Images of fractional Brownian motion with deterministic drift:
Positive Lebesgue measure and non-empty interior

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Joint work with M. Erraoui
1. Introduction

2. Preliminaries on Parabolic Hausdorff dimension

3. Main results
The images and graphs of FBM was one of the most things studied. We recall that an explicite formula for the Hausdorff dimension of the images of the FBM was given by J-P Kahane. Namely

$$\dim B_H (A) = \min \{ \dim (A), d \}.$$  

where $\dim (A)$ denote the Hausdorff dimension.

Several works of J-P Kahane give precise information on the rang set $B_H (A)$:

i) If $\dim (A) / H > d$, then $B_H (A)$ is a.s. a set of positive Lebesgue measure,

ii) If $\dim (A) / H < d$, then $B_H (A)$ is a Salem set.

Recently Peres and Sousi studied fractal properties of images and graphs of $B_H + f$ where $f : [0,1] \to \mathbb{R}$ is a Borel measurable function.
The images and graphs of FBM was one of the most things studied. We recall that an explicite formula for the Hausdorff dimension of the images of the FBM was given by J-P Kahane [3]. Namely

$$\dim B^H(A) = \min \left\{ \frac{\dim(A)}{H}, d \right\}.$$

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Introduction

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  i) If \(\dim(A)/H > d\), then \(B^H(A)\) is a.s. a **set of positive Lebesgue measure**, 
  ii) If \(\dim(A)/H < d\), then \(B^H(A)\) is a **Salem set**.
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Recently Peres and Sousi [1] studied fractal properties of images and graphs of \(B^H + f\) where \(f : [0, 1] \to \mathbb{R}^d\) is a Borel measurable function.
They expressed the dimension of the image set \((B^H + f)(A)\) in terms of the so-called **parabolic Hausdorff dimension** of the graph of \(f\) restricted to \(A\) denoted by \(\dim_{\psi,H}(Gr_A(f))\).
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- Precisely, they stated that

\[
\dim \left( (B^H + f)(A) \right) = \min \left( \frac{\dim_{\psi,H}(Gr_A(f))}{H}, d \right) \text{ a.s.}
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- Precisely, they stated that

\[
\dim \left( (B^H + f)(A) \right) = \min \left( \frac{\dim_{\psi,H}(Gr_A(f))}{H}, d \right) \quad \text{a.s.}
\]

- It is therefore quite natural to ask the following question:

  - How to extend \((i)\) and \((ii)\) for \(B^H + f\), according to the value of \(\dim_{\psi,H}(Gr_A(f))\)?

  **This issue will be the main goal of this work.**
Preliminaries on Parabolic Hausdorff dimension
For $\beta > 0$, $F \subset \mathbb{R}_+ \times \mathbb{R}^d$, and $H \in (0, 1)$, the $H$-parabolic $\beta$-dimensional Hausdorff content is defined by

$$\Psi_H^\beta(F) = \inf \left\{ \sum_j \delta_j^\beta : F \subseteq \bigcup_j [a_j, a_j + \delta_j] \times [b_{j,1}, b_{j,1} + \delta_j^H] \times \ldots \times [b_{j,d}, b_{j,d} + \delta_j^H] \right\}$$

where the infimum is taken over all covers of $F$ by rectangles of the form given above.
• For $\beta > 0$, $F \subseteq \mathbb{R}_+ \times \mathbb{R}^d$, and $H \in (0, 1)$, the $H$-parabolic $\beta$-dimensional Hausdorff content is defined by

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(1)

where the infimum is taken over all covers of $F$ by rectangles of the form given above.

• The $H$-parabolic Hausdorff dimension is then defined to be

$$\dim_{\Psi, H}(F) = \inf \left\{ \beta > 0 : \Psi^\beta_H(F) = 0 \right\}.$$
Remark

Let $\rho_H$ be the metric defined on $\mathbb{R}_+ \times \mathbb{R}^d$ by

$$\rho_H((s, x), (t, y)) = \max\{|s - t|^H, \|x - y\|_\infty\} \quad \forall (s, x), (t, y) \in \mathbb{R}_+ \times \mathbb{R}^d.$$  \hfill (2)

We define the $\beta$-dimensional Hausdorff content as

$$\mathcal{H}_\rho^\beta(F) = \inf \left\{ \sum_j \text{diam}(U_j)^\beta : F \subseteq \bigcup U_j \right\}$$  \hfill (3)

where $\{U_j\}$ is a countable cover of $F$ by any sets and $\text{diam}(U_j)$ denotes the diameter of a set $U_j$ relatively to the metric $\rho_H$. 

