Images of fractional Brownian motion with deterministic drift:
Positive Lebesgue measure and non-empty interior

Youssef Hakiki

University Cadi Ayyad. Faculty of sciences Semlalia

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Introduction

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• The images and graphs of FBM was one of the most things studied. We recall that an explicite formula for the Hausdorff dimension of the images of the FBM was given by J-P Kahane [3]. Namely

$$\dim B^H(A) = \min\left\{\frac{\dim(A)}{H}, d\right\}.$$

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i) If $\dim(A)/H > d$, then $B^H(A)$ is a.s. a **set of positive Lebesgue measure**,

ii) If $\dim(A)/H < d$, then $B^H(A)$ is a **Salem set**.

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ii) If $\dim(A)/H < d$, then $B^H(A)$ is a **Salem set**.

• Recently Peres and Sousi [1] studied fractal properties of images and graphs of $B^H + f$ where $f : [0,1] \to \mathbb{R}^d$ is a Borel measurable function.

They expressed the dimension of the image set $\left(B^H + f\right)(A)$ in terms of the so-called **parabolic Hausdorff dimension** of the graph of f restricted to A denoted by $\dim_{\Psi,H}(Gr_A(f))$.

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$$\dim\left((B^H + f)(A)\right) = \min\left(\frac{\dim_{\Psi,H}(Gr_A(f))}{H}, d\right) \text{ a.s.}$$

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$$\dim\left((B^H + f)(A)\right) = \min\left(\frac{\dim_{\Psi,H}(Gr_A(f))}{H}, d\right) \text{ a.s.}$$

- It is therefore quite natural to ask the following question:
  - How to extend (i) and (ii) for $B^H + f$, according to the value of $\dim_{\Psi,H}(Gr_A(f))$?

This issue will be the main goal of this work.

Preliminaries on Parabolic Hausdorff dimension

- For $\beta > 0$, $F \subset \mathbb{R}_+ \times \mathbb{R}^d$, and $H \in (0,1)$, the H-parabolic β-dimensional Hausdorff content is defined by

$$\Psi_H^\beta(F) = \inf \left\{ \sum_j \delta_j^\beta : F \subseteq \bigcup_j [a_j, a_j + \delta_j] \times \left[b_{j,1}, b_{j,1} + \delta_j^H\right] \times \ldots \times \left[b_{j,d}, b_{j,d} + \delta_j^H\right] \right\} \tag{1}$$

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  where the infimum is taken over all covers of F by rectangles of the form given above.
- The H-parabolic Hausdorff dimension is then defined to be

$$\dim_{\Psi,H}(F) = \inf\left\{\beta > 0 : \Psi_H^\beta(F) = 0\right\}.$$

Remark

Let ρ_H be the metric defined on $\mathbb{R}_+ \times \mathbb{R}^d$ by

$$\rho_H((s,x),(t,y)) = \max\{|s-t|^H, \|x-y\|_\infty\} \quad \forall (s,x),(t,y) \in \mathbb{R}_+ \times \mathbb{R}^d. \tag{2}$$

We define the β-dimensional Hausdorff content as

$$\mathcal{H}^\beta_{\rho_H}(F) = \inf\left\{\sum_j diam(U_j)^\beta : F \subseteq \bigcup_j U_j\right\} \tag{3}$$

where $\{U_j\}$ is a countable cover of F by any sets and $diam(U_j)$ denotes the diameter of a set U_j relatively to the metric ρ_H.

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Then, since any set U_j of the forme $U_j = [a_j, a_j + \delta_j] \times \left[b_{j,1}, b_{j,1} + \delta_j^H\right] \ldots \times \left[b_{j,d}, b_{j,d} + \delta_j^H\right]$ has a diameter "$diam\left(U_j\right) = \delta_j^H$", it can be shown that for any $\beta > 0$

$$\mathcal{H}^{\beta/H}_{\rho_H}(F) > 0 \quad \text{iff} \quad \Psi^\beta_H(F) > 0. \tag{4}$$

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Hence we obtain

$$\dim_{\Psi,H}(F) = H \times \dim_{\rho_H}(F). \tag{5}$$

The following proposition relates β-dimensional capacity to the H-parabolic Hausdorff dimension.

