Necessary and sufficient condition for the uniqueness/existence of hyperbolic Anderson equations with Gaussian noise that is fractional in time

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Introduction

In literature, the hyperbolic Anderson model (HAM) is (perhaps only symbolically) written as

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \dot{W}(t,x)u(t,x) \\ \\ u(0,x) = u_0(x) \ \text{ and } \ \dfrac{\partial u}{\partial t}(0,x) = u_1(x) \quad x \in \mathbb{R}^d \end{cases}$$

Introduction

In this talk, $\dot{W}(t,x)$ is a time-space generalized Gaussian field on $\mathbb{R}^+ \times \mathbb{R}^d$ with zero mean and the covariance

$$\mathrm{Cov}\left(\dot{W}(t,x), \dot{W}(s,y)\right) = \gamma_0(s-t)\gamma(x-y) \quad t,s \in \mathbb{R}^+, \quad x,y \in \mathbb{R}^d$$

In our main theorem,

$$\gamma_0(\cdot) = |\cdot|^{-\alpha_0}$$

for some $0 < \alpha_0 < 1$ and $\gamma(\cdot)$ is a non-negative (positively generalized) functon on $\mathbb{R}$ and on $\mathbb{R}^d$.

Introduction

With the covarince between $\dot{W}(t, x)$ and $\dot{W}(s, y)$ being given as the function of $(t - s, x - y)$, the Gaussian field $\dot{W}(t, x)$ is stationary in (t, x). It usually appears as the formal derivative

$$\dot{W}(t, x) = \frac{\partial^{d+1} W(t, x)}{\partial t \partial x_1 \cdots \partial x_d}$$

of a increment-stationary Gaussian field $W(t, x)$, such as the Brownian sheet and fractional Brownian sheet

A similar but much more understood system is the parabolic Anderson model (PAM), where the time derivative $\partial^2 u / \partial t^2$ is replaced by $\partial u / \partial t$

Introduction

The central topic of this talk is on the uniqueness and existence of the solution to HAM that will be properly defined later. More precisely, the focus is on the condition under which the uniqueness/existence holds. Our discussion is limited to the dimensions $d = 1, 2, 3$.

As the space covariance function, $\gamma(\cdot)$ has to be non-negative definite. Equivalently (Bochner theorem)

$$\gamma(x) = \int_{\mathbb{R}^d} e^{i\xi\cdot x}\mu(d\xi) \qquad x \in \mathbb{R}^d$$

for some measure $\mu(d\xi)$ on $\mathbb{R}^d$ that is known as the spectral measure of $\gamma(\cdot)$

Introduction

The spectral measure of $\gamma_0(\cdot) = c|\cdot|^{-\alpha_0}$ is $|\lambda|^{-(1-\alpha_0)}d\lambda$, i.e.,

$$\gamma_0(u) = c\frac{1}{|u|^{\alpha_0}} = \int_{\mathbb{R}} e^{i\lambda u}\frac{d\lambda}{|\lambda|^{1-\alpha_0}}$$

When $\dot{W}$ is white in time, i.e., $\gamma_0(\cdot) = c\delta_0(\cdot)$, the spectral measure of $\gamma_0(\cdot)$ is the Lebesgue measure on $\mathbb{R}$, i.e.,

$$\gamma_0(u) = c\delta_0(u) = \int_{\mathbb{R}} e^{i\lambda u}d\lambda$$

So the case $\gamma_0(\cdot) = c\delta_0(\cdot)$ can be viewed as a natural extension of $\gamma_0(\cdot) = c|\cdot|^{-\alpha_0}$ to the setting $\alpha_0 = 1$.

Introduction

It has been known (see, e.g., Balan, R. and Song, J. (2017)) that when $\dot{W}$ is white in time, the system has one and only one solution, provided that

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2}\mu(d\xi) < \infty.$$

This assumption is known as the Dalang's condition, introduced by Robert Dalang (1999) as the creteria for the solvability of the parabolic Anderson models (PAM) with Gaussian noise that is white in time.

Given the fact that $\mu(A) < \infty$ for any bounded $A \subset \mathbb{R}^d$, the Dalang's condition controls the tail of the measure $\mu(d\xi)$ at infinity, or, equivalently, the degree of the singularity of the spatial covariance $\gamma(x)$ at $x = 0$.

Introduction

The Dalang's condition has been dominating the whole area of the PAM in the sense that no solvability has been established without assuming the Dalong's condition even when $\dot{W}$ is not white in time.

Consequently, some practically interesting cases are excluded under the Dalang's condition. In the setting of PAM, for example, the noise $\dot{W}$ is not allowed to be white in space (i.e., $\gamma(\cdot) = \delta_0(\cdot)$) as soon as $d \geq 2$. The Dalang's condition is hard to be challengad as far as the model PAM is concerned.