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For any $F \subseteq \mathbb{R}_+ \times \mathbb{R}^d$, the Hausdorff dimension, in the metric $\rho_H$, of $F$ is defined by

$$\dim_{\rho_H}(F) = \inf \left\{ \beta : \mathcal{H}_\rho^\beta(F) = 0 \right\}.$$
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Then, since any set \( U_j \) of the forme \( U_j = [a_j, a_j + \delta_j] \times [b_{j,1}, b_{j,1} + \delta^H_j] \ldots \times [b_{j,d}, b_{j,d} + \delta^H_j] \) has a diameter "\( \text{diam} (U_j) = \delta^H_j \)" , it can be shown that for any \( \beta > 0 \)

\[
\mathcal{H}_{\rho_H}^{\beta/H}(F) > 0 \quad \text{iff} \quad \Psi_{\mathcal{H}}^\beta(F) > 0.
\] (4)
Remark

Let $\rho_H$ be the metric defined on $\mathbb{R}_+ \times \mathbb{R}^d$ by

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We define the $\beta$-dimensional Hausdorff content as

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$$\mathcal{H}^{\beta/H}_{\rho_H}(F) > 0 \quad \text{iff} \quad \Psi^{\beta}_H(F) > 0. \quad (4)$$

Hence we obtain

$$\dim_{\Psi, H}(F) = H \times \dim_{\rho_H}(F). \quad (5)$$
The following proposition relates $\beta$-dimensional capacity to the $H$-parabolic Hausdorff dimension.

**Proposition**

Let $F \subset \mathbb{R}_+ \times \mathbb{R}^d$ be a compact set. Then we have

$$\dim_{\Psi,H}(F) = \sup \{ \beta : C_{\rho_H,\beta}(F) > 0 \} = \inf \{ \beta : C_{\rho_H,\beta}(F) = 0 \},$$  \hspace{1cm} (6)

where $C_{\rho_H,\beta}(.)$ is the $\beta$-capacity on the metric space $(\mathbb{R}_+ \times \mathbb{R}^d, \rho_H)$ defined by

$$C_{\rho_H,\beta}(F) = \left[ \inf_{\mu \in \mathcal{P}(F)} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \int_{\mathbb{R}_+ \times \mathbb{R}^d} \mu(du) \mu(dv) (\rho_H(u,v))^\beta \right]^{-1}. \hspace{1cm} (7)$$

Here $\mathcal{P}(F)$ is the family of probability measure carried by $F$. 

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Here $\mathcal{P}(F)$ is the family of probability measure carried by $F$.

The next theorem is the analogue of **Frostman’s theorem** for parabolic Hausdorff dimension.

**Theorem (Peres and Sousi 2013 [1])**

Let $F$ a Borel set in $\mathbb{R}_+ \times \mathbb{R}^d$. If $\dim_{\psi,H}(F) > \kappa$, then there exists a Borel probability measure $\mu$ supported on $F$, and a constant $C > 0$, such that, for any $(a, b_1, \cdots, b_d) \in \mathbb{R}_+ \times \mathbb{R}^d$ and $\delta > 0$

$$\mu \left( [a, a + \delta] \times \prod_{j=1}^{d} \left[ b_j, b_j + \delta^H \right] \right) \leq C\delta^{\kappa},$$

where $C$ and $\kappa$ are constants.
Now we give a comparison result for the Hausdorff parabolic dimensions with different parameters.

**Proposition**

Let $F \subset \mathbb{R}_+ \times \mathbb{R}^d$ and $K, H \in (0, 1)$ such that $K < H$. Then we have

$$
\dim_{\psi,K}(F) \lor \left( \frac{H}{K} \dim_{\psi,K}(F) + 1 - \frac{H}{K} \right) \leq \dim_{\psi,H}(F) \leq \left( \frac{H}{K} \dim_{\psi,K}(F) \right) \land \left( \dim_{\psi,H}(F) + (H - K)d \right).
$$

**Proof:** Let us start by the first term in the lower inequality, let $K, H$ such that $K < H$, an immediate consequence of the definition is

$$
\dim_{\psi,K}(F) \leq \dim_{\psi,H}(F).
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\dim_{\psi,K}(F) \vee \left( \frac{H}{K} \dim_{\psi,K}(F) + 1 - \frac{H}{K} \right) \leq \dim_{\psi,H}(F) \leq \left( \frac{H}{K} \dim_{\psi,K}(F) \right) \wedge \left( \dim_{\psi,H}(F) + (H - K)d \right).
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\[
\dim_{\psi,K}(F) \leq \dim_{\psi,H}(F).
\]

For the second term in the lower inequality it suffice to show that

\[
\frac{\dim_{\psi,K}(F) - 1}{K} \leq \frac{\dim_{\psi,H}(F) - 1}{H}.
\]
Proof

Now let $0 < \varepsilon < 1$ and $\gamma > \frac{\dim_{\psi, H}(F) - 1}{H}$. Then $\psi_{H}^{\gamma+1}(F) = 0$, and hence there exists a cover $\left( [a_{n}, a_{n} + \delta_{n}] \times \prod_{j=1}^{d}[b_{n,j}, b_{n,j} + \delta_{n}^{H}] \right)_{n \geq 1}$ of the set $F$, such that

