Numerical Stochastic Integrations and Limit Theorems

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A joint work with Z. Selk and S. Tindel.

Outline

Presentation Outline

1. Elements of rough paths
2. Numerical methods for rough integrals
3. Numerical methods for stochastic rough integrals

Signatures

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$$\mathbf{x}^1_{st} := \int_s^t dx_u = x_t - x_s \qquad \mathbf{x}^2_{st} := \int_s^t \int_s^u dx_v \otimes dx_u$$

$$\mathbf{x}^3_{st} := \int_s^t \int_s^u \int_s^v dx_w \otimes dx_v \otimes dx_u$$

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- The n-th order signature of x:

$$S_n(x)_{st} = (\mathbf{x}^1_{st}, \mathbf{x}^2_{st}, \dots, \mathbf{x}^n_{st}).$$

It is known that for fixed s and t, the $S_n(x)_{st}$ contains all information of x for $n = \infty$.

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where $\mathcal{P}$ is the set of finite partitions of the time interval $[0, T]$.

- Denote x^ϵ: a mollification of x.
- Suppose that for $n = \lfloor p \rfloor$ the nth order signature $S_n(x^\epsilon)$ of x^ϵ converges under the p-variation norm and denote the limit by $S_n(x) := \mathbf{x}$. Then $\mathbf{x}$ is called a *p*-rough path.

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- We extend the Itô map I to p-variation rough paths. We define $\mathbf{y} = I(\mathbf{x})$ as the solution of the differential equation

$$d\mathbf{y}_t = V(y_t) d\mathbf{x}_t.$$

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- The rough paths framework provides the stability of the Itô map.
- The signature has been applied to model complex data streams.

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- Let (t_k) be a partition of $[0, T]$. For each k we consider the approximation

$$\int_{t_k}^{t_{k+1}} V(x_t)dx_t \approx \int_{t_k}^{t_{k+1}} V(x_{t_k})dx_t = V(x_{t_k})\delta x_{t_k t_{k+1}}.$$

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- Note that if $p \geq 2$ the Riemann sum $\sum_k V(x_{t_k})\delta x_{t_k t_{k+1}}$ diverges.

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- The compensated Riemann sum converges to the rough integral $\int_0^T V(x)dx$ when $n > p - 1$.
- Note that the compensated Riemann sum requires the computations of signatures of x.

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- Rough integration with respect to random fields have been studied in the following cases:
  - 2-dim Young integral: Towghi '01 and Quer-Sardanyons-Tindel '07
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- Let x and y be 2d Hölder functions on $[0, T]^2$ of order (α_1, α_2) and (β_1, β_2), respectively, and $\alpha_i + \beta_i > 1$, $i = 1, 2$. The Riemann sum

$$\sum_{(t_i),(t'_j)} y_{t_i t'_j} x^{t_i t_{i+1}}_{t'_i t'_{i+1}}$$

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- The Riemann sum is divergent when $\alpha_i + \beta_i < 1$, $i = 1$ or 2, and compensated Riemann sum is introduced.

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$$\|R\|_{\rho\text{-var}} = \sup_{(t_i),(t_j')} \left(\sum_{i,j} \left| R_{t_i t_{i+1}}^{t_j' t_{j+1}'} \right|^{\rho} \right)^{1/\rho},$$

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where (t_j) and (t'_j) are partitions on $[0, T]$.

- If R has finite ρ-variation for $\rho \in [1, 2)$, then X gives raise to a p-rough path, provided $p > 2\rho$. (Friz-Victoir '11).

- Consider a process y such that

$$\delta y_{st} = y'_s \mathbf{X}^1_{st} + y''_s \mathbf{X}^2_{st} + r^0_{st}, \qquad \delta y'_{st} = y''_s \mathbf{X}^1_{st} + r^1_{st},$$

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where y', y'', r^0, r^1 are processes satisfying some regularity conditions.

- y is called a controlled paths of X of order 2. Such processes contains most of the interesting examples. e.g. $y = V(X)$ or y is the solution of a RDE.

Define the trapezoid rule:

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Theorem (Liu-Selk-Tindel '21)

- Suppose that $\|R\|_{\rho\text{-var}} < \infty$. Then as the mesh size of the partition (t_k) goes to 0 we have

$$\text{tr-}\mathcal{J}_0^T(y, X) \to \int_0^T y_t d\mathbf{X}_t \qquad \text{in probability.}$$

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- Suppose that there exists a constant $C > 0$ such that $\|R\|_{\rho\text{-var};[s,t]\times[0,T]} \leq C|t-s|$ for all $[s,t] \subset [0,T]$. Then we have

$$\text{tr-}\mathcal{J}_0^T(y, X) \to \int_0^T y_t d\mathbf{X}_t \qquad \text{almost surely.}$$

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$$I_2 = \frac{1}{2} y'_{t_k} \mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - y'_{t_k} \cdot \mathbf{X}^2_{t_k t_{k+1}}$$

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- I_1 is the compensated Riemann sum of $\int_0^T y d\mathbf{X}$.

