Local times for systems of non-linear stochastic heat equations

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systems of non-linear stochastic heat equations

We consider the following system of non-linear stochastic heat equations

$$\frac{\partial u_k}{\partial t}(t,x) = \frac{\partial^2 u_k}{\partial x^2}(t,x) + b_k(u(t,x)) + \sum_{l=1}^{d} \sigma_{k,l}(u(t,x)) \dot{W}^l(t,x), \quad (1.1)$$

with Neumann boundary conditions

$$u_k(0,x) = 0, \qquad \frac{\partial u_k(t,0)}{\partial x} = \frac{\partial u_k(t,1)}{\partial x} = 0,$$

We put
$u := (u_1, \cdots, u_d)$, $b = (b_k)$, and $\sigma = (\sigma_{k,l})$.

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Conditions on b and σ

Let us state the following hypotheses on the coefficients $\sigma_{k,l}$ and b_k of the system of non-linear stochastic heat equations (1.1):

A1 For all $1 \leq k, l \leq d$ The functions $\sigma_{k,l}$ and b_k are bounded and infinitely differentiable such that the partial derivatives of all orders are bounded.

A2 The matrix σ is uniformly elliptic i.e., there exists $\rho > 0$ such that for all $x \in \mathbb{R}^d$ and $z \in \mathbb{R}^d$ with $\|z\| = 1$, we have $\|\sigma(x)z\|^2 \geq \rho^2$ (where$\| \cdot \|$ is the Euclidean norm on $\mathbb{R}^d$).

Mild solution of SHE

The mild solution

$$u_k(t,x) = \int_0^t \int_0^1 G_{t-r}(x,v) \sum_{l=1}^{d} \sigma_{k,l}(u(r,v)) W^l(dr,dv) + \int_0^t \int_0^1 G_{t-r}(x,v) b_k(u(r,v)) dv\, dr, \tag{1.2}$$

Malliavin derivative of the mild solution of SHE

Proposition (Bally and Pardoux (1998))

Assume **A1**. Then for any $t \in [0, T]$ and $x \in [0, 1]$ we have $u(t, x) \in (\mathbb{D}^\infty)^d$. Furthermore, its derivative satisfies for all $r < t$,

$$D^{(i)}_{r,v}(u_k(t, x)) = G_{t-r}(x, v)\sigma_{k,i}(u(r, v)) + a_k(i, r, v, t, x),$$

where

$$a_k(i, r, v, t, x) = \sum_{l=1}^{d} \int_r^t \int_0^1 G_{t-\tau}(x, z) D^{(i)}_{r,v}(\sigma_{k,l}(u(\tau, z)))W^l(d\tau, dz) + \int_r^t \int_0^1 G_{t-\tau}(x, z) D^{(i)}_{r,v}(b_k(u(\tau, z)))dz\, d\tau,$$

and

$$D^{(i)}_{r,v}(u_k(t, x)) = 0 \qquad \text{when } r > t. \tag{1.3}$$

1^{st} goal

Theorem

Let $u(t,x)$ be the solution to Eq. (1.1).

(i) *For every $x \in (0,1)$, almost surely, when $d \leq 3$, the local time $L(\xi, t)$ of the process $(u(t,x)\,;\; t \in [0,T])$ exists for any fixed t, moreover, $L(\bullet, t) \in H^\alpha(\mathbb{R}^d)$ for $\alpha < \frac{4-d}{2}$, where $H^\alpha(\mathbb{R}^d)$ is the Sobolev space of index α; when $d \geq 4$, the local time does not exist in $L^2(\mathbb{P} \otimes \lambda_d)$ for any t, here λ_d is the Lebesgue measure on $\mathbb{R}^d$.*

(ii) Assume $d \leq 3$. for each $x \in (0,1)$, the local time of the process $(u(t,x)\,;\; t \in [0,T])$ has a version, denoted by $L(\xi,t)$, which is s jointly continuous in (ξ,t) almost surely, and which is γ-Hölder continuous in t, uniformly in ξ, for all $\gamma < 1 - \frac{d}{4}$: There exist two random variables η and δ which are almost surely finite and positive such that

$$\sup_{\xi \in \mathbb{R}^d} |L(\xi, t+h) - L(\xi, t)| \leq \eta |h|^{\gamma},$$

for all $t, t+h \in [0,T]$ and all $|h| < \delta$.