$$\sum_{n \geq 1} \delta_{n}^{\gamma+1} \leq \varepsilon. \quad (10)$$

Each interval $[a_{n}, a_{n} + \delta_{n}]$ can be divided into $\left\lceil \frac{\delta_{n}^{H/K}}{K} \right\rceil$ intervals of length $\delta_{n}^{H/K}$. In this way we obtain a new cover $\left( [a', a' + \delta_{i}^{H/K}] \times \prod_{j=1}^{d}[b'_{i,j}, b'_{i,j} + \left( \delta_{i}^{H/K} \right)^{K}] \right)_{l \geq 1}$ of the set $F$ which satisfies

$$\psi_{K}^{\gamma+1}(F) \leq \sum_{l \geq 1} \left( \delta_{i}^{H/K} \right)^{\gamma+1} \leq 2 \sum_{n \geq 1} \delta_{n}^{1 - \frac{H}{K}} \left( \delta_{n}^{H/K} \right)^{\gamma+1} \leq 2\varepsilon. \quad (11)$$
Then we get

\[ \dim_{\psi,K}(F) \leq \gamma K + 1, \]

Letting \( \gamma \downarrow \frac{\dim_{\psi,H}(F) - 1}{H} \), the desired inequality follows.
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For the upper inequality, let $\kappa < \dim_{\psi,H}(F)$. Then by Frostman's theorem there exists a probability measure $\mu$ supported on $F$ such that

$$\mu \left( \prod_{j=1}^{d} [b_j, b_j + \delta^H] \right) \leq C\delta^\kappa. \quad (12)$$
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For the upper inequality, let \( \kappa < \dim_{\psi,H}(F) \). Then by Frostman's theorem there exists a probability measure \( \mu \) supported on \( F \) such that

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\]

By some covering arguments, we can deduce that

\[
\mu \left( \left[ a, a + \delta \right] \times \prod_{j=1}^{d} \left[ b_j, b_j + \delta^K \right] \right) \leq C \left( \delta^\kappa + d(K-H) \wedge \delta^{\kappa K/H} \right), \quad (13)
\]
Then we get
\[ \dim_{\Psi,K}(F) \leq \gamma K + 1, \]

Letting \( \gamma \downarrow \frac{\dim_{\Psi,H}(F) - 1}{H} \), the desired inequality follows.

For the upper inequality, let \( \kappa < \dim_{\Psi,H}(F) \). Then by Frostman's theorem there exists a probability measure \( \mu \) supported on \( F \) such that

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By some covering arguments, we can deduce that

\[ \mu \left( [a, a + \delta] \times \prod_{j=1}^{d} [b_j, b_j + \delta^K] \right) \leq C \left( \delta^{\kappa + d(K - H)} \wedge \delta^{\kappa K / H} \right), \quad (13) \]

Then the Mass Distribution Principle in the metric space \( (\mathbb{R}_+ \times \mathbb{R}^d, \rho_K) \) implies that

\[ \dim_{\rho_K}(F) \geq \frac{\kappa}{H} \vee \frac{\kappa + d(K - H)}{K}. \]
Using the fact that $\dim_{\Psi,K}(F) = K \times \dim_{\rho_K}(F)$, it follows that

$$\kappa \leq \left( \frac{H}{K} \dim_{\Psi,K}(F) \right) \land (\dim_{\Psi,K}(F) + d(H - K)).$$

Therefore letting $\kappa \uparrow \dim_{\Psi,H}(F)$ the desired inequality follows.
Using the fact that \( \dim_{\Psi,K}(F) = K \times \dim_{\rho_K}(F) \), it follows that

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\]

Therefore letting \( \kappa \uparrow \dim_{\Psi,H}(F) \) the desired inequality follows.

**Remark**

Let \( \alpha \in (0, 1) \), and let \( f : [0, 1] \to \mathbb{R}^d \) be a Borel measurable function, and \( A \) be a Borel subset of \([0, 1]\). Then by projection we can deduce that

\[
\dim(A) \leq \dim_{\Psi,\alpha}(\text{Gr}_A(f)) \quad (14)
\]

if \( f : [0, 1] \to \mathbb{R}^d \) is an \( \alpha \)-Hölder continuous function, then it can be shown from a covering argument like in previous proposition that

\[
\dim_{\Psi,\alpha}(\text{Gr}_A(f)) = \dim(A) \quad (15)
\]
The next proposition looks at the effect of the Hölder continuity of $f$ on the $H$-parabolic Hausdorff dimension of its graph $\dim_{\psi,H}(\text{Gr}_A(f))$. 

Proposition: Let $f: \mathbb{R}^d \to \mathbb{R}$ be an $\alpha$-Hölder continuous function, where $\alpha \leq H$, then we have

$$
\dim(A) \leq \dim_{\psi,H}(\text{Gr}_A(f)) \leq (H - \alpha) \dim(A) + (H - \alpha) \dim(A) + (H - \alpha) d.
$$

Especially, when $f$ is $(H - \varepsilon)$-Hölder continuous for all $\varepsilon > 0$, then

$$
\dim(A) = \dim_{\psi,H}(\text{Gr}_A(f)).
$$
The next proposition looks at the effect of the Hölder continuity of $f$ on the $H$-parabolic Hausdorff dimension of its graph $\dim_{\psi,H}(\text{Gr}_A(f))$.

