Local Times and Geometric Properties of Gaussian Random Fields

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Outline

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1. Local times: existence and joint continuity

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-vector field. For any Borel set $T \subseteq \mathbb{R}^N$, the *occupation measure* of X on T is defined by

$$\mu_T(\bullet) = \lambda_N\{t \in T : X(t) \in \bullet\}.$$

If $\mu_T \ll \lambda_d$, then X is said to have a *local time* on T, which is defined by

$$L(x, T) = \frac{d\mu_T}{d\lambda_d}(x),$$

where x is the *space variable*, and T is the *time variable*. $L(x, T)$ satisfies the following *occupation density formula*: For every Borel set $T \subseteq \mathbb{R}^N$ and for every measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\int_T f(X(t))\, dt = \int_{\mathbb{R}^d} f(x) L(x, T)\, dx. \tag{1}$$

Joint continuity

Suppose we fix an interval $I = \prod_{\ell=1}^{N}[a_\ell,\, b_\ell]$ in $\mathbb{R}^N$. Let $T = \prod_{\ell=1}^{N}[a_\ell,\, t_\ell] \subset I$. If we can choose a version of the local time, still denoted by $L(x, T)$, such that it is continuous in $(x, t_1, \ldots, t_N) \in \mathbb{R}^d \times I$, then X is said to have a *jointly continuous local time* on I.

- The smoother the local time, the rougher the sample path (Berman, 1972).
- When a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on

$$X^{-1}(x) = \{t \in I : X(t) = x\}$$

and is a useful tool for studying fractal properties of $X^{-1}(x)$.

Other applications of local times

Local times also appear naturally in limit theorems involving:

- random walks in random scenery: Kesten and Spitzer (1979), Khoshnevisan and Lewis (1998, 1999)
- random rewards and self-similar stable processes: Cohen and Samorodnitsky (2006), Owada and Samorodnitsky (2015)
- functionals of integrated and fractionally integrated time series, nonlinear cointegrating regression: Wang and Phillips (2009, 2016)

Our setting

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an (N, d)-Gaussian random field defined by

$$X(t) = \big(X_1(t), \dots, X_d(t)\big), \tag{2}$$

where $X_1, \dots, X_d$ are independent copies of a real-valued Gaussian random field X_0.

We study the following questions:

- The existence and joint continuity of local times of X.
- Hölder conditions for the local times of X and apply these results to study the sample path properties of X.

Denote

$$\sigma^2(s,t) = \mathbb{E}\big(X_0(s) - X_0(t)\big)^2.$$

Given constants $0 < H_1 \le \ldots \le H_N \le 1$, define a metric ρ on $\mathbb{R}^N$:

$$\rho(s,t) = \sum_{j=1}^N |s_j - t_j|^{H_j}, \qquad \forall\, s,t \in \mathbb{R}^N. \tag{3}$$

Let $I = \prod_{j=1}^N [a_j,\, b_j]$ be an interval in $\mathbb{R}^N$, say, $I = [0,1]^N$ or $I = [\varepsilon, 1]^N$.

Existence of local times

The problem on existence of local times of random fields can be studied in several ways. The following result can be proved by applying Theorem 21.9 in Geman and Horowitz (1980).

Theorem 1.1

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (2) such that

$$\mathbb{E}\Big[\big(X_0(s) - X_0(t)\big)^2\Big] \asymp \rho(s,t)^2, \quad \text{for } \ s,t \in T. \tag{4}$$

Then X has an $L^2(\mathbb{P} \times \lambda_d)$ local time if and only if $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$.

Joint continuity of local times

For proving joint continuity of local times, we use the properties of local nondeterminism, in addition to (4).
Here we consider the following two of them:

Condition (C-1) [sectorial local nondeterminism]

For a constant vector $H = (H_1, \ldots, H_N) \in (0, 1)^N$, there exists a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\Big(X_0(u) \,\Big|\, X_0(t^1), \ldots, X(t^n)\Big) \geq c \sum_{j=1}^{N} \min_{1 \leq k \leq n} \big|u_j - t_j^k\big|^{2H_j}. \tag{5}$$

Condition (C-2) [strong ρ-local nondeterminism]

There exists a constant $c > 0$ such that $\forall\ n \geq 1$ and $u, t^1, \ldots, t^n \in I$,

$$\mathrm{Var}\Big(X_0(u) \,\Big|\, X_0(t^1), \ldots, X_0(t^n)\Big) \geq c_{1,5} \min_{0 \leq k \leq n} \rho(u,\ t^k)^2,$$

where $t^0 = 0$ and ρ is the metric on $\mathbb{R}^N$ defined in (3).

Remark: Condition (C-2) implies (C-1). We will show that the solution to the stochastic heat equation with an additive Gaussian noise that is white in time and colored in space satisfies (C-2).

Theorem 1.2 [Ayache, Wu and X. (2008), Wu and X. (2011)]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (2) such that (4) and (C-1) hold. If $\sum_{\ell=1}^N \frac{1}{H_\ell} > d$ then the local time of X is jointly continuous on $I \times \mathbb{R}^d$.