2^{nd} Goal

Theorem

Assume **A1** and **A2**. Then we get the following:

(a) There exists a constant $c > 0$ such that for any $x \in (0,1)$ fixed, and for all $0 \leq s < t \leq T$ and $\xi \in \mathbb{R}^d$,

$$p_{s,t,x}(\xi) \geq \frac{c}{(t-s)^{d/4}} \exp\left(-\frac{\|\xi\|^2}{c(t-s)^{1/2}}\right), \tag{1.4}$$

where $p_{s,t,x}(\xi)$ is the density of the $\mathbb{R}^d$-valued random vector $(u_1(t,x) - u_1(s,x), \cdots, u_d(t,x) - u_d(s,x))$.

(b) There exists $c > 0$ such that for any $0 = t_0 < t_1 < \cdots < t_n \leq T$, $x \in (0,1)$, $m_{i,k}$ positive integer, for $i = 1, \cdots, n$ and $k = 1, \cdots, d$, and $\xi = (\xi_{1,1}, \cdots, \xi_{1,d}, \cdots, \xi_{n,1}, \cdots, \xi_{n,d}) \in \mathbb{R}^{n \times d}$,

$$\left| \partial_{\xi_{1,1}}^{m_{1,1}} \cdots \partial_{\xi_{1,d}}^{m_{1,d}} \cdots \partial_{\xi_{n,1}}^{m_{n,1}} \cdots \partial_{\xi_{n,d}}^{m_{n,d}} \, p_{t_1,\cdots,t_n,x}(\xi) \right| \leq c \prod_{i=1}^{n} \frac{1}{(t_i - t_{i-1})^{(d+\sum_{k=1}^{d} m_{i,k})/4}} \exp\left(-\frac{\|\xi_i\|^2}{c\,(t_i - t_{i-1})^{1/2}} \right), \tag{1.5}$$

where $\xi_i = (\xi_{i,1}, \cdots, \xi_{i,d})$, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$ and, $\partial_y^l := \frac{\partial^l}{\partial y^l}$.

Elements of the local times

Let $(\theta_t)_{t\in[0,T]}$ be a Borel function with values in $\mathbb{R}^d$. For any Borel set $B \subseteq [0, T]$, the occupation measure of θ on B is given by the following measure on $\mathbb{R}^d$:

$$\nu_B(\bullet) = \lambda\{t \in B\,;\ \theta_t \in \bullet\},$$

where λ is the Lebesgue measure. When ν_B is absolutely continuous with respect to λ_d (the Lebesgue measure on $\mathbb{R}^d$), we say that the local time of θ on B exists and it is defined, $L(\bullet, B)$, as the Radon-Nikodym derivative of ν_B with respect to λ_d, i.e., for almost every x,

$$L(x, B) = \frac{d\nu_B}{d\lambda_d}(x).$$

The local time satisfies the following occupation formula: for any Borel set $B \subseteq [0, T]$, and for every measurable bounded function $f : \mathbb{R}^d \to \mathbb{R}$,

$$\int_B f(\theta_s)ds = \int_{\mathbb{R}^d} f(x)L(x, B)dx.$$

- The deterministic function θ can be chosen to be the sample path of a separable stochastic process $(X_t)_{t\in[0,T]}$ with $X_0 = 0$ a.s.
- We investigate the local time via Berman's approach.

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Let us state the following hypotheses on the integrability of the characteristic function of X:

B1

$$\int_{\mathbb{R}^d}\int_0^T\int_0^T \mathbb{E}\left[e^{i\langle u, X_t - X_s\rangle}\right] dt\, ds\, du < \infty,$$

where $\langle\cdot,\cdot\rangle$ is the Euclidean inner product on $\mathbb{R}^d$.

B2 For every even integer $m \geq 2$,

$$\int_{(\mathbb{R}^d)^m}\int_{[0,T]^m}\left|\mathbb{E}\left[\exp\left(i\sum_{j=1}^m \langle u_j, X_{t_j}\rangle\right)\right]\right|\prod_{j=1}^m dt_j \prod_{j=1}^m du_j < \infty.$$

Theorem

Assume **B1**. Then the process X has a square integrable local time. Moreover, we have almost surely, for all Borel set $B \subseteq [0, T]$, and for almost every x,

$$L(x, B) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\langle u,x\rangle} \int_B e^{i\langle u,X_t\rangle} dt\, du. \tag{1.6}$$

Theorem (Berman(1969))

Assume **B1** and **B2**. Put for all integer $N \geq 1$,

$$L_N(x,t) = \frac{1}{(2\pi)^d} \int_{[-N,N]^d} e^{-i\langle u,x\rangle} \int_0^t e^{i\langle u,X_s\rangle} ds\, du.$$