**Proposition**

Let $f : [0, 1] \rightarrow \mathbb{R}^d$ be an $\alpha$-Hölder continuous function $\alpha \leq H$, then we have

$$
\dim(A) \leq \dim_{\psi,H}(\text{Gr}_A(f)) \leq \left( \frac{H}{\alpha} \dim(A) \right) \land (\dim(A) + (H - \alpha)d). \tag{16}
$$
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**Proposition**

Let $f : [0, 1] \to \mathbb{R}^d$ be an $\alpha$-Hölder continuous function $\alpha \leq H$, then we have

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Especially, when $f$ is $(H - \varepsilon)$-Hölder continuous for all $\varepsilon > 0$ then

$$\dim(A) = \dim_{\Psi,H}(\text{Gr}_A(f))$$ \hspace{1cm} (17)
A natural question that arises from the previous Proposition is whether the upper bound is optimal?
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The answer is given by using the trajectories of a fractional Brownian motion $B^\alpha$ with Hurst index $\alpha$ with $\alpha \leq H$
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\[ \text{Theorem} \]

Let \( \alpha \leq H \), \( \{B^\alpha(t) : t \in [0, 1]\} \) a \( d \)-dimensional fractional Brownian motion of Hurst index \( \alpha \) and \( A \subset [0, 1] \) a Borel set. Then we have

\[
\dim_{\psi,H}(\text{Gr}_A(B^\alpha)) = \left( \left( \frac{H}{\alpha} \dim(A) \right) \wedge (\dim(A) + d(H - \alpha)) \right) \text{ a.s.} \tag{18}
\]
A natural question that arises from the previous Proposition is whether the upper bound is optimal?

The answer is given by using the trajectories of a fractional Brownian motion $B^\alpha$ with Hurst index $\alpha$ with $\alpha \leq H$

**Theorem**

Let $\alpha \leq H$, $\{B^\alpha(t) : t \in [0, 1]\}$ a $d$-dimensional fractional Brownian motion of Hurst index $\alpha$ and $A \subset [0, 1]$ a Borel set. Then we have

$$\dim_{\psi, H}(\text{Gr}_A(B^\alpha)) = \left(\left(\frac{H}{\alpha} \dim(A)\right) \wedge (\dim(A) + d(H - \alpha))\right) \text{ a.s.} \quad (18)$$
• The upper bound of $\dim_{\psi,H}(Gr_A(B^\alpha))$ follows directly from the previous proposition.
Proof of the theorem

- The upper bound of $\dim_{\psi,H}(Gr_{A}(B^{\alpha}))$ follows directly from the previous proposition.
- For the lower bound part, there is two different cases:
  1. $\dim(A) \leq \alpha d$
  2. $\dim(A) > \alpha d$:
    - If $\dim(A) \leq \alpha d$, then
      \[
      \left(\frac{H}{\alpha} \dim(A)\right) \wedge \left(\dim(A) + d(H - \alpha)\right) = \frac{H}{\alpha} \dim(A).
      \]
      Let $\gamma < \frac{H}{\alpha} \dim(A)$. Then by the Frostman’s Theorem there exists a probability measure $\nu$ on $A$ such that
      \[
      \mathcal{E}_{\gamma/H}(\nu) := \int_{A} \int_{A} \frac{1}{|t - s|^{\gamma/H}} \nu(ds) \nu(dt) < \infty. \tag{19}
      \]
      Let $\tilde{\mu}$ be the random measure defined as the image measure of $\nu$ by the map $s \mapsto (s, B^{\alpha}(s))$
      \[
      \tilde{\mu}(E) = \nu\{s : (s, B^{\alpha}(s)) \in E\},
      \]
      where $E \subset Gr_{A}(B^{\alpha})$. We will show that
      \[
      \mathcal{E}_{\rho_{H},\gamma/H}(\tilde{\mu}) := \int_{\mathbb{R}^{+} \times \mathbb{R}^{d}} \int_{\mathbb{R}^{+} \times \mathbb{R}^{d}} \frac{\tilde{\mu}(du)\tilde{\mu}(dv)}{(\rho_{H}(u, v))^{\gamma/H}} < \infty \text{ a.s.}
      \]
\[
\mathbb{E} \left[ \mathcal{E}_{\rho H, \gamma/H}(\tilde{\mu}) \right] = \int_A \int_A \mathbb{E} \left[ \frac{1}{(\max(|s-t|^H, \|B^{\alpha}(t) - B^{\alpha}(s)\|_\infty))^{\gamma/H}} \right] \nu(ds)\nu(dt).
\] (20)

Since \( \gamma < H \) we deduce from the previous lemma that

\[
\mathbb{E} \left[ \mathcal{E}_{\rho H, \gamma/H}(\tilde{\mu}) \right] \leq C \int_A \int_A \frac{1}{(\max(|s-t|^H, \|B^{\alpha}(t) - B^{\alpha}(s)\|_\infty))^{\gamma/H}} \nu(ds)\nu(dt) < \infty.
\]
\[ E \left[ \mathcal{E}_{\rho, \gamma/H}(\bar{\mu}) \right] = \int_A \int_A E \left[ \frac{1}{(\max(|s-t|^H, \|B^\alpha(t) - B^\alpha(s)\|_\infty))^{\gamma/H}} \right] \nu(ds)\nu(dt). \] (20)

