Numerical Stochastic Integrations and Limit Theorems

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CBMS Conference, August 2-6, 2021, UAH.

A joint work with Z. Selk and S. Tindel.
Outline

1. Elements of rough paths

2. Numerical methods for rough integrals

3. Numerical methods for stochastic rough integrals
1. Elements of rough paths

2. Numerical methods for rough integrals

3. Numerical methods for stochastic rough integrals
We recall some elements of rough paths.

Consider a $\mathbb{R}^d$-valued smooth path $x = (x_1, \ldots, x_d)$.

Consider multiple integrals of $x$:

- $x_1$-st: $\int_t^s dx_u = x_t - x_s$
- $x_2$-st: $\int_t^s \int_u^s dx_v \otimes dx_u$
- $x_3$-st: $\int_t^s \int_u^s \int_v^s dx_w \otimes dx_v \otimes dx_u$

and similar definition for $m$-th order multiple integral $x_m$.

The $n$-th order signature of $x$:

$$S_n(x)_{st} = (x_{1 \text{st}}, x_{2 \text{st}}, \ldots, x_{n \text{st}}).$$

It is known that for fixed $s$ and $t$, the $S_n(x)_{st}$ contains all information of $x$ for $n = \infty$. 
Signatures

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and similar definition for m-th order multiple integral $\mathbf{x}^m$. 
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  and similar definition for m-th order multiple integral $x^m$.

- The $n$-th order signature of $x$:
  
  $$S_n(x)_{st} = (x_{st}^1, x_{st}^2, \ldots, x_{st}^n).$$

It is known that for fixed $s$ and $t$, the $S_n(x)_{st}$ contains all information of $x$ for $n = \infty$. 
Consider a \( p \)-variation continuous path \( x \) in \( \mathbb{R}^d \). We denote \( \delta x_{st} = x_t - x_s \).

Recall that the \( p \)-variation norm is defined as
\[
\|x\|_{p\text{-var}} = \left( \sup_{(t_k)} \sum_{k} |\delta x_{t_k}|^{\frac{1}{p}} \right)^{\frac{1}{p}},
\]
where \( P \) is the set of finite partitions of the time interval \([0, T] \).

Denote \( x_\epsilon \) a mollification of \( x \).

Suppose that for \( n = \lfloor p \rfloor \) the \( n \)th order signature \( S_n(x_\epsilon) \) of \( x_\epsilon \) converges under the \( p \)-variation norm and denote the limit by \( S_n(x) := x \).

Then \( x \) is called a \( p \)-rough path.
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Rough paths

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Consider differential equation for a smooth function $x$:

$$y(t) = y_0 + \int_0^t V(y(u)) \, du,$$

$t \in [0, T]$.

- $S_n(x)$: signature of $x$.
- $S_n(y)$: signature of solution $y$.

Consider the Itô map $I$:

$$S_n(x) \rightarrow S_n(y).$$

$I$ is continuous under $p$-var norm with $p < n + 1$. (Lyons '98)

We extend the Itô map $I$ to $p$-variation rough paths. We define $y = I(x)$ as the solution of the differential equation

$$dy_t = V(y_t) \, dx_t.$$
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Applications

• The rough paths theory provides a framework for differential equations driven by an arbitrary irregular noise.
• Such solutions are path-wise solutions. No probability structure is required (e.g. Markovian or martingale).
• The rough paths framework provides the stability of the Itô map.
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Consider a $p$-rough path $x$. Denote $x \in \epsilon$: a mollification of $x$.