Main theorem

In the setting of HAM, the system is solved under a different condition

Theorem

In the setting of fractional time, i.e., $\gamma_0(\cdot) = |\cdot|^{-\alpha_0}$ *for* $0 < \alpha_0 < 1$, *and of* $1 \leq \mathrm{d} \leq 3$, *the HAM has a unique solution (in a proper sense) if and only if*

$$\int_{\mathbb{R}^{\mathrm{d}}} \frac{1}{1 + |\xi|^{3-\alpha_0}} \mu(\mathrm{d}\xi) < \infty$$

Under the spatial homogenity

$$\gamma(\mathrm{cx}) = \mathrm{c}^{-\alpha}\gamma(\mathrm{x}) \quad \mathrm{c} > 0 \;\; \mathrm{x} \in \mathbb{R}^{\mathrm{d}},$$

in particular, the condition becomes $\alpha_0 + \alpha < 3$.

Remarks

Remark 1. Our condition agrees with the Dalang's condition when $\alpha_0 = 1$. Consequently, our result consistent with the one by Balan and Song for time-white noise.

Remark 2. As $0 < \alpha_0 < 1$ and $\gamma(\cdot) = \delta_0(\cdot)$ (i.e., the Gaussian noise is white in space), the spatial homogenity holds with $\alpha = \mathrm{d}$ and there the condition is $\alpha_0 + \mathrm{d} < 3$. Consequently, our main theorem embraces the interesting case when $\dot{\mathrm{W}}$ is a $(1+2)$-dimensional Gaussian noise that is white in space (i.e., $\gamma(\cdot) = \delta_0(\cdot)$), a setting tht is not included under the Dalang's condition, and is excluded (for a good reason) in the context of the PAM.

Mathematical set-up of HAM

Given $d \geq 1$, the $(1+d)$-stochastic wave equation

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \dot{W}(t,x)u(t,x) \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^d \\ u(0,x) = u_0(x) \text{ and } \dfrac{\partial u}{\partial t}(0,x) = u_1(x) \quad x \in \mathbb{R}^d \end{cases}$$

is defined by the integral equation

Mathematical set-up of HAM

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, y-x) u(s,y) W(dsdy)$$

where $G(t,x)$ is the solution of

$$\int_{\mathbb{R}^d} e^{i\xi\cdot x} G(t,x) dx = \frac{\sin(|\xi| t)}{|\xi|} \quad \xi \in \mathbb{R}^d$$

and

$$w(t,x) = \frac{\partial}{\partial t}\big(G(t,\cdot) * u_0\big)(x) + \big(G(t,\cdot) * u_1\big)(x)$$

In this talk, the stochastic integral on the right hand is defined in Ito-Skorodhod sense.

Fundamental solution

In the dimensions $d = 1, 2, 3$, $G(t, x)$ can be written explicitly as

$$G(t,x) = \begin{cases} \frac{1}{2} 1_{\{|x| \le t\}} & d = 1 \\ \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x| \le t\}} & d = 2 \\ \frac{1}{4\pi t} \sigma_t(dx) & d = 3 \end{cases}$$

where $\sigma_t(dx)$ is the surface measure on $\{x \in \mathbb{R}^3;\ |x| = t\}$.

Fundamental solution

Among other things, one can see that $G(t, x) \geq 0$ in $d = 1, 2, 3$. When $d \geq 4$, the situation is more complicated. In particular. $G(t, x)$ is sign-switch—that is a main reason we limit our investigation to $d = 1, 2, 3$.

Ito-Wiener chaos expansion

For simplicity, we consider the initial condition $u_0(x) = 1$ and $u_1(x) = 0$ which makes $w(t,x) \equiv 1$. So the equation takes the form

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, y-x) u(s,y) W(dsdy)$$

Iterating the equation we may expect

$$u(t,x) = \sum_{n=0}^{\infty} I_n(t,x)$$

with $I_0(t,x) = 1$.

Ito-Wiener chaos expansion

The above decomposition is orthogonal in $L_2(\Omega, \mathcal{A}, \mathbb{P})$. Formally,

$$\mathbb{E}\, u^2(t,x) = \mathbb{E}\, u^2(t,0) = 1 + \sum_{n=1}^{\infty} \mathbb{E}\left[I_n\left(\tilde{f}_n(t,0,\cdot)\right]^2\right.$$

$$= \mathbb{E}\, u^2(t,0) = 1 + \sum_{n=1}^{\infty} n! \|I_n\left(\tilde{f}_n(t,0,\cdot)\right)\|_{\mathcal{H}^{\otimes n}}^2$$

where the first equality follows from the fact that $u(t,x)$ is stationary in x under our initial condition, and