Then there exists a stochastic process $\tilde{L}(x,t)$ separable in the x-variable, such that for each even integer $m \geq 2$,

$$\lim_{N\to\infty} \sup_{(x,t)\in\mathbb{R}^d\times[0,T]} \mathbb{E}\left[\left|L_N(x,t) - \tilde{L}(x,t)\right|^m\right] = 0. \qquad (1.7)$$

Theorem (Berman(1969))

Let $\tilde{L}(x,t)$ be given by (1.7). If $\tilde{L}(x,t)$ is almost surely continuous in x, then it is a (continuous in x) version of the local time on $[0,t]$.

$$\mathbb{E}[\tilde{L}(x+k,t+h) - \tilde{L}(x,t+h) - \tilde{L}(x+k,t) + \tilde{L}(x,t)]^m$$

$$= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{[t,t+h]^m} \prod_{j=1}^{m} \left(e^{-i\left\langle v_j - v_{j+1}, x+k \right\rangle} - e^{-i\left\langle v_j - v_{j+1}, x \right\rangle} \right)$$

$$\times \mathbb{E}\left[e^{i\sum_{j=1}^{m}\left\langle v_j, X_{t_j} - X_{t_{j-1}} \right\rangle} \right] \prod_{j=1}^{m} dt_j \prod_{j=1}^{m} dv_j, \tag{1.8}$$

and

$$\mathbb{E}[\tilde{L}(x,t+h) - \tilde{L}(x,t)]^m$$

$$= \frac{1}{(2\pi)^{md}} \int_{(\mathbb{R}^d)^m} \int_{[t,t+h]^m} e^{-i\langle v_1, x\rangle} \mathbb{E}\left[e^{i\sum_{j=1}^{m}\left\langle v_j, X_{t_j} - X_{t_{j-1}} \right\rangle} \right] \prod_{j=1}^{m} dt_j \prod_{j=1}^{m} dv_j, \tag{1.9}$$

For the Gaussian case

Assume $d = 1$.
We know that

$$\mathbb{E}\left[e^{i\sum_{j=1}^m v_j(X_{t_j}-X_{t_{j-1}})}\right] = \exp\left(-\frac{\mathrm{Var}(\sum_{j=1}^m v_j(X_{t_j}-X_{t_{j-1}}))}{2}\right).$$

That is why we define the local nondeterminism (LND) as:

Lemma (Berman 1973)

For any $m \geq 2$, there exist two positive constants c_m and ε such that for every ordered points $0 = t_0 \leq t_1 < \cdots < t_m \leq 1$ with $t_m - t_1 < \varepsilon$, and $(v_1, \cdots, v_m) \in \mathbb{R}^m \setminus \{0\}$,

$$\mathrm{Var}\left(\sum_{j=1}^m v_j(X_{t_j} - X_{t_{j-1}})\right) \geq c_m \sum_{j=1}^m v_j^2 \,\mathrm{Var}\left(X_{t_j} - X_{t_{j-1}}\right). \qquad (1.10)$$

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For the non-Gaussian case

- For non-Gaussian processes, the unknown form of the characteristic functions caused the difficulty in extending the LND condition outside the Gaussian framework.
- We introduce a new concept of the LND (which is called the α-LND) that deals directly with the characteristic function.

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The α-LND

Definition

Let $X = (X_t)_{t\in[0,T]}$ be a stochastic process with values in $\mathbb{R}^d$ and J a subinterval of $[0, T]$. We say that X is α-LND on J, if for every nonnegative integers $m \geq 2$, $k_{1,1}, \cdots, k_{1,d}, \cdots, k_{m,1}, \cdots, k_{m,d}$, there exist two positive constants c and ε (both may depend on $m, k_{1,1}, \cdots, k_{1,d}, \cdots, k_{m,1}, \cdots, k_{m,d}$) such that

$$\left|\mathbb{E}\left[e^{i\sum_{j=1}^{m}\left\langle v_j, X_{t_j} - X_{t_{j-1}}\right\rangle}\right]\right| \leq \frac{c}{\prod_{j=1}^{m}\prod_{l=1}^{d}|v_{j,l}|^{k_{j,l}}(t_j - t_{j-1})^{\alpha k_{j,l}}}, \quad (1.11)$$

for all $v_j = (v_{j,1}, \cdots, v_{j,d}) \in \mathbb{R}^d$ with $v_{j,l} \neq 0$ ($j = 1, \cdots, m$ and $l = 1, \cdots, d$), and for every ordered points $t_1 < \cdots < t_m$ in J with $t_m - t_1 < \varepsilon$ and $t_0 = \inf(J)$.