For $V \in C^\infty$ we define the rough integral:

$$\int_0^T V(x) \, dx = \lim_{\epsilon \to 0} \int_0^T V(x \epsilon) \, dx \epsilon.$$

Let $(t_k)$ be a partition of $[0, T]$. For each $k$ we consider the approximation

$$\int_{t_k+1}^{t_{k+1}} V(x) \, dx \approx \int_{t_k+1}^{t_{k+1}} V(x_{t_k}) \, dx = V(x_{t_k}) \delta x_{t_k} t_{k+1} - t_k.$$

This leads to the Riemann sum approximation

$$\int_0^T V(x) \, dx \approx \sum_k \int_{t_k+1}^{t_{k+1}} V(x) \, dx \approx \sum_k V(x_{t_k}) \delta x_{t_k} t_{k+1} - t_k.$$

Note that if $p \geq 2$, the Riemann sum $\sum_k V(x_{t_k}) \delta x_{t_k}$ diverges.
Rough integrations

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- Note that if $p \geq 2$ the Riemann sum $\sum_k V(x_{t_k})\delta x_{t_k}t_{k+1}$ diverges.
Denote $L_V = \partial V$ and $L_n = L \circ \cdots \circ L$.

We consider an improved approximation

$$\int_{t_k}^{t_{k+1}} V(x_t) \, dx_t \approx \int_{t_k}^{t_{k+1}} \left( V(x_t) + (L_V)(x_t) \right) x_{k+1} \, dx_t + \cdots + (L_{n-1}V)(x_t) x_n \, dx_t + \cdots$$

The first $n$ terms of Taylor expansion of $V(x_t)$ at $t_k$.

The compensated Riemann sum:

$$\int_{T_0}^T V(x_t) \, dx_t \approx \sum_{k} V(x_{k+1}) \, dx_t + \cdots + (L_{n-1}V)(x_t) x_n \, dx_t.$$
Denote $\mathcal{L} V = \partial V V$ and $\mathcal{L}^n = \mathcal{L} \circ \cdots \circ \mathcal{L}$. 

The first $n$ terms of Taylor expansion of $V(\mathbf{x}_t)$ at $t_0$. 

The compensated Riemann sum converges to the rough integral $\int T_0 V(\mathbf{x}_t) \, dt$ when $n > p - 1$. 

Note that the compensated Riemann sum requires the computations of signatures of $\mathbf{x}$. 


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• We obtain the compensated Riemann sum:

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\int_0^T V(x_t) dt \approx \sum_k V(x_{t_k}) \delta x_{t_k t_{k+1}} + (\mathcal{L}V)(x_{t_k})x_{t_k t_{k+1}}^2 + \cdots + (\mathcal{L}^{n-1}V)(x_{t_k})x_{t_k t_{k+1}}^n.
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Multi-dimensional rough integral

Rough integration with respect to random fields have been studied in the following cases:

- 2-dim Young integral: Towghi '01 and Quer-Sardanyons-Tindel '07
- 2-dim rough integral of order $\frac{1}{3}$: Chouk-Gubinelli '18
- Multi-dim Young integral: Harang '18

Let $x$ and $y$ be 2d Hölder functions on $[0,T]^2$ of order $(\alpha_1, \alpha_2)$ and $(\beta_1, \beta_2)$, respectively, and $\alpha_i + \beta_i > 1$, $i = 1, 2$. The Riemann sum $\sum (t_i), (t'_j) y(t_i) x(t_i+1) t'_i t'_i+1$ converges to the Young integral $\int_{[0,T]^2} y(st) ds t$.

The Riemann sum is divergent when $\alpha_i + \beta_i < 1$, $i = 1$ or 2, and compensated Riemann sum is introduced.
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Presentation Outline

1. Elements of rough paths
2. Numerical methods for rough integrals
3. Numerical methods for stochastic rough integrals
Let $X = (X_1, \ldots, X_d)$ be a continuous centered Gaussian process with i.i.d. components.

The covariance function of $X$ is defined as follows:

$$R(s, t) = \mathbb{E}[X_j s X_j t].$$

The information concerning $X$ is mostly encoded in the rectangular increments of $R$:

$$R_{stuv} := R(t, v) - R(t, u) - R(s, v) + R(s, u) = \mathbb{E}[\delta X_{st} \delta X_{uv}].$$

For $\rho \geq 1$ we define the $\rho$-variation of $R$ as

$$\|R\|_{\rho-\text{var}} = \sup_{(t_i), (t'_j)} \left( \sum_{i, j} \left| R_{t'_j t'_j + 1} t_i t_i + 1 \right|^\rho \right)^{1/\rho},$$

where $(t_j)$ and $(t'_j)$ are partitions on $[0, T]$.