The proof of this theorem relies on Kolmogorov's continuity theorem and the moment estimates for $L(t, D)$ and $L(x, D) - L(y, D)$.

In fact, one can prove the joint continuity of local times under weaker conditions than (C-1). One of the advantages of (C-1) and (C-2) is to allow us to prove more precise results.

2. Hölder conditions for local times

Assume $0 < H_1 \le \ldots \le H_N < 1$ and $Q = \sum_{\ell=1}^N \frac{1}{H_\ell} > d$. There exists a unique $\tau \in \{1, \ldots, N\}$ such that $\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \le d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}$.

Under (C-1), Wu and X. (2011) studied Hölder conditions for the maximum local time $L^*(D) = \sup_{x\in\mathbb{R}^d} L(x, D)$. They distinguished three cases:

$$\text{Case 1:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} < d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell},$$

$$\text{Case 2:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} = d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell} \quad \text{and } H_{\tau-1} = H_\tau,$$

$$\text{Case 3:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} = d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell} \quad \text{and } H_{\tau-1} < H_\tau.$$

The following result is on local Hölder condition of $L^*(\overline{B}_\rho(a, r$
where $\overline{B}_\rho(a, r) \subset I$ denotes a ρ-ball of radius r.

Theorem 2.1 [Wu and X., 2011]

Under Condition (4) and (C-1), there exists a constant $c_{2,1} > 0$ such that for every $a \in I$,

$$\limsup_{r\to 0} \frac{L^*(\overline{B}_\rho(a, r))}{\varphi_1^\rho(r)} \le c_{2,1}, \qquad \text{a.s.} \quad \text{Cases 1 \& 2,}$$

$$\limsup_{r\to 0} \frac{L^*(\overline{B}_\rho(a, r))}{\varphi_2^\rho(r)} \le c_{2,1}, \qquad \text{a.s.} \quad \text{Case 3.}$$

In the above,

$$\varphi_1^\rho(r) = r^\alpha \left(\log\log(1/r)\right)^{\eta_\tau},$$
$$\varphi_2^\rho(r) = r^\alpha \left(\log\log(1/r)\right)^{\tau-1} \log\log\log\left(1/r\right),$$

where $\alpha = Q - d$ and $\eta_\tau = \tau + H_\tau d - \sum_{\ell=1}^{\tau} \frac{H_\tau}{H_\ell}$.

To state the uniform Hölder condition, let

$$\Phi_1^\rho(r) = r^\alpha \left(\log(1/r)\right)^{\eta_\tau},$$
$$\Phi_2^\rho(r) = r^\alpha \left(\log(1/r)\right)^{\tau-1} \log\log\left(1/r\right).$$

The following is a uniform Hölder condition for $L^*(\overline{B}_\rho(a,r))$.

Theorem 2.2 [Wu and X., 2011]

$$\limsup_{r\to 0}\sup_{a\in I}\frac{L^*(\overline{B}_\rho(a,r))}{\Phi_1^\rho(r)}\le c_{2,2}, \qquad \text{a.s.} \quad \text{Cases 1 \& 2},$$

$$\limsup_{r\to 0}\sup_{a\in I}\frac{L^*(\overline{B}_\rho(a,r))}{\Phi_2^\rho(r)}\le c_{2,2}, \qquad \text{a.s.} \quad \text{Case 3}.$$

3. Optimal Hölder conditions

We can establish optimal Hölder conditions for the local times under strong local nondeterminism (C-2).

Lemma 3.1 [Khoshnevisan, Lee, and X. (2022)]

Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (2) such that (4) and (C-2) hold. If $Q > d$, then there exists a finite constant C such that for all Borel sets $D \subset I$, all $x \in \mathbb{R}^d$ and all integers $n \geq 1$, we have

$$\mathbb{E}\big[L(x,D)^n\big] \leq C^n (n!)^{d/Q} \lambda_N(D)^{n(1-d/Q)}.$$

In particular, for all $a \in T$ and $r \in (0,1)$ with $B_\rho(a,r) \subset I$, we have

$$\mathbb{E}\big[L(x,B_\rho(a,r))^n\big] \leq C^n (n!)^{d/Q} r^{n(Q-d)}.$$

Proof Recall that [e.g., Geman and Horowitz (1980)] for all $x \in \mathbb{R}^d$, any Borel set $D \subseteq \mathbb{R}^N$ and integer $n \geq 1$,

$$\mathbb{E}\Big[L(x, D)^n\Big] = (2\pi)^{-nd} \int_{D^n} \int_{\mathbb{R}^{nd}} \exp\bigg(- i \sum_{j=1}^{n} \langle u^j, x\rangle \bigg) \times \mathbb{E} \exp\bigg(i \sum_{j=1}^{n} \langle u^j, X(t^j)\rangle \bigg)\, d\bar{u}\, d\bar{t},$$

where

$$\bar{u} = (u^1, \ldots, u^n) \in \mathbb{R}^{nd}, \quad \bar{t} = (t^1, \ldots, t^n) \in D^n.$$