\( B^\alpha \) has a stationary increments, then we have

\[ E \left[ \mathcal{E}_{\rho, \gamma/H}(\bar{\mu}) \right] = \int_A \int_A E \left[ \frac{1}{(\max(|s-t|^H, \|B^\alpha(|s-t|)\|_\infty))^{\gamma/H}} \right] \nu(ds)\nu(dt). \]
\[
\mathbb{E} \left[ \mathcal{E}_{\rho, \gamma/H}(\tilde{\mu}) \right] = \int_A \int_A \mathbb{E} \left[ \frac{1}{\left( \max\left( |s-t|^H, \|B^\alpha(t) - B^\alpha(s)\|_\infty \right) \right)^{\gamma/H}} \right] \nu(ds) \nu(dt).
\]

(20)

\(B^\alpha\) has a stationary increments, then we have

\[
\mathbb{E} \left[ \mathcal{E}_{\rho, \gamma/H}(\tilde{\mu}) \right] = \int_A \int_A \mathbb{E} \left[ \frac{1}{\left( \max\left( |s-t|^H, \|B^\alpha(s-t)\|_\infty \right) \right)^{\gamma/H}} \right] \nu(ds) \nu(dt).
\]

So we need the following lemma

We need the following lemma for the proof of the last theorem

**Lemma**

*There exists a constants \(C\) such that, for all \(s, t \in (0, 1]\) with \(s \neq t\) we have*

\[
\mathbb{E} \left[ \frac{1}{\left( \max\left\{ |t-s|^H, \|B^\alpha(t-s)\| \right\} \right)^{\gamma/H}} \right] \leq \begin{cases} 
C |t-s|^{-\gamma\alpha/H} & \text{if } \gamma < H\alpha, \\
C |t-s|^{d(H-\alpha)-\gamma} & \text{if } \gamma > H\alpha.
\end{cases}
\]

(21)

Since \(\gamma < H\alpha\) we deduce from the previous lemma that

\[
\mathbb{E} \left[ \mathcal{E}_{\rho, \gamma/H}(\tilde{\mu}) \right] \leq C \int_A \int_A \frac{1}{|t-s|^{\gamma\alpha/H}} \nu(ds) \nu(dt) < \infty
\]
Hence $C_{\rho H,\gamma/H}(Gr_A(B^\alpha)) > 0$ a. s. and then $\dim_{\psi,H}(Gr_A(B^\alpha)) \geq \gamma$ a.s.
Hence $C_{\rho H, \gamma/H}(Gr_A(B^\alpha)) > 0 \text{ a.e.}$ and then $\dim_{\Psi,H}(Gr_A(B^\alpha)) \geq \gamma \text{ a.s.}$

ii) If $\dim(A) > \alpha d$, then

$$\left(\frac{H}{\alpha} \dim(A)\right) \land (\dim(A) + d(H - \alpha)) = \dim(A) + d(H - \alpha).$$
Hence $C_{\rho_H, \gamma/H}(Gr_A(B^\alpha)) > 0$ a. s. and then $\dim_{\Psi,H}(Gr_A(B^\alpha)) \geq \gamma$ a.s.

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Let $\gamma < \dim(A) + d(H - \alpha)$. Then, we proceed as in (i) to prove that $C_{\rho_H, \gamma/H}(Gr_A(B^\alpha)) > 0$ a.s via the probability measure $\nu$ satisfying

$$\mathcal{E}_{\gamma-d(H-\alpha)}(\nu) < \infty.$$

Letting $\gamma \uparrow (\frac{H}{\alpha} \dim(A)) \land (\dim(A) + d(H - \alpha))$ finishes the proof.
Hence $C_{\rho, H, \gamma/H} (Gr_A (B^\alpha)) > 0$ a. s. and then $\dim_{\Psi, H} (Gr_A (B^\alpha)) \geq \gamma$ a.s.

ii) If $\dim(A) > \alpha d$, then

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Letting $\gamma \uparrow \left( \frac{H}{\alpha} \dim(A) \right) \wedge (\dim(A) + d(H - \alpha))$ finishes the proof.