If $R$ has finite $\rho$-variation for $\rho \in [1, 2)$, then $X$ gives raise to a $p$-rough path, provided $p > 2/\rho$. (Friz-Victoir '11).
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$$R^{st}_{uv} := R(t, v) - R(t, u) - R(s, v) + R(s, u) = \mathbb{E}[\delta X^i_{st} \delta X^i_{uv}].$$

For $\rho \geq 1$ we define the $\rho$-variation of $R$ as

$$\|R\|_{\rho\text{-var}} = \sup_{(t_i), (t'_i)} \left( \sum_{i,j} \left| R^{t'_i t'_{i+1}}_{t_i t_{i+1}} \right|^\rho \right)^{1/\rho},$$

where $(t_j)$ and $(t'_j)$ are partitions on $[0, T]$. 
• Let $X = (X^1, \ldots, X^d)$ be a continuous centered Gaussian process with i.i.d. components.

• The covariance function of $X$ is defined as follows

$$R(s, t) = \mathbb{E}[X^i_s X^i_t].$$

• The information concerning $X$ is mostly encoded in the rectangular increments of $R$:

$$R_{uv}^{st} := R(t, v) - R(t, u) - R(s, v) + R(s, u) = \mathbb{E}[\delta X^i_s \delta X^i_{uv}].$$

• For $\rho \geq 1$ we define the $\rho$-variation of $R$ as

$$\|R\|_{\rho\text{-var}} = \sup_{(t_i), (t'_i)} \left( \sum_{i,j} \left| \int_{t_i}^{t'_{i+1}} R^i_{t_i t_{i+1}} \right|^\rho \right)^{1/\rho},$$

where $(t_j)$ and $(t'_j)$ are partitions on $[0, T]$.

• If $R$ has finite $\rho$-variation for $\rho \in [1, 2)$, then $X$ gives raise to a $p$-rough path, provided $p > 2$.$\rho$. (Friz-Victoir ’11).
Consider a process \( y \) such that
\[
\delta y_{st} = y'_s X_{1st} + y''_s X_{2st} + r_{0st},
\]
\[
\delta y'_{st} = y''_s X_{1st} + r_{1st},
\]
where \( y'_{st}, y''_{st}, r_{0st}, r_{1st} \) are processes satisfying some regularity conditions.

\( y \) is called a controlled paths of \( X \) of order 2. Such processes contains most of the interesting examples. e.g. \( y = V(X) \) or \( y \) is the solution of a RDE.
Consider a process \( y \) such that

\[
\delta y_{st} = y'_s x^1_{st} + y''_s x^2_{st} + r^0_{st}, \quad \delta y'_{st} = y''_s x^1_{st} + r^1_{st},
\]

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where $y'$, $y''$, $r^0$, $r^1$ are processes satisfying some regularity conditions.

$y$ is called a controlled paths of $X$ of order 2. Such processes contains most of the interesting examples. e.g. $y = V(X)$ or $y$ is the solution of a RDE.
Define the trapezoid rule:

\[ \text{tr-} \mathcal{J}_0^T (y, X) = \sum_k \frac{y_{t_k} + y_{t_{k+1}}}{2} \cdot \delta X_{t_k t_{k+1}}. \]
Define the trapezoid rule:

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**Theorem (Liu-Selk-Tindel ’21)**

- Suppose that $\| R \|_{\rho \text{-var}} < \infty$. Then as the mesh size of the partition $(t_k)$ goes to 0 we have $\text{tr-} \mathcal{J}_0^T (y, X) \to \int_0^T y_t dX_t$ in probability.