Ito-Wiener chaos expansion

$$
\begin{aligned}
&\left\| I_n\left(\tilde{f}_n(0,t,\cdot)\right)\right\|^2_{\mathcal{H}^{\otimes n}} \\
&= \frac{1}{(n!)^2}\int_{([0,t]^n_<)^2} d\mathbf{s} d\mathbf{r} \int_{(\mathbb{R}^d)^{2n}} d\mathbf{x} d\mathbf{y} \left(\prod_{k=1}^n \gamma_0(s_k - r_k)\gamma(y'_k - y'_k)\right) \\
&\times \left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n G(s_k - s_{k-1},\, x_{\sigma(k)} - x_{\sigma(k-1)})\right) \\
&\times \left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n G(r_k - r_{k-1},\, y_{\sigma(k)} - y_{\sigma(k-1)})\right)
\end{aligned}
$$

where Σ_n is the permutation group on $\{1, 2, \cdots, n\}$,

$$
[0,t]^n_< = \{(s_1, \cdots, s_n]^n; \;\; s_1 < s_2 < \cdots < s_n\}
$$

and we use the covention $s_0 = r_0 = 0$ and $x_{\sigma(0)} = y_{\sigma(0)} = 0$.

Here is what we mean by solving the equation

Thus, the existence/uniqueness (in $\mathcal{L}^2(\Omega)$) is equivalent to

$$\sum_{n=1}^{\infty} n! \| I_n\big(\tilde{f}_n(0,t,\cdot)\big) \|^2_{\mathcal{H}^{\otimes n}} < \infty \quad \text{for some/every } t > 0$$

Our job is to show that it holds if and only if

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^{3-\alpha_0}} \mu(d\xi) < \infty$$

Our condition is necessary

The "only if" part is to show that the condition is necessary for the first term of the series to be finite, i.e.,

$$\|I_1(\tilde{f}_1(0,t,\cdot))\|_{\mathcal{H}}^2 < \infty \quad \text{for some } t > 0$$

By Parseval identity,

$$\begin{aligned}
&\|I_1(\tilde{f}_1(0,t,\cdot))\|_{\mathcal{H}}^2 \\
&= \int_{[0,t]^2} dsdt \int_{\mathbb{R}^d\times\mathbb{R}^d} dxdy\gamma_0(s-r)\gamma(x-y)G(s,x)G(r,y) \\
&= \int_{\mathbb{R}^{d+1}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}}\left|\int_0^t e^{i\lambda s}\frac{\sin(|\xi|s)}{|\xi|}ds\right|^2 \\
&\geq \int_{\mathbb{R}^{d+1}} \frac{\mu(d\xi)}{|\xi|^2}\frac{d\lambda}{|\lambda|^{1-\alpha_0}}\left(\int_0^t \cos(\lambda s)\sin(|\xi|s)ds\right)^2
\end{aligned}$$

Our condition is necessary

The right hand side is equal to

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_{\mathbb{R}} \frac{d\lambda}{|\lambda|^{1-\alpha_0}} \left(\frac{1-\cos\left((|\xi|)t\right)}{2(|\xi|+\lambda)} + \frac{1-\cos\left((|\xi|-\lambda)t\right)}{2(|\xi|-\lambda)} \right)^2$$

$$\geq \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^2} \int_0^{|\xi|} \frac{d\lambda}{|\lambda|^{1-\alpha_0}} \left(\frac{1-\cos\left((|\xi|-\lambda)t\right)}{2(|\xi|-\lambda)} \right)^2$$

$$\geq \int_{\mathbb{R}^d} \frac{\mu(d\xi)}{|\xi|^{3-\alpha_0}} \int_0^{|\xi|} \left(\frac{1-\cos\left((|\xi|-\lambda)t\right)}{2(|\xi|-\lambda)} \right)^2 d\lambda$$

Our condition is necessary

By variable substitution

$$\int_0^{|\xi|} \left(\frac{1 - \cos\left((|\xi| - \lambda) t \right)}{2(|\xi| - \lambda)} \right)^2 \mathrm{d}\lambda = \int_0^{|\xi|} \left(\frac{1 - \cos(\lambda t)}{2\lambda} \right)^2 \mathrm{d}\lambda$$

In summary,

$$\begin{aligned} &\| I_1 \left(\tilde{f}_1(0, t, \cdot) \right) \|_{\mathcal{H}}^2 \\ &\geq \int_{\mathbb{R}^d} \frac{\mu(\mathrm{d}\xi)}{|\xi|^{3-\alpha_0}} \int_0^{|\xi|} \left(\frac{1 - \cos(\lambda t)}{2\lambda} \right)^2 \mathrm{d}\lambda \\ &\geq \left(\int_0^1 \left(\frac{1 - \cos(\lambda t)}{2\lambda} \right)^2 \mathrm{d}\lambda \right) \int_{\{|\xi| \geq 1\}} \frac{\mu(\mathrm{d}\xi)}{|\xi|^{3-\alpha_0}} \end{aligned}$$

Our condition is necessary

Therefore, $\|I_1(\tilde{f}_1(0,t,\cdot))\|_{\mathcal{H}}^2 < \infty$ implies

$$\int_{\{|\xi|\geq 1\}} \frac{\mu(d\xi)}{|\xi|^{3-\alpha_0}} < \infty$$

Notice that the $\mu(A)$ for any bounded $A \subset \mathbb{R}^d$. The above integrability is equivalent to our condition