Proposition

Let $F \subset \mathbb{R}_+ \times \mathbb{R}^d$ be a compact set. Then we have

$$\dim_{\Psi,H}(F) = \sup\{\beta : \mathcal{C}_{\rho_H,\beta/H}(F) > 0\} = \inf\{\beta : \mathcal{C}_{\rho_H,\beta/H}(F) = 0\}, \tag{6}$$

where $\mathcal{C}_{\rho_H,\beta}(.)$ is the β-capacity on the metric space $(\mathbb{R}_+ \times \mathbb{R}^d, \rho_H)$ defined by

$$\mathcal{C}_{\rho_H,\beta}(F) = \left[\inf_{\mu\in\mathcal{P}(F)} \int_{\mathbb{R}_+\times\mathbb{R}^d} \int_{\mathbb{R}_+\times\mathbb{R}^d} \frac{\mu(du)\mu(dv)}{(\rho_H(u,v))^\beta}\right]^{-1}. \tag{7}$$

Here $\mathcal{P}(F)$ is the family of probability measure carried by F.

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The next theorem is the analogue of **Frostman's theorem** for parabolic Hausdorff dimension.

Theorem (Peres and Sousi 2013 [1])

Let F a Borel set in $\mathbb{R}_+ \times \mathbb{R}^d$. If $\dim_{\Psi,H}(F) > \kappa$, then there exists a Borel probability measure μ supported on F, and a constant $C > 0$, such that, for any $(a, b_1, \cdots, b_d) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\delta > 0$

$$\mu\left([a, a+\delta] \times \prod_{j=1}^{d}\left[b_j, b_j + \delta^H\right]\right) \le C\delta^\kappa, \tag{8}$$

Now we give a comparison result for the Hausdorff parabolic dimensions with different parameters.

Proposition

Let $F \subset \mathbb{R}_+ \times \mathbb{R}^d$ and $K, H \in (0,1)$ such that $K < H$. Then we have

$$\begin{aligned} \dim_{\Psi,K}(F) \vee \left(\frac{H}{K}\dim_{\Psi,K}(F) + 1 - \frac{H}{K}\right) \leq \dim_{\Psi,H}(F) \leq \\ \left(\frac{H}{K}\dim_{\Psi,K}(F)\right) \wedge \left(\dim_{\Psi,H}(F) + (H-K)d\right). \end{aligned} \tag{9}$$

Proof: Let us start by the first term in the lower inequality, let K, H such that $K < H$, an immediate consequence of the definition is

$$\dim_{\Psi,K}(F) \leq \dim_{\Psi,H}(F).$$

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$$\dim_{\Psi,K}(F) \leq \dim_{\Psi,H}(F).$$

For the second term in the lower inequality it suffice to show that

$$\frac{\dim_{\Psi,K}(F) - 1}{K} \leq \frac{\dim_{\Psi,H}(F) - 1}{H}$$

Proof

Now let $0 < \varepsilon < 1$ and $\gamma > \dfrac{\dim_{\Psi,H}(F) - 1}{H}$. Then $\Psi_H^{\gamma H+1}(F) = 0$, and hence there exists a cover $\left([a_n, a_n + \delta_n] \times \prod_{j=1}^{d}[b_{n,j}, b_{n,j} + \delta_n^H]\right)_{n\geq 1}$ of the set F, such that

$$\sum_{n\geq 1} \delta_n^{\gamma H+1} \leq \varepsilon. \tag{10}$$

Each interval $[a_n, a_n + \delta_n]$ can be divided into $\left\lceil \delta_n^{1-\frac{H}{K}} \right\rceil$ intervals of length $\delta_n^{H/K}$. In this way we obtain a new cover $\left(\left[a'_l, a'_l + \delta_l^{H/K}\right] \times \prod_{j=1}^{d}\left[b'_{l,j}, b'_{l,j} + \left(\delta_l^{H/K}\right)^K\right]\right)_{l\geq 1}$ of the set F which satisfies

$$\Psi_K^{\gamma K+1}(F) \leq \sum_{l\geq 1} \left(\delta_l^{H/K}\right)^{\gamma K+1} \leq 2\sum_{n\geq 1} \delta_n^{1-\frac{H}{K}} \left(\delta_n^{H/K}\right)^{\gamma K+1} \leq 2\varepsilon. \tag{11}$$

Then we get

$$\dim_{\Psi,K}(F) \leq \gamma K + 1,$$

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For the upper inequality, let $\kappa < \dim_{\Psi,H}(F)$. Then by **Frostman's theorem** there exists a probability measure μ supported on F such that

$$\mu\left([a, a+\delta] \times \prod_{j=1}^{d} \left[b_j, b_j + \delta^H\right]\right) \leq C\delta^{\kappa}. \tag{12}$$