- Suppose that there exists a constant $C > 0$ such that $\| R \|_{\rho \text{-var}}; [s, t] \times [0, T] \leq C |t - s|$ for all $[s, t] \subset [0, T]$. Then we have $\text{tr-} \mathcal{J}_0^T (y, X) \to \int_0^T y_t dX_t$ almost surely.
Define the trapezoid rule:

\[ \text{tr-} \mathcal{J}_0^T (y, X) = \sum_k \frac{y_{t_k} + y_{t_{k+1}}}{2} \cdot \delta X_{t_k t_{k+1}}. \]

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**Theorem (Liu-Selk-Tindel ’21)**

- Suppose that \( \|R\|_{\rho-\text{var}} < \infty \). Then as the mesh size of the partition \((t_k)\) goes to 0 we have

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- Suppose that there exists a constant \( C > 0 \) such that

\[ \|R\|_{\rho-\text{var},[s,t] \times [0,T]} \leq C|t - s| \quad \text{for all } [s, t] \subset [0, T]. \]

Then we have

\[ \text{tr-} J_0^T(y, X) \to \int_0^T y_t dX_t \quad \text{almost surely.} \]
Ingredients of proof.

With a careful rearrangement of the trapezoid rule one can recast it as

\[ T_0(y, X) = \sum I_1 + I_2 + I_3 + I_4, \]

where

\[ I_1 = y^t_k \cdot X_1^t_k + y'^t_k \cdot X_2^t_k + y''^t_k \cdot X_3^t_k, \]

\[ I_2 = \frac{1}{2} y'^t_k \cdot X_1^t_k - y'^t_k \cdot X_2^t_k, \]

\[ I_3 = \frac{1}{2} y''^t_k \cdot X_2^t_k - y''^t_k \cdot X_3^t_k, \]

\[ I_4 = \frac{1}{2} r_0^t_k \cdot X_1^t_k. \]

- \( I_1 \) is the compensated Riemann sum of \( \int T_0 y dX \).
- \( I_2 \) and \( I_3 \) are weighted random sums of the forms: \( \sum k y'^t_k h_n^t_k \) and \( \sum k y''^t_k \tilde{h}_n^t_k \).
- The convergences of \( I_i, i = 2, 3, 4 \) can be shown based on a transfer principle combined with some 2d young-type estimates.
Ingredients of proof.
With a careful rearrangement of the trapezoid rule one can recast it as

\[ tr - J_0^T(y, X) = \sum_k I_1 + I_2 + I_3 + I_4, \]

- \( I_1 \) is the compensated Riemann sum of \( \int T_0 y dX \).
- \( I_2 \) and \( I_3 \) are weighted random sums of the forms:
  \[ \sum_k y' t_k h_n t_k t_k + 1 \] and \[ \sum_k y'' t_k \tilde{h}_n t_k t_k + 1 \].
- The convergences of \( I_i, i = 2, 3, 4 \) can be shown based on a transfer principle combined with some 2d young-type estimates.
Ingredients of proof.
With a careful rearrangement of the trapezoid rule one can recast it as

$$\text{tr-} \mathcal{J}_{0}^{T}(y, X) = \sum_{k} l_{1} + l_{2} + l_{3} + l_{4},$$

where

$$l_{1} = y_{t_{k}} \cdot x_{t_{k}t_{k+1}}^{1} + y'_{t_{k}} \cdot x_{t_{k}t_{k+1}}^{2} + y''_{t_{k}} \cdot x_{t_{k}t_{k+1}}^{3},$$

$$l_{2} = \frac{1}{2} y'_{t_{k}} x_{t_{k}t_{k+1}}^{1} \cdot x_{t_{k}t_{k+1}}^{1} - y'_{t_{k}} \cdot x_{t_{k}t_{k+1}}^{2},$$

$$l_{3} = \frac{1}{2} y''_{t_{k}} x_{t_{k}t_{k+1}}^{2} \cdot x_{t_{k}t_{k+1}}^{1} - y''_{t_{k}} \cdot x_{t_{k}t_{k+1}}^{3},$$

$$l_{4} = \frac{1}{2} r_{t_{k}t_{k+1}}^{0} \cdot x_{t_{k}t_{k+1}}^{1}. $$
Ingredients of proof.
With a careful rearrangement of the trapezoid rule one can recast it as
\[
\text{tr- } J_0^T(y, X) = \sum_k I_1 + I_2 + I_3 + I_4,
\]
where