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^{3-\alpha_0}} \mu(d\xi) < \infty$$

Our condition is sufficient

To establish

$$\sum_{n=1}^{\infty} n! \| I_n \left(\tilde{f}_n(t, 0, \cdot) \right) \|_{\mathcal{H}^{\otimes n}}^2 < \infty$$

all we need is to show that for any $t > 0$,

$$\lim_{n \to \infty} \frac{1}{n} \log n! \| I_n \left(\tilde{f}_n(t, 0, \cdot) \right) \|_{\mathcal{H}^{\otimes n}}^2 = -\infty$$

under the condition

$$\int_{\mathbb{R}^d} \frac{1}{1 + |\xi|^{3-\alpha_0}} \mu(d\xi) < \infty$$

Time exponentiation

Recall the representation

$$\|I_n(\tilde{f}_n(0,t,\cdot))\|^2_{\mathcal{H}^{\otimes n}}$$

$$= \frac{1}{(n!)^2}\int_{([0,t]^n_<)^2} d\mathbf{s}dr \int_{(\mathbb{R}^d)^{2n}} d\mathbf{x}dy \left(\prod_{k=1}^n \gamma_0(s_k - r_k)\gamma(y'_k - y'_k)\right)$$

$$\times \left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n G(s_k - s_{k-1},\, x_{\sigma(k)} - x_{\sigma(k-1)})\right)$$

$$\times \left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n G(r_k - r_{k-1},\, y_{\sigma(k)} - y_{\sigma(k-1)})\right)$$

Let τ and $\tilde{\tau}$ be two independent exponential times with the parameter n. Then $\tau \wedge \tilde{\tau}$ is exponential with parameter $2n$.

Time exponentiation

$$
\begin{aligned}
&2n\int_0^\infty dt e^{-2nt}\|I_n(\tilde{f}_n(0,t,\cdot))\|^2_{\mathcal{H}^{\otimes n}} \\
&= n^2\int_0^\infty\int_0^\infty dt d\tilde{t} e^{-n(t+\tilde{t})}\|I_n(\tilde{f}_n(0,t\wedge\tilde{t},\cdot))\|^2_{\mathcal{H}^{\otimes n}} \\
&= \frac{n^2}{(n!)^2}\int_{(\mathbb{R}^d)^{2n}} d\mathbf{x}d\mathbf{y}\int_0^\infty\int_0^\infty dt d\tilde{t} e^{-n(t+\tilde{t})} \\
&\times\int_{([0,t\wedge\tilde{t}]^n_<)^2} d\mathbf{s}d\mathbf{r}\left(\prod_{k=1}^n \gamma_0(s_k-r_k)\gamma(x_k-y_k)\right) \\
&\times\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n G(s_k-s_{k-1},\, x_{\sigma(k)}-x_{\sigma(k-1)})\right) \\
&\times\left(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n G(r_k-r_{k-1},\, y_{\sigma(k)}-y_{\sigma(k-1)})\right)
\end{aligned}
$$

Time exponentiation

By monotonicity in t and $\tilde{t}$, the right hand side is no greater than

$$\frac{n^2}{(n!)^2}\int_{(\mathbb{R}^d)^{2n}} d\mathbf{x}d\mathbf{y}\int_0^\infty\int_0^\infty dtd\tilde{t}e^{-n(t+\tilde{t})}$$
$$\times\int_{[0,t]^n_<\times[0,\tilde{t}]^n_<} d\mathbf{s}d\mathbf{r}\Big(\prod_{k=1}^n\gamma_0(s_k-r_k)\gamma(x_k-y_k)\Big)$$
$$\times\Big(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n G(s_k-s_{k-1},\, x_{\sigma(k)}-x_{\sigma(k-1)})\Big)$$
$$\times\Big(\sum_{\sigma\in\Sigma_n}\prod_{k=1}^n G(r_k-r_{k-1},\, y_{\sigma(k)}-y_{\sigma(k-1)})\Big)$$

Time exponentiation

where the inequality relies on the fact that $\gamma_0(\cdot), \gamma(\cdot), G(\cdot,\cdot) \geq 0$.

