)\cos\beta(x - x_0) + \frac{1}{\beta}[v'(x_0) - \alpha v(x_0)]\sin\beta(x - x_0)\right\}[1 + R(x, x_0)],$$

where $\lim_{x_0\to-\infty}\sup_{x<x_0}|R(x, x_0)| = 0$. (See the proof of Lemma 4.4 in [22] for details.) This asymptotic expression shows that $v(x)$ changes the signs infinitely many times as $x \to -\infty$, a contradiction. This shows (ii), thereby completing the proof of Theorem 2.1. □

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