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I_4 &= \frac{1}{2} r^0_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}}.
\end{aligned}$$

- I_1 is the compensated Riemann sum of $\int_0^T y d\mathbf{X}$.
- I_2 and I_3 are weighted random sums of the forms: $\sum_k y'_{t_k} h^n_{t_k t_{k+1}}$ and $\sum_k y''_{t_k} \tilde{h}^n_{t_k t_{k+1}}$.

Ingredients of proof.

With a careful rearrangement of the trapezoid rule one can recast it as

$$\text{tr-}\mathcal{J}_0^T(y,X) = \sum_k I_1 + I_2 + I_3 + I_4,$$

where

$$\begin{aligned} I_1 &= y_{t_k} \cdot \mathbf{X}^1_{t_k t_{k+1}} + y'_{t_k} \cdot \mathbf{X}^2_{t_k t_{k+1}} + y''_{t_k} \cdot \mathbf{X}^3_{t_k t_{k+1}} \\ I_2 &= \frac{1}{2} y'_{t_k} \mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - y'_{t_k} \cdot \mathbf{X}^2_{t_k t_{k+1}} \\ I_3 &= \frac{1}{2} y''_{t_k} \mathbf{X}^2_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - y''_{t_k} \cdot \mathbf{X}^3_{t_k t_{k+1}} \\ I_4 &= \frac{1}{2} r^0_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}}. \end{aligned}$$

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- The convergences of I_i, $i = 2, 3, 4$ can be shown based on a transfer principle combined with some 2d young-type estimates.

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- In order to bound a weighted sum $\sum_{k=0}^{n-1} y_{t_k} h^n_{t_k t_{k+1}}$ it suffices to consider the following elementary weighted sums:

$$\sum_{s\le t_k<t} h^n_{t_k t_{k+1}}, \quad \sum_{s\le t_k<t} \mathbf{X}^1_{st_k} h^n_{t_k t_{k+1}}, \cdots \sum_{s\le t_k<t} \mathbf{X}^\ell_{st_k} h^n_{t_k t_{k+1}},$$

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- For example, in order to estimate $I_2 = \frac{1}{2}\sum_k y'_{t_k}(\mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - \cdot \mathbf{X}^2_{t_k t_{k+1}})$ it suffices to bound

$$\sum_k (\mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - \mathbf{X}^2_{t_k t_{k+1}}) \quad \text{and} \quad \sum_k \mathbf{X}^1_{t_k}(\mathbf{X}^1_{t_k t_{k+1}} \cdot \mathbf{X}^1_{t_k t_{k+1}} - \mathbf{X}^2_{t_k t_{k+1}})$$

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- Such transfer principle for limit theorems of weighted sums are obtained and has been applied to very general weighted sum.

Let f be a smooth function on $\mathbb{R}^m$. Define the midpoint rule:

$$\text{m-}\mathcal{J}_0^T(f(X), X) = \sum_{k=0}^{n-1} f\Big(\frac{X_{t_k} + X_{t_{k+1}}}{2}\Big) \cdot \delta X_{t_k t_{k+1}}.$$

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Corollary

- Suppose that $\|R\|_{\rho\text{-var}} < \infty$. Then as the mesh size of the partition (t_k) goes to 0 we have

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- If we assume further that $\|R\|_{\rho\text{-var};[s,t]\times[0,T]} \leq C|t-s|$. Then the convergence holds almost surely.

Ingredients of proof:

For $a, b \in \mathbb{R}^d$ we consider the following mean value identity

$$\frac{f(a) + f(b)}{2} - f\left(\frac{a+b}{2}\right) = \frac{1}{2}\partial^2 f(c)\left(\frac{b-a}{2}\right)^{\otimes 2},$$

where $c \in \mathbb{R}^3$ satisfies $c = a + \theta(b-a)$ for some $\theta \in [0,1]$.

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- Apply the mean value identity with $a = X_{t_k}$ and $b = X_{t_{k+1}}$ to the difference

$$\text{tr-}\mathcal{J}_0^T(f(X), X) - \text{m-}\mathcal{J}_0^T(f(X), X).$$

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- We conclude that the two numerical integral methods converge to the same limit.

Thank you very much for your attention!