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By some covering arguments, we can deduce that

$$\mu\left([a, a+\delta] \times \prod_{j=1}^{d} \left[b_j, b_j + \delta^K\right]\right) \leq C\left(\delta^{\kappa + d(K-H)} \wedge \delta^{\kappa K/H}\right), \tag{13}$$

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Then the **Mass Distribution Principle** in the metric space $(\mathbb{R}_+ \times \mathbb{R}^d, \rho_K)$ implies that

$$\dim_{\rho_K}(F) \geq \frac{\kappa}{H} \vee \frac{\kappa + d(K-H)}{K}.$$

Using the fact that $\dim_{\Psi,K}(F) = K \times \dim_{\rho_K}(F)$, it follows that

$$\kappa \leq \left(\frac{H}{K}\dim_{\Psi,K}(F)\right) \wedge \left(\dim_{\Psi,K}(F) + d(H-K)\right).$$

Therefore letting $\kappa \uparrow \dim_{\psi,H}(F)$ the desired inequality follows.

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Therefore letting $\kappa \uparrow \dim_{\psi,H}(F)$ the desired inequality follows.

Remark

Let $\alpha \in (0,1)$, and let $f : [0,1] \to \mathbb{R}^d$ be a Borel measurable function, and A be a Borel subset of $[0,1]$. Then by projection we can deduce that

$$\dim(A) \leq dim_{\Psi,\alpha}(Gr_A(f)) \tag{14}$$

if $f : [0,1] \to \mathbb{R}^d$ is an α-Hôlder continuous function, then it can be shown from a covering argument like in previous proposition that

$$dim_{\Psi,\alpha}(Gr_A(f)) = \dim(A) \tag{15}$$

The next proposition looks at the effect of the Hölder continuity of f on the H-parabolic Hausdorff dimension of its graph $\dim_{\Psi,H}(Gr_A(f))$

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Proposition

Let $f : [0,1] \to \mathbb{R}^d$ be an α-Hölder continuous function $\alpha \leq H$, then we have

$$\dim(A) \leq \dim_{\Psi,H}(Gr_A(f)) \leq \left(\frac{H}{\alpha}\dim(A)\right) \wedge (\dim(A) + (H-\alpha)d)\,. \tag{16}$$

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Especially, when f is $(H-\varepsilon)$-Hölder continuous for all $\varepsilon > 0$ then

$$\dim(A) = \dim_{\Psi,H}(Gr_A(f)) \tag{17}$$

A natural question that arises from the previous Proposition is whether the upper bound is optimal ?

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Theorem

Let $\alpha \leq H$, $\{B^{\alpha}(t) : t \in [0,1]\}$ a d-dimensional fractional Brownian motion of Hurst index α and $A \subset [0,1]$ a Borel set. Then we have

$$\dim_{\Psi,H}(Gr_A(B^{\alpha})) = \left(\left(\frac{H}{\alpha} \dim(A) \right) \wedge (\dim(A) + d(H-\alpha)) \right) \; a.s. \tag{18}$$

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Proof of the theorem

- The upper bound of $\dim_{\Psi,H}(Gr_A(B^\alpha))$ follows directly from the previous proposition.

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- The upper bound of $\dim_{\Psi,H}(Gr_A(B^\alpha))$ follows directly from the previous proposition.
- For the lower bound part, there is two different cases *i*) $\dim(A) \le \alpha d$ and *ii*) $\dim(A) > \alpha d$:

i) If $\dim(A) \le \alpha d$, then

$$\left(\frac{H}{\alpha}\dim(A)\right) \wedge (\dim(A) + d(H-\alpha)) = \frac{H}{\alpha}\dim(A).$$

Let $\gamma < \frac{H}{\alpha}\dim(A)$. Then by the **Frostman's Theorem** there exists a probability measure ν on A such that

$$\mathcal{E}_{\gamma\alpha/H}(\nu) := \int_A\int_A \frac{1}{|t-s|^{\gamma\alpha/H}}\nu(ds)\nu(dt) < \infty. \tag{19}$$