Performing Fourier transform on the right hand side we have

$$
\begin{aligned}
&2n \int_0^\infty dt e^{-2nt} \| I_n\left(\tilde{f}_n(0,t,\cdot)\right) \|^2_{\mathcal{H}^{\otimes n}} \\
&\leq \frac{1}{(n!)^2} \int_{(\mathbb{R}^{d+1})^n} \prod_{k=1}^n \mu(d\xi_k) \frac{d\lambda_k}{|\lambda_k|^{1-\alpha_0}} \\
&\times \Bigg| \sum_{\sigma \in \Sigma_n} n \int_0^\infty dt e^{-nt} \int_{[0,t]^n_<} d\mathbf{s} \exp\left\{ i \sum_{k=1}^n \lambda_{\sigma(k)} s_k \right\} \\
&\times \prod_{k=1}^n \frac{\sin\left((s_k - s_{k-1})(\xi_{\sigma(k)} + \cdots + \xi_{\sigma(n)}) \right)}{|\xi_{\sigma(k)} + \cdots + \xi_{\sigma(n)}|} \Bigg|^2
\end{aligned}
$$

Time exponentiation

By the identities (in the convention $s_0 = 0$)

$$n \int_0^\infty dt e^{-nt} \int_{[0,t]^n_<} d\mathbf{s} \prod_{k=1}^n \varphi_k(s_k - s_{k-1}) = \prod_{k=1}^n \int_0^\infty e^{-nt} \varphi_k(t) dt$$

and

$$\int_0^\infty e^{-(n-i\lambda)t} \sin(bt) dt = \frac{b}{(n - i\lambda)^2 + b^2}$$

Time exponentiation

$$
\begin{aligned}
& n \int_0^\infty dt e^{-nt} \int_{[0,t]^n_<} d\mathbf{s} \exp\left\{ i \sum_{k=1}^n \lambda_{\sigma(k)} s_k \right\} \\
& \times \prod_{k=1}^n \frac{\sin\left((s_k - s_{k-1})(\xi_{\sigma(k)} + \cdots + \xi_{\sigma(n)}) \right)}{|\xi_{\sigma(k)} + \cdots + \xi_{\sigma(n)}|} \\
& = \int_0^\infty dt e^{-nt} \int_{[0,t]^n_<} d\mathbf{s} \exp\left\{ i \sum_{k=1}^n \left(\sum_{j=k}^n \lambda_{\sigma(j)} \right) (s_k - s_{k-1} \right\} \\
& \times \prod_{k=1}^n \frac{\sin\left((s_k - s_{k-1})(\xi_{\sigma(k)} + \cdots + \xi_{\sigma(n)}) \right)}{|\xi_{\sigma(k)} + \cdots + \xi_{\sigma(n)}|} \\
& = \prod_{k=1}^n \int_0^\infty \exp\left\{ - \left(n - i \sum_{j=k}^n \lambda_{\sigma(j)} \right) t \right\} \frac{\sin\left((t(\xi_{\sigma(k)} + \cdots + \xi_{\sigma(n)}) \right)}{|\xi_{\sigma(k)} + \cdots + \xi_{\sigma(n)}|} dt
\end{aligned}
$$

Time exponentiation

$$= \prod_{k=1}^{n} \left\{ \left(n - i \sum_{j=k}^{n} \lambda_{\sigma(j)} \right)^2 + \left| \sum_{j=k}^{n} \xi_{\sigma(j)} \right|^2 \right\}^{-1}$$

Hence,

$$2n \int_0^\infty dt e^{-2nt} \| I_n\left(\tilde{f}_n(0, t, \cdot)\right) \|^2_{\mathcal{H}^{\otimes n}}$$

$$\leq \frac{1}{(n!)^2} \int_{(\mathbb{R}^{d+1})^n} \prod_{k=1}^{n} \mu(d\xi_k) \frac{d\lambda_k}{|\lambda_k|^{1-\alpha_0}}$$

$$\times \left| \sum_{\sigma \in \Sigma_n} \prod_{k=1}^{n} \left\{ \left(n - i \sum_{j=k}^{n} \lambda_{\sigma(j)} \right)^2 + \left| \sum_{j=k}^{n} \xi_{\sigma(j)} \right|^2 \right\}^{-1} \right|^2$$

Removal of permutation

By Jensen's inequality, the right hand side is bounded by

$$\frac{1}{(n!)^2} n! \int_{(\mathbb{R}^{d+1})^n} \prod_{k=1}^{n} \mu(d\xi_k) \frac{d\lambda_k}{|\lambda_k|^{1-\alpha_0}}$$
$$\times \sum_{\sigma \in \Sigma_n} \left| \prod_{k=1}^{n} \left\{ \left(n - i \sum_{j=k}^{n} \lambda_{\sigma(j)} \right)^2 + \left| \sum_{j=k}^{n} \xi_{\sigma(j)} \right|^2 \right\}^{-1} \right|^2$$
$$= \int_{(\mathbb{R}^{d+1})^n} \prod_{k=1}^{n} \mu(d\xi_k) \frac{d\lambda_k}{|\lambda_k|^{1-\alpha_0}} \prod_{k=1}^{n} \left| \left(n - i \sum_{j=k}^{n} \lambda_j \right)^2 + \left| \sum_{j=k}^{n} \xi_j \right|^2 \right|^{-2}$$

Our next step is to separate the high multiple integral on the right hand side.