Remark

1. Let $d = 1$ and $Y = (Y_t)_{t\in[0,T]}$ be a centred Gaussian process verifies the classical local nondeterminism (LND) property on J, we have fore any $m \geq 2$, there exist two positive constants c_m and ε such that for every ordered points $t_1 < \cdots < t_m$ in J with $t_m - t_1 < \varepsilon$, and $(v_1, \cdots, v_m) \in \mathbb{R}^m$ with $(v_1, \cdots, v_m) \neq (0, \cdots, 0)$,

$$\operatorname{Var}\left(\sum_{j=1}^{m} v_j (Y_{t_j} - Y_{t_{j-1}})\right) \geq c_m \sum_{j=1}^{m} v_j^2 \operatorname{Var}\left(Y_{t_j} - Y_{t_{j-1}}\right). \qquad (1.12)$$

Assume also that there exists a positive constant K, such that for every $s, t \in J$ with $s < t$,

$$K(t-s)^{2\alpha} \leq \operatorname{Var}\left(Y_t - Y_s\right). \qquad (1.13)$$

Hence Y is α-LND on J.

1. Let $d > 1$ and $Y^0 = (Y_t^0)_{t\in[0,T]}$ be a real-valued centred Gaussian process verifies the classical local nondeterminism (LND) property on J (i.e. (1.12)) and (1.13). Define

$$Y_t = (Y_t^1, \cdots, Y_t^d),$$

where $Y^1, \cdots, Y^d$ are independent copies of Y^0. Then Y is α-LND on J.

Elements of Malliavin calculus

Let $(W^i(t,x),\ t\in[0,T],\ x\in[0,1]),\ i=1,\cdots,d$, be d-independent space-time white noises defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, and put $W=(W^1,\cdots,W^d)$. For any $h\in\mathcal{H}:=L^2([0,T]\times[0,1],\mathbb{R}^d)$, we set $W(h)=\sum_{i=1}^d\int_0^T\int_0^1 h^i(t,x)W^i(dx,dt)$ the Wiener integral. Denote by $\mathcal{S}$ the class of cylindrical random variables of the form
$F=\varphi(W(h_1),\cdots,W(h_n))$.
$F\in\mathcal{S}$

$$D^{(l)}_{t,x}F=\sum_{i=1}^n\frac{\partial\varphi}{\partial x_i}(W(h_1),\cdots,W(h_n))h^l_i(t,x).$$

$\mathcal{H} := L^2([0,T] \times [0,1], \mathbb{R}^d)$
$\mathcal{H}_{s,t} := L^2([s,t] \times [0,1], \mathbb{R}^d)$.
For any $F, G \in \mathbb{D}^{1,p}$ we point out that

$$\langle DF\,,\, DG\rangle_{\mathcal{H}} = \sum_{l=1}^{d} \int_0^T \int_0^1 D_{r,x}^{(l)} F\, D_{r,x}^{(l)} G\, drdx,$$

and

$$\langle DF\,,\, DG\rangle_{\mathcal{H}_{s,t}} = \sum_{l=1}^{d} \int_s^t \int_0^1 D_{r,x}^{(l)} F\, D_{r,x}^{(l)} G\, drdx.$$

New definition for the Malliavin matrix

Definition

Let $0 = t_0 < t_1 < \cdots < t_n \leq T$ and $F = (F_1, \cdots, F_n)$ be a $\mathbb{R}^{n\times d}$-valued random vector such that for $i = 1, \cdots, n$, $F_i = (F_{i,1}, \cdots, F_{i,d})$ where $F_{i,k} \in \mathbb{D}^{1,p}$ for any $1 \leq i \leq n$ and $1 \leq k \leq d$. We define the following matrices, for every $1 \leq i, j \leq n$,

$$\Gamma^{i,j} = \left(\Gamma^{i,j}_{k,l}\right)_{1\leq k,l\leq d} \quad \text{where} \quad \Gamma^{i,j}_{k,l} = \langle DF_{i,k}\,,\, DF_{j,l}\rangle_{\mathcal{H}_{t_{i-1},t_i}}.$$

where $\mathcal{H}_{t_{i-1},t_i} = L^2([t_{i-1}, t_i] \times [0,1])$ we write $\Gamma_{F\,,\,t_1,\cdots,t_n}$ for the Malliavin matrix of F with respect to $t_1, \cdots, t_n$. That is, the following block matrix

$$\Gamma_{F\,,\,t_1,\cdots,t_n} = \left(\Gamma^{i,j}\right)_{1\leq i,j\leq n}.$$

For $d = 1$.