As a consequence, we have the following result

**Corollary**

Let $\alpha \leq H$, $\{B^\alpha (t) : t \in [0, 1]\}$ a fractional Brownian motion of Hurst index $\alpha$ and $A \subset [0, 1]$ a Borel set. Then we have

$$\dim_{\Psi, H} (Gr_A (B^\alpha)) > Hd \text{ a.s } \iff \dim(A) > \alpha d. \quad (22)$$
First, we recall some results for the Brownian motion.
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Let \((B_t)_{t \in [0,1]}\) be the BM plane, and \(\lambda_2\) be the Lebesgue measure on \(\mathbb{R}^2\), then:

**Theorem 1 (Lévy 1940)**

\[ \mathbb{P}.a.s \text{ we have } \lambda_2(B([0,1])) = 0. \]
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Let now \((B_t)_{t \geq 0}\) the \(d\)-dimensional BM such that \(d \geq 2\), then:

**Theorem 2 (Taylor 1952)**

\[\mathbb{P}.a.s. \text{ pour tout } A \subset [0, \infty) \text{ we have } \mathcal{H}^2(B(A)) = 0.\]
Let $f : [0, 1] \mapsto \mathbb{R}^2$ be a continuous function and $(B_t)_{t \geq 0}$ be the BM on $\mathbb{R}^2$. We denote by $D[0, 1]$ the Dirichlet space defined by

$$D[0, 1] = \left\{ f \in C[0, 1] : \exists g \in L^2[0, 1] \text{ t.q. } f(t) = \int_0^t g(s)\,ds, \forall t \in [0, 1] \right\}$$
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Let us denote by $\mathbb{L}_B$ and $\mathbb{L}_{B+f}$ the laws of $B$ and $B + f$ respectively.

**Theorem (Cameron-Martin 1944)**

If $f \in D[0, 1]$, then the distribution $\mathbb{L}_B$ and $\mathbb{L}_{B+f}$ are equivalent.

Hence, if $f \in D[0, 1]$, then the results of *Levy* and *Taylor* still hold for $B + f$ also.
Theorem (Gravensen 1982[4])

For all $0 < \alpha < 1/2$, there is an $\alpha$-Hölder continuous function $f : \mathbb{R}_+ \to \mathbb{R}^2$ such that

$$\mathbb{P}\{\lambda_2(B + f)[0, 1]) > 0\} > 0.$$
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Let \(A\) be a Borel subset of \([0,1]\), if \(\dim(A) > Hd\) then
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Question:

- Can we extend the previous result of Kahane to $B^H + f$ under the weak condition $\dim_{\psi,H}(Gr_A(f)) > H d$?
Theorem

Let \( \{B^H_t : t \in [0, 1] \} \) be a d-dimensional FBM of Hurst index \( H \in (0, 1) \). Let \( f : [0, 1] \to \mathbb{R}^d \) be a Borel measurable function and let \( A \subset [0, 1] \) be a Borel set. If \( \dim_{\Psi,H}(\text{Gr}_A(f)) > H d \), then

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The idea is to find an appropriate random probability measure \( \mu_\omega \) supported on \( (B^H_\omega + f)(A) \) and

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\mu_\omega \ll \lambda_d,
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Let $Y = (Y(t))_{t \in [0,1]}$ be an $\mathbb{R}^d$-valued stochastic process and $\nu$ is a positive measure on $[0,1]$. The occupation measure of the sample path $[0,1] \ni t \mapsto Y(t)(w) \in \mathbb{R}^d$ is defined by

$$\mu_Y(E) := \nu \{ t \in [0,1] : Y(t) \in E \},$$

where $E \subset \mathbb{R}^d$ is a Borel set.
Differentiation’s method

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Differentiation’s method

Let \( Y = (Y(t))_{t \in [0,1]} \) be an \( \mathbb{R}^d \)-valued stochastic process and \( \nu \) is a positive measure on \([0, 1]\). The occupation measure of the sample path \([0, 1] \ni t \mapsto Y(t)(w) \in \mathbb{R}^d \) is defined by

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A simple modification of the differentiation’s method allows to give the following Lemma:

**Lemma**

The following assertions are equivalent:

1. \( \mu_Y \ll \lambda_d \) a.s. with \( \frac{d\nu_Y}{d\lambda_d}(.) \in L^2(\lambda_d \otimes \mathbb{P}). \)

2. \( \liminf_{r \downarrow 0} r^{-d} \int_A \int_A \mathbb{P} \{ \| Y(s) - Y(t) \| < r \} \ d\nu(s) d\nu(t) < \infty. \)
Let $A \subset [0, 1]$ and assume that $\eta := \dim_{\Psi, H}(\text{Gr}_A(f)) > Hd$. It follows from the parabolic Frostman’s theorem that for $\kappa \in (Hd, \eta)$ there exists a Borel probability measure $\sigma$ supported on $\text{Gr}_A(f)$ and $C > 0$ such that

$$\sigma \left( [a, a + \delta] \times \prod_{j=1}^{d} [b_j, b_j + \delta^H] \right) \leq C \delta^\kappa,$$

for all $a \in A, b_1, \ldots, b_n \in \mathbb{R}^d$ and all $\delta \in (0, 1)$. 
Proof of our Theorem

• Let $A \subset [0, 1]$ and assume that $\eta := \dim_{\psi, H}(\text{Gr}_A(f)) > Hd$. It follows from the parabolic Frostman’s theorem that for $\kappa \in (Hd, \eta)$ there exists a Borel probability measure $\sigma$ supported on $\text{Gr}_A(f)$ and $C > 0$ such that

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• Let $\nu$ be the measure defined on $A$ by $\nu = \sigma \circ P_1^{-1}$, where $P_1$ is the projection mapping on $A$, i.e. $P_1(s, f(s)) = s$. 
Proof of our Theorem

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• Let $\nu$ be the measure defined on $A$ by $\nu = \sigma \circ P_1^{-1}$, where $P_1$ is the projection mapping on $A$, i.e. $P_1(s, f(s)) = s$.