Variable-separation

Notice that

$$
\begin{aligned}
&\left\{(n - i\lambda)^2 + |\xi|^2\right\}^{-1} \\
&= \int_0^\infty dt e^{-nt} \int_{\mathbb{R}^d} dx \exp\{i(\lambda t + \xi \cdot x)\} G(t, x) \\
&= \int_{\mathbb{R}^{d+1}} \exp\left\{i(\lambda u + \xi \cdot x)\right\} 1_{[0,\infty)}(u) e^{-nu} G(u, x) dx dt \\
&= \int_{\mathbb{R}^{d+1}} \exp\left\{i(\lambda u + \xi \cdot x)\right\} \varphi(u, x) du dx \quad \text{(say)}
\end{aligned}
$$

with $\varphi(u, x) \geq 0$.

Variable-separation

So we have

$$\left|(n - i\lambda)^2 + |\xi|^2\right|^{-2}$$
$$= \int_{\mathbb{R}^{d+1}} \exp\{i(\lambda u + \xi \cdot x)\}\tilde{\varphi}(u, x)dudx$$

where $\tilde{\varphi}(\cdot) = \varphi(\cdot) * \varphi(-\cdot)$ is non-negative and symmetric.

Variable-separation

For any $(\lambda_0, \xi_0) \in \mathbb{R}^{d+1}$, by Parseval identity,

$$\int_{\mathbb{R}^{d+1}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}}\left|\big(n - i(\lambda_0+\lambda)\big)^2 + |\xi_0+\xi|^2\right|^{-2}$$
$$= \int_{\mathbb{R}^{d+1}} \gamma_0(u)\gamma(x)\tilde{\varphi}(u,x)\exp\Big\{i(\lambda_0 u + \xi_0\cdot x)\Big\}dudx$$
$$\le \int_{\mathbb{R}^{d+1}} \gamma_0(u)\gamma(x)\tilde{\varphi}(u,x)dudx$$
$$= \int_{\mathbb{R}^{d+1}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}}\left|\big(n - i\lambda\big)^2 + |\xi|^2\right|^{-2}$$

Variable-separation

Apply it with Fubini theorem,

$$\int_{(\mathbb{R}^{d+1})^n} \prod_{k=1}^{n} \mu(d\xi_k) \frac{d\lambda_k}{|\lambda_k|^{1-\alpha_0}} \prod_{k=1}^{n} \left| \left(n - i\sum_{j=k}^{n} \lambda_j\right)^2 + \left|\sum_{j=k}^{n} \xi_j\right|^2 \right|^{-2}$$

$$= \int_{(\mathbb{R}^{d+1})^{n-1}} \prod_{k=2}^{n} \mu(d\xi_k) \frac{d\lambda_k}{|\lambda_k|^{1-\alpha_0}} \prod_{k=2}^{n} \left| \left(n - i\sum_{j=k}^{n} \lambda_j\right)^2 + \left|\sum_{j=k}^{n} \xi_j\right|^2 \right|^{-2}$$

$$\times \int_{\mathbb{R}^{d+1}} \mu(d\xi_1) \frac{d\lambda_1}{|\lambda_1|^{1-\alpha_0}} \left| \left(n - i\sum_{j=1}^{n} \lambda_j\right)^2 + \left|\sum_{j=1}^{n} \xi_j\right|^2 \right|^{-2}$$

$$\leq \left(\int_{\mathbb{R}^{d+1}} \mu(d\xi) \frac{d\lambda}{|\lambda|^{1-\alpha_0}} \left| (n - i\lambda)^2 + |\xi|^2 \right|^{-2} \right)$$

$$\times \int_{(\mathbb{R}^{d+1})^{n-1}} \prod_{k=2}^{n} \mu(d\xi_k) \frac{d\lambda_k}{|\lambda_k|^{1-\alpha_0}} \prod_{k=2}^{n} \left| \left(n - i\sum_{j=k}^{n} \lambda_j\right)^2 + \left|\sum_{j=k}^{n} \xi_j\right|^2 \right|^{-2}$$

Variable-separation

Repeating this game,

$$\int_{(\mathbb{R}^{d+1})^n} \prod_{k=1}^{n} \mu(d\xi_k) \frac{d\lambda_k}{|\lambda_k|^{1-\alpha_0}} \prod_{k=1}^{n} \left| \left(n - i \sum_{j=k}^{n} \lambda_j\right)^2 + \left|\sum_{j=k}^{n} \xi_j\right|^2 \right|^{-2}$$
$$\leq \left(\int_{\mathbb{R}^{d+1}} \mu(d\xi) \frac{d\lambda}{|\lambda|^{1-\alpha_0}} \left| (n - i\lambda)^2 + |\xi|^2 \right|^{-2} \right)^n$$

In summary,

$$2n \int_0^\infty dt e^{-2nt} \| I_n\left(\tilde{f}_n(0,t,\cdot)\right) \|^2_{\mathcal{H}^{\otimes n}}$$
$$\leq \left(\int_{\mathbb{R}^{d+1}} \mu(d\xi) \frac{d\lambda}{|\lambda|^{1-\alpha_0}} \left| (n - i\lambda)^2 + |\xi|^2 \right|^{-2} \right)^n$$