Let $\widetilde{\mu}$ be the random measure defined as the image measure of ν by the map $s \mapsto (s, B^\alpha(s))$

$$\widetilde{\mu}(E) = \nu\{s : (s, B^\alpha(s)) \in E\},$$

where $E \subset Gr_A(B^\alpha)$. We will show that

$$\mathcal{E}_{\rho_H,\gamma/H}(\widetilde{\mu}) := \int_{\mathbb{R}_+\times\mathbb{R}^d}\int_{\mathbb{R}_+\times\mathbb{R}^d} \frac{\widetilde{\mu}(du)\widetilde{\mu}(dv)}{(\rho_H(u,v))^{\gamma/H}} < \infty \text{ a.s.}$$

$$\mathbb{E}\left[\mathcal{E}_{\rho_H,\gamma/H}(\widetilde{\mu})\right] = \int_A \int_A \mathbb{E}\left[\frac{1}{\left(\max\left(|s-t|^H, \|B^\alpha(t) - B^\alpha(s)\|_\infty\right)\right)^{\gamma/H}}\right] \nu(ds)\nu(dt). \tag{20}$$

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B^α has a stationary increments, then we have

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Lemma

There exists a constants C such that, for all $s,t\in(0,1]$ with $s\neq t$ we have

$$\mathbb{E}\left[\frac{1}{\left(\max\{|t-s|^H, \|B^\alpha(t-s)\|\}\right)^{\gamma/H}}\right] \leq \begin{cases} C\,|t-s|^{-\gamma\alpha/H} & \text{if} \quad \gamma < Hd, \\ C\,|t-s|^{d(H-\alpha)-\gamma} & \text{if} \quad \gamma > Hd. \end{cases} \tag{21}$$

Since $\gamma < Hd$ we deduce from the previous lemma that

$$\mathbb{E}\left[\mathcal{E}_{\rho_H,\gamma/H}(\widetilde{\mu})\right] \leq C\int_A\int_A \frac{1}{|t-s|^{\gamma\alpha/H}}\nu(ds)\nu(dt) < \infty$$

Hence $\mathcal{C}_{\rho_H,\gamma/H}(Gr_A(B^\alpha)) > 0$ a. s. and then $\dim_{\Psi,H}(Gr_A(B^\alpha)) \geq \gamma$ *a.s.*

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ii) If $\dim(A) > \alpha d$, then

$$\left(\frac{H}{\alpha}\dim(A)\right) \wedge (\dim(A) + d(H-\alpha)) = \dim(A) + d(H-\alpha).$$

Hence $\mathcal{C}_{\rho_H,\gamma/H}(Gr_A(B^\alpha)) > 0$ a. s. and then $\dim_{\Psi,H}(Gr_A(B^\alpha)) \geq \gamma$ *a.s.*

ii) If $\dim(A) > \alpha d$, then

$$\left(\frac{H}{\alpha}\dim(A)\right) \wedge (\dim(A) + d(H-\alpha)) = \dim(A) + d(H-\alpha).$$

Let $\gamma < \dim(A) + d(H-\alpha)$. Then, we proceed as in (*i*) to prove that $\mathcal{C}_{\rho_H,\gamma/H}(Gr_A(B^\alpha)) > 0$ *a.s* via the probability measure ν satisfying

$$\mathcal{E}_{\gamma-d(H-\alpha)}(\nu) < \infty.$$

Letting $\gamma \uparrow (\frac{H}{\alpha}\dim(A)) \wedge (\dim(A) + d(H-\alpha))$ finishes the proof.

Hence $\mathcal{C}_{\rho_H,\gamma/H}(Gr_A(B^\alpha)) > 0$ a. s. and then $\dim_{\Psi,H}(Gr_A(B^\alpha)) \geq \gamma$ *a.s.*

ii) If $\dim(A) > \alpha d$, then

$$\left(\frac{H}{\alpha}\dim(A)\right) \wedge (\dim(A) + d(H-\alpha)) = \dim(A) + d(H-\alpha).$$

Let $\gamma < \dim(A) + d(H-\alpha)$. Then, we proceed as in (i) to prove that $\mathcal{C}_{\rho_H,\gamma/H}(Gr_A(B^\alpha)) > 0$ *a.s* via the probability measure ν satisfying

$$\mathcal{E}_{\gamma-d(H-\alpha)}(\nu) < \infty.$$

Letting $\gamma \uparrow (\frac{H}{\alpha}\dim(A)) \wedge (\dim(A) + d(H-\alpha))$ finishes the proof.