\[
I_1 = y_{tk} \cdot X_{tk,tk+1}^1 + y'_t \cdot X_{tk,tk+1}^2 + y''_t \cdot X_{tk,tk+1}^3,
\]

\[
I_2 = \frac{1}{2} y'_t \cdot X_{tk,tk+1}^1 \cdot X_{tk,tk+1}^1 - y'_t \cdot X_{tk,tk+1}^2,
\]

\[
I_3 = \frac{1}{2} y''_t \cdot X_{tk,tk+1}^2 \cdot X_{tk,tk+1}^1 - y''_t \cdot X_{tk,tk+1}^3,
\]

\[
I_4 = \frac{1}{2} r_{tk,tk+1}^0 \cdot X_{tk,tk+1}^1.
\]

- \(I_1\) is the compensated Riemann sum of \(\int_0^T y dX\).
Ingredients of proof.
With a careful rearrangement of the trapezoid rule one can recast it as

$$\text{tr} \cdot \mathcal{J}_0^T(y, X) = \sum_k l_1 + l_2 + l_3 + l_4,$$

where

$$l_1 = y_{tk} \cdot X_{tk tk+1}^1 + y'_{tk} \cdot X_{tk tk+1}^2 + y''_{tk} \cdot X_{tk tk+1}^3,$$

$$l_2 = \frac{1}{2} y'_{tk} X_{tk tk+1}^1 \cdot X_{tk tk+1}^1 - y'_{tk} \cdot X_{tk tk+1}^2,$$

$$l_3 = \frac{1}{2} y''_{tk} X_{tk tk+1}^2 \cdot X_{tk tk+1}^1 - y''_{tk} \cdot X_{tk tk+1}^3,$$

$$l_4 = \frac{1}{2} r_{tk tk+1}^0 \cdot X_{tk tk+1}^1.$$

- $l_1$ is the compensated Riemann sum of $\int_0^T y \, dX$.
- $l_2$ and $l_3$ are weighted random sums of the forms: $\sum_k y'_{tk} h_{tk tk+1}^n$ and $\sum_k y''_{tk} \tilde{h}_{tk tk+1}^n$. 
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With a careful rearrangement of the trapezoid rule one can recast it as

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\begin{align*}
l_1 &= y_{tk} \cdot X_{tk_{k+1}}^1 + y'_{tk} \cdot X_{tk_{k+1}}^2 + y''_{tk} \cdot X_{tk_{k+1}}^3 \\
l_2 &= \frac{1}{2} y'_{tk} X_{tk_{k+1}}^1 \cdot X_{tk_{k+1}}^1 - y'_{tk} \cdot X_{tk_{k+1}}^2 \\
l_3 &= \frac{1}{2} y''_{tk} X_{tk_{k+1}}^2 \cdot X_{tk_{k+1}}^1 - y''_{tk} \cdot X_{tk_{k+1}}^3 \\
l_4 &= \frac{1}{2} r_{tk_{k+1}}^0 \cdot X_{tk_{k+1}}^1.
\end{align*}

- $l_1$ is the compensated Riemann sum of $\int_0^T y \mathrm{d}X$.
- $l_2$ and $l_3$ are weighted random sums of the forms: $\sum_k y'_{tk} h^n_{tk_{k+1}}$ and $\sum_k y''_{tk} \tilde{h}^n_{tk_{k+1}}$.
- The convergences of $l_i$, $i = 2, 3, 4$ can be shown based on a transfer principle combined with some 2d young-type estimates.
Transfer principle (Liu-Tindel ’19)

• In order to bound a weighted sum
\[
\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} y_{t_k} h_{n_{t_k}} t_k + 1
\]

it suffices to consider the following elementary weighted sums:

\[
\sum_{s \leq t_k < t} h_{n_{t_k}} t_k + 1,
\]

\[
\sum_{s \leq t_k < t} \text{X}^1_{st} h_{n_{t_k}} t_k + 1,
\]

\[
\sum_{s \leq t_k < t} \text{X}^{\ell}_{st} h_{n_{t_k}} t_k + 1,
\]

where \( \ell \) is an integer depending on \( \text{X} \) and \( h_{n_{t_k}} \).