$\Gamma_{F\,,t_1,\cdots,t_n} =$

$$\begin{pmatrix} \left\langle DF^1, DF^1 \right\rangle_{L^2([t_0,t_1]\times[0,1])} & \cdots & \left\langle DF^1, DF^n \right\rangle_{L^2([t_0,t_1]\times[0,1])} \\ \left\langle DF^2, DF^1 \right\rangle_{L^2([t_1,t_2]\times[0,1])} & \cdots & \left\langle DF^2, DF^n \right\rangle_{L^2([t_1,t_2]\times[0,1])} \\ \vdots & \ddots & \vdots \\ \left\langle DF^n, DF^1 \right\rangle_{L^2([t_{n-1},t_n]\times[0,1])} & \cdots & \left\langle DF^n, DF^n \right\rangle_{L^2([t_{n-1},t_n]\times[0,1])} \end{pmatrix}$$

Definition

Let $0 = t_0 < t_1 < \cdots < t_n \leq T$ and $F = (F_1, \cdots, F_n)$ be a $\mathbb{R}^{n \times d}$-valued random vector such that $F_i = (F_{i,1}, \cdots, F_{i,d})$, for $i = 1, \cdots, n$. F is said to be nondegenerate with respect to $t_1, \cdots, t_n$, if it satisfies the following three conditions:

(i) For all $i = 1, \cdots, n$, and $k = 1, \cdots, d$, $F_{i,k} \in \mathbb{D}^{\infty}$.

(ii) $\Gamma_{F, t_1, \cdots, t_n}$ is invertible a.s. and we denote by $\Gamma^{-1}_{F, t_1, \cdots, t_n}$ its inverse.

(iii) $\left(\det \Gamma_{F, t_1, \cdots, t_n}\right)^{-1} \in L^p$ for all $p \geq 1$.

Integration by parts

Proposition

Let $0 = t_0 < t_1 < \cdots < t_n \leq T$ *and* $F = (F_1, \cdots, F_n)$ *be a nondegenerate with respect to* $t_1, \cdots, t_n$, $\mathbb{R}^{n\times d}$*-valued random vector such that,* $F_i = (F_{i,1}, \cdots, F_{i,d})$, *for* $i = 1, \cdots, n$. *Let* $G \in \mathbb{D}^\infty$ *and let* $g(x_{1,1}, \cdots, x_{1,d}, \cdots, x_{n,1}, \cdots, x_{n,d}) \in C_P^\infty(\mathbb{R}^{n\times d})$. *Fix* $m \geq 1$. *For any multi-index* $\beta = (\beta_1, \cdots, \beta_m)$ *with* $\beta_\theta = (i_\theta, k_\theta) \in \{1, \cdots, n\} \times \{1, \cdots, d\}$, *for* $\theta = 1, \cdots, m$, *we introduce the following notations:*

$$\partial_{\beta_\theta} := \frac{\partial}{\partial x_{i_\theta, k_\theta}} \quad \text{and} \quad \partial_\beta := \partial_{\beta_1} \cdots \partial_{\beta_m}, \quad \text{for } \theta = 1, \cdots, m.$$

Then for any $\beta = (\beta_1, \cdots, \beta_m)$, *there exists* $H^\beta_{t_1,\cdots,t_n}(F\,,\,G) \in \mathbb{D}^\infty$ *such that*

$$\mathbb{E}\left[(\partial_\beta g)(F)G\right] = \mathbb{E}\left[g(F)H^\beta_{t_1,\cdots,t_n}(F\,,\,G)\right], \tag{1.14}$$

where the random variables $H^{\beta}_{t_1,\cdots,t_n}(F, G)$ are recursively given by

$$H^{(i,k)}_{t_1,\cdots,t_n}(F\,,\;G) = \sum_{j=1}^{n}\sum_{l=1}^{d}\delta\left(G\left(\Gamma^{-1}_{F\,,\,t_1,\cdots,t_n}\right)^{i,j}_{k,l} DF_{j,l}\mathbb{1}_{[t_{j-1}\,,\,t_j]\times[0\,,\,1]}\right), \tag{1.15}$$

$$H^{\beta}_{t_1,\cdots,t_n}(F\,,\;G) = H^{\beta_m}_{t_1,\cdots,t_n}\left(F\,,\;H^{(\beta_1,\cdots,\beta_{m-1})}_{t_1,\cdots,t_n}(F\,,\;G)\right). \tag{1.16}$$