• Then to achieve our purpose it is enough to verify the second assertion of the last Lemma for the process $Y = B^H + f$. 
Indeed for \( s, t \in A \) and \( r > 0 \), we have

\[
\mathbb{P} \left\{ \| (B^H + f)(s) - (B^H + f)(t) \| < r \right\} = \\
\frac{1}{(2\pi)^{d/2}|t - s|^{Hd}} \int_{\|y\| < r} \exp \left( - \frac{\|y - f(t) + f(s)\|^2}{2|t - s|^{2H}} \right) dy.
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Indeed for $s, t \in A$ and $r > 0$, we have

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Then using Fubini’s theorem we obtain, for any fixed $t \in A$ and $r > 0$, that

$$\int_A \mathbb{P}\left\{ \| (B^H + f)(s) - (B^H + f)(t) \| < r \right\} \, d\nu(s) = \frac{1}{(2\pi)^{d/2}} \int_{\|y\| < r} \int_A \frac{1}{|t - s|^{Hd}} \exp\left(-\frac{\|y - f(t) + f(s)\|^2}{2|t - s|^{2H}}\right) \, d\nu(s) \, dy$$

$$\leq C r^d \sup_{\|y\| < r} \int_A \frac{1}{|t - s|^{Hd}} \exp\left(-\frac{\|y - f(t) + f(s)\|^2}{2|t - s|^{2H}}\right) \, d\nu(s),$$

where $C$ is a positive constant depending only on $d$. 

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For any fixed \( y \in \{ \|y\| < r \} \) and \((t, f(t)) \in Gr_A(f)\), we have the following decomposition

\[
l(y, (t, f(t))) = I_1(y, (t, f(t))) + I_2(y, (t, f(t))),
\]

where

\[
l_1(y, (t, f(t))) = \int_{\{s \in A: \|y - f(t) + f(s)\| \leq c_1 |s - t|^H \sqrt{\log |s - t|} \}} \frac{1}{|s - t|^{Hd}} \exp \left( - \frac{\|y - f(t) + f(s)\|^2}{2 |s - t|^{2H}} \right) d\nu(s) =
\]

\[
\int_{\{(s, f(s)) \in Gr_A(f): \|y - f(t) + f(s)\| \leq c_1 |s - t|^H \sqrt{\log |s - t|} \}} \frac{1}{|s - t|^{Hd}} \exp \left( - \frac{\|y - f(t) + f(s)\|^2}{2 |s - t|^{2H}} \right) d\mu(s, f(s)).
\]
and

\[ I_2 (y, (t, f(t))) = \]

\[ \int \left\{ s \in A : \|y-f(t)+f(s)\| > C_1 \right\} \frac{1}{|s-t|^H} \exp \left( -\frac{\|y-f(t)+f(s)\|^2}{2|s-t|^{2H}} \right) \, \nu(s) = \]

\[ \int \left\{ (s, f(s)) \in \text{Gr}_A(f) : \|y-f(t)+f(s)\| > C_1 \right\} \frac{1}{|s-t|^H} \exp \left( -\frac{\|y-f(t)+f(s)\|^2}{2|s-t|^{2H}} \right) \, \mu(s, f(s)), \]

where \( C_1 \) is a positive constant which will be chosen later.
We first show that

\[ \sup_{(y, (t, f(t))) \in \{ \|y\| < r \} \times \text{Gr}_A(f)} l_1 (y, (t, f(t))) < +\infty. \]
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We first show that

\[ \sup_{(y, (t, f(t))) \in \{ \|y\| < r \} \times \text{Gr}_A(f)} l_1(y, (t, f(t))) < +\infty. \]

By the Frostman’s inequality (23) we can verify that the measure \( \nu \) is no atomic.

Then we split the integral \( l_1(y, (t, f(t))) \) into the regions \( \{ s \in A : 2^{-n} < |t - s| \leq 2^{1-n} \} \), we obtain that

\[
l_1(y, (t, f(t))) \leq \sum_{n=1}^{\infty} 2^{nHd} \sigma \left\{ (s, f(s)) : \begin{array}{l}
\|f(s) - f(t) + y\| \leq C_1 |s - t|^H \sqrt{\log |s - t|} \\
2^{-n} < |s - t| \leq 2^{1-n}
\end{array} \right\}
\]

\[
\leq \sum_{n=1}^{\infty} 2^{nHd} \sigma \left\{ (s, f(s)) : \begin{array}{l}
\|f(s) - f(t) + y\| \leq C_2 2^{-nH} \sqrt{n} \\
2^{-n} < |s - t| \leq 2^{1-n}
\end{array} \right\},
\]

where \( C_2 = C_1 2^H \sqrt{\log(2)} \).
Now, for $n \geq 1$, we set

$$S_n(y, (t, f(t))) = \begin{cases} (s, f(s)) \in \text{Gr}_A(f) : & \|y - f(t) + f(s)\| \leq C_2 2^{-nH} \sqrt{n} \\ 2^{-n} < |s - t| \leq 2^{1-n} \end{cases}$$
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Applying the **Frostman's condition** (23) and some covering arguments, we can deduce that

$$\sup_{(y,(t,f(t))) \in \{\|y\| < r\} \times Gr_A(f)} \sigma(S_n(y, (t, f(t)))) \leq C_3 n^{d/2} 2^{-\kappa n}.$$
Now, for \( n \geq 1 \), we set

\[
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\]