High moment bounds

As the final step we shall prove that

$$\int_{\mathbb{R}^{d+1}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}}\Big|(n - i\lambda)^2 + |\xi|^2\Big|^{-2} = o\Big(\frac{1}{n}\Big) \quad (n \to \infty)$$

under our condition. Together with the established moment bound, this leads to

$$\lim_{n\to\infty} \frac{1}{n} \log n^n(2n) \int_0^\infty ds e^{-2ns} \|I_n\big(\tilde{f}_n(0, s, \cdot)\big)\|^2_{\mathcal{H}^{\otimes n}} = -\infty$$

High moment bounds

For any $t > 0$, on the other hand,

$$2n\int_0^\infty dse^{-2ns}\|I_n(\tilde{f}_n(0,s,\cdot))\|^2_{\mathcal{H}^{\otimes n}} \geq 2n\int_t^\infty dse^{-2ns}\|I_n(\tilde{f}_n(0,s,\cdot))\|^2_{\mathcal{H}^{\otimes n}}$$
$$\geq \|I_n(\tilde{f}_n(0,t,\cdot))\|^2_{\mathcal{H}^{\otimes n}} 2n\int_t^\infty e^{-2ns}ds = e^{-2nt}\|I_n(\tilde{f}_n(0,t,\cdot))\|^2_{\mathcal{H}^{\otimes n}}$$

This proves that

$$\lim_{n\to\infty}\frac{1}{n}\log n^n\|I_n(\tilde{f}_n(0,t,\cdot))\|^2_{\mathcal{H}^{\otimes n}} = -\infty$$

So our claim follows from the relation $n^n \geq n!$. □

Here is where the condition is used

It remains to prove

Lemma

In the assumption

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^{3-\alpha_0}} \mu(d\xi) < \infty$$

we have

$$\lim_{n\to\infty} n \int_{\mathbb{R}^{d+1}} \mu(d\xi) \frac{d\lambda}{|\lambda|^{1-\alpha_0}} \left| (n - i\lambda)^2 + |\xi|^2 \right|^{-2} = 0$$

Here is where the condition is used

Proof. A direct algebra gives

$$\begin{aligned}
&\left|\left(n - i\lambda\right)^2 + |\xi|^2\right|^2 \\
&= n^4 + (|\xi| + |\lambda|)^2(|\xi| - |\lambda|)^2 + 2n^2(|\xi|^2 + \lambda^2) \\
&\geq n^4 + (|\xi| + |\lambda|)^2\left(n^2 + (|\xi| - |\lambda|)^2\right)
\end{aligned}$$

So it suffices to show

$$\lim_{n\to\infty} \int_{\mathbb{R}^{d+1}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}} \frac{n}{n^4 + (|\xi| + |\lambda|)^2\left(n^2 + (|\xi| - |\lambda|)^2\right)} = 0$$

Here is where the condition is used

Let $a > 0$ be fixed but arbitrary.

$$\int_{\{|\xi|\le a\}\times\mathbb{R}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}} \frac{n}{n^4 + (|\xi|+|\lambda|)^2\big(n^2+(|\xi|-|\lambda|)^2\big)}$$
$$\le \mu(\{|\xi|\le a\})\int_{\mathbb{R}} \frac{d\lambda}{|\lambda|^{1-\alpha_0}}\frac{n}{n^4+n^2|\lambda|^2} \longrightarrow 0 \quad (n\to\infty)$$

In addition, consider the decomposition

$$\int_{\{|\xi|> a\}\times\mathbb{R}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}} \frac{n}{n^4 + (|\xi|+|\lambda|)^2\big(n^2+(|\xi|-|\lambda|)^2\big)}$$
$$= \left\{\int_{\{|\xi|>a,\ |\lambda|\le 2^{-1}|\xi|\}} + \int_{\{|\xi|>a,\ |\lambda|> 2^{-1}|\xi|\}}\right\}\mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}}$$
$$\times \frac{n}{n^4 + (|\xi|+|\lambda|)^2\big(n^2+(|\xi|-|\lambda|)^2\big)}$$

Here is where the condition is used

For the first term on the right,

$$\int_{\{|\xi|>a,\ |\lambda|\le 2^{-1}|\xi|\}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}}\frac{n}{n^4+(|\xi|+|\lambda|)^2\big(n^2+(|\xi|-|\lambda|)^2\big)}$$

$$\le \int_{\mathbb{R}^d} \mu(d\xi)\frac{n}{n^4+|\xi|^2\big(n^2+4^{-1}|\xi|^2\big)}\int_{-\frac{1}{2}|\xi|}^{\frac{1}{2}|\xi|}\frac{d\lambda}{|\lambda|^{1-\alpha_0}}$$

$$=\frac{1}{\alpha_0}\int_{\mathbb{R}^d}\mu(d\xi)\frac{n|\xi|^{\alpha_0}}{n^4+|\xi|^2\big(n^2+4^{-1}|\xi|^2\big)}$$

$$\le\frac{1}{\alpha_0}\int_{\mathbb{R}^d}\mu(d\xi)\frac{n|\xi|^{\alpha_0}}{n^4+|\xi|^2\cdot n|\xi|}=\frac{1}{\alpha_0}\int_{\mathbb{R}^d}\mu(d\xi)\frac{|\xi|^{\alpha_0}}{n^3+|\xi|^3}\longrightarrow 0$$

as $n\to\infty$.