For $d = 1$. Put $F = (X_{t_1} - X_{t_0}, \cdots, X_{t_m} - X_{t_{m-1}})$ and

$$\partial_\beta = \frac{\partial}{\partial x_{\beta_1}} \cdots \frac{\partial}{\partial x_{\beta_n}}.$$

On one hand, we have by integration by parts

$$\mathbb{E}[\partial_\beta e^{i \sum_{j=1}^m u_j (X_{t_j} - X_{t_{j-1}})}] = \mathbb{E}[e^{i \sum_{j=1}^m u_j (X_{t_j} - X_{t_{j-1}})} H_{\pi_m}^\beta (F, 1)] \qquad (1.17)$$

On the other hand we have

$$\mathbb{E}[\partial_\beta e^{i \sum_{j=1}^m u_j (X_{t_j} - X_{t_{j-1}})}] = (iu_1)^{k_1} \cdots (iu_m)^{k_m} \mathbb{E}[e^{i \sum_{j=1}^m u_j (X_{t_j} - X_{t_{j-1}})}] \quad (1.18)$$

Combining (1.17) and (1.18), we get

$$|\mathbb{E}[e^{i \sum_{j=1}^m u_j (X_{t_j} - X_{t_{j-1}})}]| \leq |u_1|^{-k_1} \cdots |u_m|^{-k_m} \| H_{\pi_m}^\beta (F, 1) \|_{L^1} \qquad (1.19)$$

Then, to establish α-LND, we need to estimate $\| H_{\pi_m}^\beta (F, 1) \|_{L^1}$.

For $d=1$. Put $F=(X_{t_1}-X_{t_0},\cdots,X_{t_m}-X_{t_{m-1}})$ and

$$\partial_\beta=\frac{\partial}{\partial x_{\beta_1}}\cdots\frac{\partial}{\partial x_{\beta_n}}.$$

On one hand, we have by integration by parts

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$$\mathbb{E}[\partial_\beta e^{i\sum_{j=1}^m u_j(X_{t_j}-X_{t_{j-1}})}]=(iu_1)^{k_1}\cdots(iu_m)^{k_m}\mathbb{E}[e^{i\sum_{j=1}^m u_j(X_{t_j}-X_{t_{j-1}})}] \quad (1.18)$$

Combining (1.17) and (1.18), we get

$$|\mathbb{E}[e^{i\sum_{j=1}^m u_j(X_{t_j}-X_{t_{j-1}})}]|\le|u_1|^{-k_1}\cdots|u_m|^{-k_m}\|H^\beta_{\pi_m}(F,1)\|_{L^1} \qquad (1.19)$$

Then, to establish α-LND, we need to estimate $\|H^\beta_{\pi_m}(F,1)\|_{L^1}$.

Theorem (Nualart (The Malliavin Calculus and Related Topics))

There exist constants $\beta, \gamma > 1$ and integers n, m such that

$$\|H_\alpha(F,G)\|_p \leq \|\det \gamma_F^{-1}\|_\beta^m \|DF\|_{k,\gamma}^n \|G\|_{k,q}.$$

Lemma

$0 = t_0 < t_1 < \cdots < t_n$ and $\beta = (\beta_1, \cdots, \beta_m)$ with $\beta_\theta = (i_\theta, k_\theta) \in \{1, \cdots, n\} \times \{1, \cdots, d\}$, for $\theta = 1, \cdots, m$, then there exists a constant $C > 0$ such that

$$\left\| H^{\beta}_{t_1,\cdots,t_n}(F\,,\,1) \right\|_{0,2} \leq C \left\| \left(\det \Gamma_{F\,,\,t_1,\cdots,t_n}\right)^{-1} \right\|^{m}_{m,2^{m+2}} \cdot \prod_{\theta=1}^{m} \| DF_{i_\theta,k_\theta} \|_{m,2^{2(m+n\times d)},\mathcal{H}} \prod_{(i_0,k_0)\in O_{(i_\theta,k_\theta)}} \| DF_{i_0,k_0} \|^2_{m,2^{2(m+n\times d)},\mathcal{H}}\,, \tag{1.20}$$

where $O_{(i_\theta,k_\theta)} = \{(i_0,k_0) \in \{1, \cdots, n\} \times \{1, \cdots, d\}\,;\ (i_0,k_0) \neq (i_\theta,k_\theta)\}$.