As a consequence, we have the following result

Corollary

Let $\alpha \leq H$, $\{B^\alpha(t) : t \in [0,1]\}$ a fractional Brownian motion of Hurst index α and $A \subset [0,1]$ a Borel set. Then we have

$$\dim_{\Psi,H}(Gr_A(B^\alpha)) > Hd \text{ a.s } \iff \dim(A) > \alpha d. \qquad (22)$$

Fractional Brownian motion with deterministic drift

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Let $(B_t)_{t\in[0,1]}$ be the BM plane, and λ_2 be the Lebesgue measure on $\mathbb{R}^2$, then:

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Let now $(B_t)_{t\geq 0}$ the d-dimensional BM such that $d \geq 2$, then:

Theorem 2 (Taylor 1952)

$\mathbb{P}$.a.s. pour tout $A \subset [0,\infty)$ we have $\mathcal{H}^2(B(A)) = 0$.

Cameron martin theorem

Let $f : [0,1] \mapsto \mathbb{R}^2$ be a continuous function and $(B_t)_{t\geq 0}$ be the BM on $\mathbb{R}^2$. we denote by $D[0,1]$ the Dirichlet space defined by

$$D[0,1] = \left\{ f \in C[0,1] : \exists g \in \mathbf{L}^2[0,1] \text{ t.q. } f(t) = \int_0^t g(s)ds, \forall t \in [0,1] \right\}$$

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Let us denote by $\mathbb{L}_B$ and $\mathbb{L}_{B+f}$ the laws of B and $B+f$ respectively.

Theorem (Cameron-Martin 1944)

If $f \in D[0,1]$, then the distributon $\mathbb{L}_B$ and $\mathbb{L}_{B+f}$ are equivalent.

Hence, if $f \in D[0,1]$, then the results of **Levy** and **Taylor** still hold for $B+f$ also.

Theorem (Gravensen 1982[4])

For all $0 < \alpha < 1/2$, there is an α-Hölder continuous function $f : \mathbb{R}_+ \to \mathbb{R}^2$ such that

$$\mathbb{P}\{\lambda_2(B + f)[0, 1]] > 0\} > 0.$$

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Question:

• Can we extend the previous result of Kahane to $B^H + f$ under the weak condition $\dim_{\Psi,H}(Gr_A(f)) > H\,d$?

Theorem

Let $\{B^H(t) : t \in [0,1]\}$ *be a d-dimensional FBM of Hurst index* $H \in (0,1)$*. Let* $f : [0,1] \to \mathbb{R}^d$ *be a Borel measurable function and let* $A \subset [0,1]$ *be a Borel set. If* $\dim_{\Psi,H}(Gr_A(f)) > Hd$*, then*

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The idea is to find an appropriate **random probability measure** μ_ω supported on $(B^H_\omega + f)(A)$ and

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Differentiation’s method

Let $Y = (Y(t))_{t\in[0,1]}$ be an $\mathbb{R}^d$-valued stochastic process and ν is a positive measure on $[0,1]$. The occupation measure of the sample path $[0,1] \ni t \longrightarrow Y(t)(w) \in \mathbb{R}^d$ is defined by

$$\mu_Y(E) := \nu\left\{t \in [0,1] : Y(t) \in E\right\},$$

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Lemma

The following assertions are equivalent:

1. $$\mu_Y << \lambda_d \quad \text{a.s.} \quad \text{with } \frac{d\nu_Y}{d\lambda_d}(.) \in L^2(\lambda_d \otimes \mathbb{P}).$$

2. $$\liminf_{r\downarrow 0} r^{-d} \int_A \int_A \mathbb{P}\{\|Y(s) - Y(t)\| < r\}\, d\nu(s)\, d\nu(t) < \infty.$$

Proof of our Theorem

- Let $A \subset [0,1]$ and assume that $\eta := \dim_{\Psi,H}(Gr_A(f)) > Hd$. It follows from th parabolic Frostman's theorem that for $\kappa \in (Hd, \eta)$ there exists a Borel probability measure σ supported on $Gr_A(f)$ and $C > 0$ such that

$$\sigma\left([a, a+\delta] \times \prod_{j=1}^{d}\left[b_j, b_j + \delta^H\right]\right) \leq C\delta^\kappa, \tag{23}$$

for all $a \in A$, $b_1, ..., b_n \in \mathbb{R}^d$ and all $\delta \in (0,1]$.

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• Let ν be the measure defined on A by $\nu = \sigma \circ P_1^{-1}$, where P_1 is the projection mapping on A, i.e. $P_1(s, f(s)) = s$.