• These special weighted sums belong to finite Wiener chaos and are easier to handle.

• For example, in order to estimate
\[
I_2 = \frac{1}{2} \sum_{k} y'_{t_k} (X^1_{t_k} t_k + 1 \cdot X^1_{t_k} t_k + 1 - X^2_{t_k} t_k + 1)
\]

it suffices to bound

\[
\sum_{k} (X^1_{t_k} t_k + 1 \cdot X^1_{t_k} t_k + 1 - X^2_{t_k} t_k + 1)
\]

and

\[
\sum_{k} X^1_{t_k} (X^1_{t_k} t_k + 1 \cdot X^1_{t_k} t_k + 1 - X^2_{t_k} t_k + 1)
\]

Such transfer principle for limit theorems of weighted sums are obtained and has been applied to very general weighted sum.
Transfer principle (Liu-Tindel ’19)

- In order to bound a weighted sum $\sum_{k=0}^{n-1} y_{tk} h_{tk}^{n_{tk+1}}$ it suffices to consider the following elementary weighted sums:

$$\sum_{s \leq t_{k} < t} h_{tk}^{n_{tk+1}}, \quad \sum_{s \leq t_{k} < t} x_{st_{k}}^{1} h_{tk}^{n_{tk+1}}, \cdots \quad \sum_{s \leq t_{k} < t} x_{st_{k}}^{\ell} h_{tk}^{n_{tk+1}},$$

where $\ell$ is an integer depending on $X$ and $h^{n}$. 
Transfer principle (Liu-Tindel ‘19)

- In order to bound a weighted sum $\sum_{k=0}^{n-1} y_k h^n_{t_k t_{k+1}}$, it suffices to consider the following elementary weighted sums:

$$
\sum_{s \leq t_k < t} h^n_{t_k t_{k+1}}, \quad \sum_{s \leq t_k < t} x^{1}_{st_k} h^n_{t_k t_{k+1}}, \quad \ldots \quad \sum_{s \leq t_k < t} x^{\ell}_{st_k} h^n_{t_k t_{k+1}},
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$$\sum_k (x_{t_k t_{k+1}}^1 \cdot x_{t_k t_{k+1}}^1 - x_{t_k t_{k+1}}^2) \quad \text{and} \quad \sum_k x_{t_k}^1 (x_{t_k t_{k+1}}^1 \cdot x_{t_k t_{k+1}}^1 - x_{t_k t_{k+1}}^2)$$
Transfer principle (Liu-Tindel ’19)

• In order to bound a weighted sum \( \sum_{k=0}^{n-1} y_{t_k} h^n_{t_k t_{k+1}} \) it suffices to consider the following elementary weighted sums:

\[
\sum_{s \leq t_k < t} h^n_{t_k t_{k+1}}, \quad \sum_{s \leq t_k < t} x^{1}_{st_k} h^n_{t_k t_{k+1}}, \cdots \sum_{s \leq t_k < t} x^{\ell}_{st_k} h^n_{t_k t_{k+1}},
\]

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• These special weighted sums belong to finite Wiener chaos and are easier to handle.

• For example, in order to estimate \( l_2 = \frac{1}{2} \sum_k y'_{t_k} (x^{1}_{t_k t_{k+1}} \cdot x^{1}_{t_k t_{k+1}} - x^{2}_{t_k t_{k+1}}) \) it suffices to bound

\[
\sum_k (x^{1}_{t_k t_{k+1}} \cdot x^{1}_{t_k t_{k+1}} - x^{2}_{t_k t_{k+1}}) \quad \text{and} \quad \sum_k x^{1}_{t_k} (x^{1}_{t_k t_{k+1}} \cdot x^{1}_{t_k t_{k+1}} - x^{2}_{t_k t_{k+1}})
\]