Estimation of $\|DF_{i,k}\|_{k,p}$

Proposition (Dalang, Khoshnevisan, and E. Nualart (2009))

Assume **A1**. Then for any $s, t \in [0, T]$ with $s \leq t$, $x \in [0, 1]$, $p > 1$, and $m \geq 1$,

$$\mathbb{E}\left[\|D^m(u_k(t,x) - u_k(s,x))\|^p_{\mathcal{H}^{\otimes m}}\right] \leq C_T\,|t-s|^{p/4}\,, \qquad k = 1, \cdots, d.$$

Estimation of $\| (\det \Gamma_{F\,,\,t_1,\cdots,t_n})^{-1} \|_{k,p}$

$0 = t_0 < t_1 < \cdots < t_n \le T$

$$Z = (u(t_1,x) - u(t_0,x), \cdots, u(t_n,x) - u(t_{n-1},x)), \tag{1.21}$$

where
$u(t_i,x) - u(t_{i-1},x) = (u_1(t_i,x) - u_1(t_{i-1},x), \cdots, u_d(t_i,x) - u_d(t_{i-1},x))$

Let $\Gamma_{Z\,,\,t_1,\cdots,t_n}$ be the Malliavin matrix of Z with respect to $t_1, \cdots, t_n$ (Z is given by (1.21)). Note that $\Gamma_{Z\,,\,t_1,\cdots,t_n} = \left(\Gamma^{i,j}\right)_{1\le i,j\le n}$ is the random block matrix, of the form

$$\Gamma^{i,j}_{k,l} = \langle D(u_k(t_i,x) - u_k(t_{i-1},x))\,,\, D(u_l(t_j,x) - u_l(t_{j-1},x))\rangle_{\mathcal{H}_{t_{i-1},t_i}}\,,$$

where $\mathcal{H}_{t_{i-1},t_i} = L^2\left([t_{i-1},t_i] \times [0,1], \mathbb{R}^d\right)$.

For $d = 1$.

$\Gamma_{Z\,,t_1,\cdots,t_n} =$

$$\begin{pmatrix} \left\langle DF^1, DF^1 \right\rangle_{L^2([t_0,t_1]\times[0,1])} & \cdots & \left\langle DF^1, DF^n \right\rangle_{L^2([t_0,t_1]\times[0,1])} \\ \left\langle DF^2, DF^1 \right\rangle_{L^2([t_1,t_2]\times[0,1])} & \cdots & \left\langle DF^2, DF^n \right\rangle_{L^2([t_1,t_2]\times[0,1])} \\ \vdots & \ddots & \vdots \\ \left\langle DF^n, DF^1 \right\rangle_{L^2([t_{n-1},t_n]\times[0,1])} & \cdots & \left\langle DF^n, DF^n \right\rangle_{L^2([t_{n-1},t_n]\times[0,1])} \end{pmatrix}$$

where $F^i = u(t_i, x) - u(t_{i-1}, x)$.

Recall that

$$D_{r,v}(u(t,x)) = 0 \qquad \text{when } r > t. \tag{1.22}$$

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i.e. $\Gamma_{Z\,,\,t_1,\cdots,t_n}$ is a triangular matrix.
Then

$$\det\left(\Gamma_{Z\,,\,t_1,\cdots,t_n}\right) = \prod_{i=1}^{n} \|DF^i\|^2_{L^2([t_{i-1},t_i]\times[0,1])}. \tag{1.23}$$

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Theorem

Assume **A1** and **A2**. Let $0 = t_0 < t_1 < \cdots < t_n \le T$, $x \in (0,1)$, and Z given by (1.21), then for any $k \ge 0$, $p > 1$,

$$\| (\det \Gamma_{Z\,,\, t_1, \cdots, t_n})^{-1} \|_{k,p} \le K \prod_{i=1}^{n} (t_i - t_{i-1})^{-d/2},$$

where K is a positive constant.