Here is where the condition is used

As for the second term

$$\int_{\{|\xi|>a,\ |\lambda|>2^{-1}|\xi|\}} \mu(d\xi)\frac{d\lambda}{|\lambda|^{1-\alpha_0}}\frac{n}{n^4+(|\xi|+|\lambda|)^2\big(n^2+(|\xi|-|\lambda|)^2\big)}$$

$$\le \int_{\{|\xi|>a\}} \mu(d\xi)\int_{|\lambda|>2^{-1}|\xi|\}} \frac{d\lambda}{|\lambda|^{1-\alpha_0}}\frac{n}{(|\xi|+|\lambda|)^2\big(n^2+(|\xi|-|\lambda|)^2\big)}$$

$$\le 2^{1-\alpha_0}\int_{\{|\xi|>a\}} \frac{\mu(d\xi)}{|\xi|^{3-\alpha_0}}\int_{\mathbb{R}} \frac{n d\lambda}{n^2+(|\xi|-|\lambda|)^2}$$

Here is where the condition is used

Notice that

$$\int_{\mathbb{R}} \frac{\mathrm{n} d\lambda}{\mathrm{n}^2 + (|\xi| - |\lambda|)^2} = 2 \int_0^\infty \frac{\mathrm{n} d\lambda}{\mathrm{n}^2 + (\lambda - |\xi|)^2}$$

$$\leq 2 \int_{\mathbb{R}} \frac{\mathrm{n} d\lambda}{\mathrm{n}^2 + \lambda^2} = 2\pi$$

So we have

$$\int_{\{|\xi|>\mathrm{a},\ |\lambda|>2^{-1}|\xi|\}} \mu(d\xi) \frac{d\lambda}{|\lambda|^{1-\alpha_0}} \frac{\mathrm{n}}{\mathrm{n}^4 + (|\xi| + |\lambda|)^2 \left(\mathrm{n}^2 + (|\xi| - |\lambda|)^2\right)}$$

$$\leq 2^{2-\alpha_0} \pi \int_{\{|\xi|>\mathrm{a}\}} \frac{\mu(d\xi)}{|\xi|^{3-\alpha_0}}$$

Here is where the condition is used

In summary, we have reached

$$\limsup_{n\to\infty} \int_{\mathbb{R}^{d+1}} \mu(d\xi) \frac{d\lambda}{|\lambda|^{1-\alpha_0}} \frac{n}{n^4 + (|\xi|+|\lambda|)^2 \left(n^2 + (|\xi|-|\lambda|)^2\right)}$$

$$\leq 2^{2-\alpha_0} \pi \int_{\{|\xi|>a\}} \frac{\mu(d\xi)}{|\xi|^{3-\alpha_0}}$$

Letting $a \to \infty$ on the right hand side complete the proof. □

Open problems

Right conditions for the setting of $\mathrm{d} \geq 4$ or rough Gaussian noise where the non-negativity or definite non-negativity is absent?

See the recent work by Song, J., Song, X. and Xu, F. (2020) for a partial result where the system with rough Gaussian noise is solved undr the Dalang's condition.

Open problems

Our key estimate is

$$\lim_{n\to\infty} \frac{1}{n!} \log n! \| I_n(\tilde{f}_n(0,t,\cdot)) \|^2_{\mathcal{H}^{\otimes n}} = -\infty$$

It should be pointed out this is not a sharp bound. Under the space homogenity $\gamma(cx) = c^{-\alpha}\gamma(x)$. our condition becomes $\alpha_0 + \alpha < 3$. In this case one can show (by modifying our computation) that

$$\limsup_{n\to\infty} \frac{1}{n!} \log(n!)^{4-\alpha-\alpha_0} \| I_n(\tilde{f}_n(0,t,\cdot)) \|^2_{\mathcal{H}^{\otimes n}} < \infty$$

which is sharper as $4 - \alpha - \alpha_0 > 1$ under our condition.

Open problems

Further, under the Dalang's condition $\alpha < 2$, it is known (e.g., Balan-Conus (2016)) that

$$\limsup_{n\to\infty} \frac{1}{n!} \log(n!)^{4-\alpha} \| I_n\big(\tilde{f}_n(0, t, \cdot)\big) \|^2_{\mathcal{H}^{\otimes n}} < \infty$$

The evidence suggests that the exponent $4 - \alpha$ is the right one. The question is: Can this be extended to the setting $\alpha + \alpha_0 < 3$?

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Thank you!