• Such transfer principle for limit theorems of weighted sums are obtained and has been applied to very general weighted sum.
Let $f$ be a smooth function on $\mathbb{R}^m$. Define the midpoint rule:

$$m-J_0^T(f(X), X) = \sum_{k=0}^{n-1} f\left(\frac{X_{t_k} + X_{t_{k+1}}}{2}\right) \cdot \delta X_{t_k t_{k+1}}.$$
Let $f$ be a smooth function on $\mathbb{R}^m$. Define the midpoint rule:

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**Corollary**

Suppose that $\|\mathcal{J}\|_{\rho \text{-var}} < \infty$. Then as the mesh size of the partition $(t_k)$ goes to 0 we have

$$m-J_0^T(f(X), X) \to \int_{0}^{T} f(X_t) \, dX_t$$

in probability.

If we assume further that $\|\mathcal{J}\|_{\rho \text{-var}}; [s, t] \times [0, T] \leq C |t - s|$. Then the convergence holds almost surely.
Let $f$ be a smooth function on $\mathbb{R}^m$. Define the midpoint rule:

$$m-J^T_0(f(X), X) = \sum_{k=0}^{n-1} f\left(\frac{X_{t_k} + X_{t_{k+1}}}{2}\right) \cdot \delta X_{t_k} X_{t_{k+1}}.$$ 

**Corollary**

- Suppose that $\|R\|_{\rho\text{-var}} < \infty$. Then as the mesh size of the partition $(t_k)$ goes to 0 we have

$$m-J^T_0(f(X), X) \rightarrow \int_0^T f(X_t) dX_t \quad \text{in probability.}$$
Let $f$ be a smooth function on $\mathbb{R}^m$. Define the midpoint rule:

$$m-J_0^T(f(X), X) = \sum_{k=0}^{n-1} f\left( \frac{X_{t_k} + X_{t_{k+1}}}{2} \right) \cdot \delta X_{t_k t_{k+1}}.$$

**Corollary**

- Suppose that $\| R \|_{\rho\text{-var}} < \infty$. Then as the mesh size of the partition $(t_k)$ goes to 0 we have

$$m-J_0^T(f(X), X) \to \int_0^T f(X_t) dX_t \quad \text{in probability.}$$

- If we assume further that $\| R \|_{\rho\text{-var};[s,t] \times [0,T]} \leq C |t - s|$. Then the convergence holds almost surely.
Ingredients of proof:

For $a, b \in \mathbb{R}^d$ we consider the following mean value identity

$$\frac{f(a) + f(b)}{2} - f\left(\frac{a + b}{2}\right) = \frac{1}{2} \partial^2 f(c) \left(\frac{b - a}{2}\right)^\otimes 2,$$

where $c \in \mathbb{R}^3$ satisfies $c = a + \theta(b - a)$ for some $\theta \in [0, 1]$. 

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where $c \in \mathbb{R}^3$ satisfies $c = a + \theta(b - a)$ for some $\theta \in [0, 1]$.

- Apply the mean value identity with $a = X_{t_k}$ and $b = X_{t_{k+1}}$ to the difference
  $$\text{tr-}\mathcal{J}_0^T(f(X), X) - \text{m-}\mathcal{J}_0^T(f(X), X).$$

We will obtain some weighted sums similar to $l_3$ and $l_4$ in the previous proof.
Ingredients of proof:

For \( a, b \in \mathbb{R}^d \) we consider the following mean value identity

\[
\frac{f(a) + f(b)}{2} - f\left(\frac{a + b}{2}\right) = \frac{1}{2} \partial^2 f(c) \left(\frac{b - a}{2}\right)^\otimes 2,
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where \( c \in \mathbb{R}^3 \) satisfies \( c = a + \theta(b - a) \) for some \( \theta \in [0, 1] \).

- Apply the mean value identity with \( a = X_{t_k} \) and \( b = X_{t_{k+1}} \) to the difference

  \[
  \text{tr-} \mathcal{J}_0^T(f(X), X) - \text{m-} \mathcal{J}_0^T(f(X), X).
  \]

We will obtain some weighted sums similar to \( l_3 \) and \( l_4 \) in the previous proof.

- We conclude that the two numerical integral methods converge to the same limit.
Thank you very much for your attention!