• Then to achieve our purpose it is enough to verify the second assertion of the last Lemma for the process $Y = B^H + f$.

Indeed for $s, t \in A$ and $r > 0$, we have

$$\mathbb{P}\left\{\|(B^H + f)(s) - (B^H + f)(t)\| < r\right\} = \frac{1}{(2\pi)^{d/2}|t-s|^{Hd}} \int_{\|y\|<r} \exp\left(-\frac{\|y - f(t) + f(s)\|^2}{2|t-s|^{2H}}\right) dy. \tag{24}$$

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Then using Fubini's theorem we obtain, for any fixed $t \in A$ and $r > 0$, that

$$\int_A \mathbb{P}\left\{\|(B^H + f)(s) - (B^H + f)(t)\| < r\right\} d\nu(s) = \frac{1}{(2\pi)^{d/2}} \int_{\|y\|<r} \int_A \frac{1}{|t-s|^{Hd}} \exp\left(-\frac{\|y - f(t) + f(s)\|^2}{2|t-s|^{2H}}\right) d\nu(s)\, dy$$

$$\leq \mathbf{C}\, r^d \sup_{\|y\|<r} \underbrace{\int_A \frac{1}{|t-s|^{Hd}} \exp\left(-\frac{\|y - f(t) + f(s)\|^2}{2|t-s|^{2H}}\right) d\nu(s)}_{=I(y,(t,f(t)))},$$

where $\mathbf{C}$ is a positive constant depending only on d.

For any fixed $y \in \{\|y\| < r\}$ and $(t, f(t)) \in Gr_A(f)$, we have the following decomposition

$$I\left(y, (t, f(t))\right) = I_1\left(y, (t, f(t))\right) + I_2\left(y, (t, f(t))\right),$$

where

$$I_1\left(y, (t, f(t))\right) =$$

$$\int_{\left\{s \in A : \|y - f(t) + f(s)\| \leq \mathbf{C}_1 \, |s-t|^H \sqrt{|\log|s-t||}\right\}} \frac{1}{|s-t|^{Hd}} \exp\left(-\frac{\|y-f(t)+f(s)\|^2}{2|s-t|^{2H}}\right) d\nu(s) =$$

$$\int_{\left\{(s, f(s)) \in Gr_A(f) : \|y - f(t) + f(s)\| \leq \mathbf{C}_1 \, |s-t|^H \sqrt{|\log|s-t||}\right\}} \frac{1}{|s-t|^{Hd}} \exp\left(-\frac{\|y-f(t)+f(s)\|^2}{2|s-t|^{2H}}\right) d\mu(s, f(s)).$$

and

$$I_2\left(y,(t,f(t))\right) =$$

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where $\mathbf{C}_1$ is a positive constant which will be chosen later.

We first show that

$$\sup_{(y,(t,f(t)))\in\{\|y\|<r\}\times Gr_A(f)} I_1\,(y,(t,f(t))) < +\infty.$$

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Then we split the integral $I_1(y,(t,f(t)))$ into the regions $\{s \in A : 2^{-n} < |t-s| \le 2^{1-n}\}$, we obtain that

$$\begin{aligned} I_1(y,(t,f(t))) &\le \sum_{n=1}^{\infty} 2^{nHd}\,\sigma\left\{(s,f(s)) : \begin{array}{l} \|f(s)-f(t)+y\| \le C_1\,|s-t|^H\sqrt{|\log|s-t||} \\ 2^{-n} < |s-t| \le 2^{1-n} \end{array}\right\} \\ &\le \sum_{n=1}^{\infty} 2^{nHd}\,\sigma\left\{(s,f(s)) : \begin{array}{l} \|f(s)-f(t)+y\| \le C_2\,2^{-nH}\sqrt{n} \\ 2^{-n} < |s-t| \le 2^{1-n} \end{array}\right\}, \end{aligned}$$

where $C_2 = C_1\,2^H\sqrt{\log(2)}$.