Therefore we get

\[
\sup_{(y, (t, f(t))) \in \{\|y\| < r\} \times Gr_A(f)} l_1 (y, (t, f(t))) \leq C_4 \sum_{n=1}^{\infty} 2^{-(\kappa-Hd)n} n^{d/2} < \infty,
\]

where \( C_4 \) depends on \( d \) and \( H \) only.
For the second term $l_2 (y, (t, f(t)))$ we have
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$$\leq \sum_{n=1}^{\infty} 2^{nHd} \exp \left( -\frac{C_1^2}{2} (n-1) \ln 2 \right) \times$$

$$\sigma \left\{ (s, f(s)) : \ |f(s) - f(t) + y| > C_1 |s-t|^H \sqrt{|\log |s-t||} \right\}$$

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$$\exp \left( -\frac{\| f(s) - f(t) + y \|^2}{2|t-s|^{2H}} \right) d\sigma \left( s, f(s) \right)$$

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Thus, for $C_1 > \sqrt{2Hd}$ we obtain

$$\sup_{(y, (t, f(t))) \in \{ \| y \| < r \} \times Gr_A(f)} l_2 (y, (t, f(t))) \leq e^{C_1^2 \ln 2/2} \sum_{n=1}^{\infty} 2^{-n(C_1^2/2 - Hd)} < +\infty. \quad (26)$$
Now putting all this together yields

\[
\sup_{(y,(t,f(t))) \in \{\|y\| < r\} \times Gr_A(f)} \int_A \frac{1}{|s-t|^{Hd}} \exp \left( - \frac{\|y - f(t) + f(s)\|^2}{2|s-t|^{2H}} \right) d\nu(s) < \infty.
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\]

Thus we get from (24) that

\[
\lim \inf_{r \downarrow 0} r^{-d} \int_A \int_A \mathbb{P} \left\{ \|(B^H + f)(s) - (B^H + f)(t)\| < r \right\} d\nu(s) d\nu(t) < \infty.
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\]

We can therefore state from Lemma 22 that the occupation measure \( \nu_{B^H + f} \) is absolutely continuous with respect to the Lebesgue measure \( \lambda_d \) a.s. Hence \( \lambda_d (B^H + f)(A) > 0 \) a.s. which finishes the proof.
Let $\alpha \in (0, 1)$ such that $\alpha < H$. Let $B^\alpha$ be a fractional Brownian motion independent to $B^H$. Then for any Borel set $A \subset [0, 1]$ such that $\dim(A) > \alpha d$, it is known from (22) that all the trajectories of $B^\alpha$ are satisfying the condition

$$\dim_{\psi,H}(\text{Gr}_A(B^\alpha)) > H d.$$ 

This gives some examples of $d$
Another criterion due to Berman (often easier to apply) tells us that:

- $\nu_Y$ has a density $\frac{d\nu_Y}{d\lambda_d}(.) \in L^2(\lambda_d \otimes \mathbb{P})$ if and only if

$$\int_{\mathbb{R}^d} \int_A \int_A \mathbb{E} \left( e^{i\langle \theta, (Y(s) - Y(t)) \rangle} \right) d\nu(s) d\nu(t) d\theta < \infty.$$  (27)
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However, we were unable to apply it. Indeed, a simple calculation using characteristic function of Gaussian random vector yields

$$\mathbb{E}\left(e^{i\langle \theta, (B^H+f)(s)-(B^H+f)(t) \rangle}\right) = e^{i\langle \theta, f(s)-f(t) \rangle} \exp\left(-\frac{|s-t|^{2H}||\theta||^2}{2}\right).$$
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Now integrating the modulus we have

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\int_{\mathbb{R}^d} \left| \mathbb{E} \left( e^{i\langle \theta, (B^H + f)(s) - (B^H + f)(t) \rangle} \right) \right| d\theta = \frac{(2\pi)^{d/2}}{|s - t|^{Hd}}.
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However, the difficulty stems from the lack of information contained in the measure $\nu$. We are unable to ensure finiteness of the integral,

\[
\int_A \int_A \frac{(2\pi)^{d/2}}{|s-t|^{Hd}} \, d\nu(s) \, d\nu(t).
\]
Thank you for your attention

[2] M. Erraoui and Y. Hakiki. **Images of fractional Brownian motion with deterministic drift: Positive Lebesgue measure and non-empty interior.** (Submitted for publication)