Lemma

Assume **A1** and **A2**. Let $0 = t_0 < t_1 < \cdots < t_n \le T$, $x \in (0,1)$, Z given by (1.21), and $\beta = (\beta_1, \cdots, \beta_m)$ with $\beta_\theta = (i_\theta, k_\theta) \in \{1, \cdots, n\} \times \{1, \cdots, d\}$, for $\theta = 1, \cdots, m$, then there exists a constant $C > 0$ such that

$$\left\| H^{\beta}_{t_1, \cdots, t_n}(Z\,,\, 1) \right\|_{0,2} \le C \prod_{\theta=1}^{m} (t_{i_\theta} - t_{i_\theta - 1})^{-1/4}. \tag{1.24}$$

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The non-linear SHE is $\frac{1}{4}$-LND

Theorem

Let $u(t,x)$ be given by (1.2). Hence, for each fixed $x \in (0,1)$, for every nonnegative integers $m \geq 2$, $k_{1,1}, \cdots, k_{1,d}, \cdots, k_{m,1}, \cdots, k_{m,d}$, there exists a constant $c = c(m, k_{1,1}, \cdots, k_{1,d}, \cdots, k_{m,1}, \cdots, k_{m,d})$ such that

$$\left| \mathbb{E}\left[e^{i \sum_{j=1}^{m} \langle v_j, u(t_j,x) - u(t_{j-1},x) \rangle} \right] \right| \leq \frac{c}{\prod_{j=1}^{m} \prod_{l=1}^{d} |v_{j,l}|^{k_{j,l}} (t_j - t_{j-1})^{\frac{1}{4} k_{j,l}}}, \tag{1.25}$$

for all $v_j = (v_{j,1}, \cdots, v_{j,d}) \in \mathbb{R}^d$ with $v_{j,l} \neq 0$ ($j = 1, \cdots, m$ and $l = 1, \cdots, d$), for each $t \in [0, T)$, and for every ordered points $t = t_0 < t_1 < \cdots < t_m$ in $[0, T]$. We said that the process $(u(t,x),\ t \in [0,T])$ is $\frac{1}{4}$-LND on $[t, T]$ for each fixed $t \in [0, T)$.

Existence of local time when $d \leq 3$

Let $\alpha \geq 0$, we define the Sobolev space $H^\alpha(\mathbb{R}^d)$ as:

$$H^\alpha(\mathbb{R}^d) = \left\{ g \in L^2(\mathbb{R}^d) \,;\; (1 + \|\xi\|^2)^{\frac{\alpha}{2}} \hat{g} \in L^2(\mathbb{R}^d) \right\},$$

Theorem

Let $u(t,x)$ be given by (1.2). Assume that $d \leq 3$, then for each $x \in (0,1)$, the process $(u(t,x)\,;\; t \in [0,T])$ has local time $L(\xi,t)$. Moreover, for every fixed t, $L(\bullet,t) \in H^\alpha(\mathbb{R}^d)$ for $\alpha < \frac{4-d}{2}$.

The local time does not exist for $d \geq 4$

Theorem

Let $u(t,x)$ be given by (1.2). Assume that $d \geq 4$, then for each $x \in (0,1)$, the process $(u(t,x)\,;\ t \in [0,T])$ does not have a local time $L(\xi,t)$ in $L^2(\mathbb{P} \otimes \lambda_d)$ for any $t \in [0,T]$.

Lemma

Let $u(t,x)$ be given by (1.2). Assume $d \leq 3$. Let $\tilde{L}(\xi, t)$ be given as in (1.7), therefore, for every $\xi, y \in \mathbb{R}^d$, $t, t+h \in [0, T]$, and even integer $m \geq 2$,

$$\mathbb{E}\left[\tilde{L}(\xi, t+h) - \tilde{L}(\xi, t)\right]^m \leq C_m |h|^{m(1-\frac{d}{4})}. \tag{1.26}$$

$$\mathbb{E}\left[\tilde{L}(\xi+y, t+h) - \tilde{L}(\xi, t+h) - \tilde{L}(\xi+y, t) + \tilde{L}(\xi, t)\right]^m \leq C_{m,\theta} \|y\|^{m\theta} |h|^{m(1-\frac{d}{4}-\frac{\theta}{4})}, \tag{1.27}$$

where $0 < \theta < (\frac{4-d}{2}) \wedge 1$.

Theorem

Assume $d \leq 3$*. for each* $x \in (0,1)$*, the local time of the process* $(u(t,x)\,;\; t \in [0,T])$ *has a version, denoted by* $L(\xi,t)$*, which is s jointly continuous in* (ξ,t) *almost surely.*

Sample paths regularity of SHE

Corollary

Let $u(t,x)$ be given by (1.2). Assume $d \leq 3$. Then for each $x \in (0,1)$, all coordinate functions of $(u(t,x),\ t \in [0,T])$ are nowhere Hölder continuous of order greater than $\frac{1}{4}$.

Upcoming works

Investigation of the local time of the following two processes:

- $(u(t,x)\,,\ x \in [0,1])$
- $(u(t,x)\,,\ (t,x) \in [0,T] \times [0,1])$.

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Thank you
for your attention!