Now, for $n \geq 1$, we set

$$S_n\left(y,(t,f(t))\right) = \left\{ (s,f(s)) \in Gr_A(f) : \begin{array}{l} \|y - f(t) + f(s)\| \leq C_2\, 2^{-nH}\sqrt{n} \\ 2^{-n} < |s-t| \leq 2^{1-n} \end{array} \right\}.$$

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Applying the **Frostman's condition** (23) and some covering arguments, we can deduce that

$$\sup_{(y,(t,f(t)))\in\{\|y\|<r\}\times Gr_A(f)} \sigma(S_n(y,(t,f(t))) \leq C_3\, n^{d/2}\, 2^{-\kappa n}.$$

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Therefore we get

$$\sup_{(y,(t,f(t))) \in \{\|y\|<r\} \times Gr_A(f)} I_1(y,(t,f(t))) \leq C_4 \sum_{n=1}^{\infty} 2^{-(\kappa - Hd)n}\, n^{d/2} < \infty, \qquad (25)$$

where C_4 depends on d and H only.

For the second term $I_2\left(y, (t, f(t))\right)$ we have

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$$\le \sum_{n=1}^{\infty} 2^{nHd} \exp\left(-\frac{\mathbf{C}_1^2}{2}(n-1)\ln 2\right) \times \sigma\left\{(s,f(s)) : \begin{array}{l} \|f(s)-f(t)+y\| > \mathbf{C}_1|s-t|^H\sqrt{|\log|s-t||} \\ 2^{-n}<|s-t|\le 2^{1-n} \end{array}\right\}$$

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Thus, for $\mathbf{C}_1 > \sqrt{2Hd}$ we obtain

$$\sup_{(y,(t,f(t)))\in\{\|y\|<r\}\times Gr_A(f)} I_2\left(y,(t,f(t))\right) \leq e^{\mathbf{C}_1^2\ln 2/2}\sum_{n=1}^{\infty} 2^{-n\left(\mathbf{C}_1^2/2-Hd\right)} < +\infty. \qquad (26)$$

Now putting all this together yields

$$\sup_{(y,(t,f(t)))\in\{\|y\|<r\}\times Gr_A(f)} \int_A \frac{1}{|s-t|^{Hd}} \exp\left(-\frac{\|y-f(t)+f(s)\|^2}{2|s-t|^{2H}}\right) d\nu(s) < \infty.$$

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Thus we get from (24) that

$$\liminf_{r\downarrow 0} r^{-d} \int_A \int_A \mathbb{P}\left\{\|(B^H+f)(s)-(B^H+f)(t)\| < r\right\} d\nu(s)\, d\nu(t) < \infty.$$

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We can therefore state from **Lemma 22** that the occupation measure ν_{B^H+f} is **absolutely continuous** with respect to the Lebesgue measure λ_d **a.s.** Hence $\lambda_d\left(B^H+f\right)(A) > 0$ **a.s.** which finishes the proof.

Example

Let $\alpha \in (0,1)$ such that $\alpha < H$. Let B^α be a fractional Brownian motion independent to B^H. Then for any Borel set $A \subset [0,1]$ such that $\dim(A) > \alpha\, d$, it is known from (22) that **all the trajectories of** B^α are satisfying the condition

$$\dim_{\Psi,H}\left(Gr_A\left(B^\alpha\right)\right) > H\, d.$$

This gives some examples of d

Remark

Another criterion due to Berman (**often easier to apply**) tells us that:

- ν_Y has a density $\frac{d\nu_Y}{d\lambda_d}(.) \in L^2(\lambda_d \otimes \mathbb{P})$ **if and only if**

$$\int_{\mathbb{R}^d} \int_A \int_A \mathbb{E}\left(e^{i\langle \theta, (Y(s)-Y(t)) \rangle}\right) d\nu(s)\, d\nu(t)\, d\theta < \infty. \tag{27}$$

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However, we were unable to apply it. Indeed, a simple calculation using characteristic function of Gaussian random vector yields

$$\mathbb{E}\left(e^{i\langle \theta, (B^H+f)(s)-(B^H+f)(t) \rangle}\right) = e^{i\langle \theta, f(s)-f(t) \rangle} \exp\left(-\frac{|s-t|^{2H} \|\theta\|^2}{2}\right).$$

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Now integrating the modulus we have

$$\int_{\mathbb{R}^d} \left|\mathbb{E}\left(e^{i\langle \theta, (B^H+f)(s)-(B^H+f)(t)\rangle}\right)\right| d\theta = \frac{(2\pi)^{d/2}}{|s-t|^{Hd}}.$$

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However, the difficulty stems from the lack of information contained in the measure ν. We are unable to ensure finiteness of the integral,

$$\int_A \int_A \frac{(2\pi)^{d/2}}{|s-t|^{Hd}}\, d\nu(s)\, d\nu(t).$$

Thank you for your attention

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