We see that $\mathbb{E}\left[L(x,D)^n\right]$ is at most

$$\int_{D^n} \prod_{k=1}^{d} \left\{ \int_{\mathbb{R}^n} \exp\left[-\frac{1}{2} \mathrm{Var}\Big(\sum_{j=1}^{n} u_k^j X_0(t^j) \Big) \right] d\overline{u}_k \right\} d\overline{t}$$

$$= \int_{D^n} \Big[\det\mathrm{Cov}\left(X_0(t^1), \ldots, X_0(t^n) \right) \Big]^{-\frac{d}{2}} d\overline{t},$$

where $\overline{u}_k = (u_k^1, \ldots, u_k^n) \in \mathbb{R}^n$, $\overline{t} = (t^1, \ldots, t^n)$ and the equality follows from the fact that for any positive definite $n \times n$ matrix Γ,

$$\int_{\mathbb{R}^n} \frac{[\det(\Gamma)]^{1/2}}{(2\pi)^{n/2}} \exp\Big(-\frac{1}{2} x'\Gamma x \Big)\, dx = 1.$$

By using the fact that for any Gaussian vector $(Z_1, \ldots, Z_n)$

$$\det \mathrm{Cov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{j=1}^{n} \mathrm{Var}(Z_j | Z_1, \ldots, Z_{j-1}).$$

and Condition (C-2) we derive

$$\mathbb{E}\big[L(x, D)^n\big] \le c^n \int_{D^n} \prod_{m=1}^{n} \Big[\min_{0 \le k \le m-1} \rho(t^m, t^k)\Big]^{-d} d\bar{t}. \tag{6}$$

To estimate the last integral in (6), we will use the following technical lemma.

Lemma 3.2

Let $0 < d \le \beta_0 < Q$ be given constants. There is a finite constant $C = C(N, d, Q, \beta_0)$ such that for all subsets $D \subset I$, $\beta \in [d, \beta_0]$, all integers $m \ge 1$, and $t^1, \ldots, t^m \in D$, we have

$$\int_D \Big[\min_{0 \le k \le m-1} \rho(t, t^k) \Big]^{-\beta} dt \le C m^{\beta/Q} \lambda_N(D)^{1-\beta/Q}. \qquad (7)$$

In particular, for all $a \in \mathbb{R}^N$ and $0 < r < 1$ with $t^1, \ldots, t^m \in B_\rho(a, r) \subset I$, we have

$$\int_{B_\rho(a,r)} \Big[\min_{0 \le k \le m-1} \rho(t, t^k) \Big]^{-\beta} dt \le C m^{\beta/Q} r^{Q-\beta}. \qquad (8)$$

If we integrate (6) in the order of $dt^n, dt^{n-1}, \ldots, dt^1$, and apply Lemma 3.2 with $\beta = d$, repeatedly, we deduce that

$$\mathbb{E}[L(x,D)^n] \le C^n (n!)^{d/Q} \lambda_N(D)^{n(1-d/Q)}.$$

This completes the proof of Lemma 3.1.

Lemma 3.3

Under the conditions of Lemma 3.1, there exist constants C and K such that for all $\gamma \in (0,1)$ small enough, for all Borel sets $D \subseteq T$, for all $x, y \in \mathbb{R}^d$, for all even integers $n \geq 2$, we have

$$\mathbb{E}\Big[\big(L(x,D) - L(y,D)\big)^n\Big] \\ \leq C^n |x-y|^{n\gamma} (n!)^{d/Q+K\gamma} \lambda_N(D)^{n(1-(d+\gamma)/Q)}.$$

In particular, for all $a \in I$, $0 < r < 1$ with $B_\rho(a,r) \subset I$, we have

$$\mathbb{E}\Big[\big(L(x,B_\rho(a,r)) - L(y,B_\rho(a,r))\big)^n\Big] \\ \leq C^n |x-y|^{n\gamma} (n!)^{d/Q+K\gamma} r^{n(Q-d-\gamma)}.$$

By Lemmas 3.1 and 3.3, and a chaining argument, we have

Theorem 3.1 [Khoshnevisan, Lee, and X. (2022)]

Under the conditions of Lemma 3.1, there exist finite constants C and C' such that for any $t \in I$,

$$\limsup_{r\to 0} \frac{L^*(B_\rho(t,r))}{\varphi_3^\rho(r)} \le C \quad \text{a.s.} \tag{9}$$

and

$$\limsup_{r\to 0} \sup_{t\in I} \frac{L^*(B_\rho(t,r))}{\Phi_3^\rho(r)} \le C' \quad \text{a.s.,} \tag{10}$$

where $\varphi_3^\rho(r) = r^\alpha (\log\log(1/r))^{d/Q}$ and $\Phi_3^\rho(r) = r^\alpha (\log(1/r))^{d/Q}$.

Remark The exponent of $\log\log(1/r)$ in φ_3^ρ is different from that in φ_1^ρ and φ_2^ρ .

Optimality of (11) and (12)

Recall that, under Conditions (4), (C-2) and another condition (omitted here), Chung's LIL for X holds at $t \in I$ and X has an exact modulus of non-differentiability.
By using these results and the following inequality: For any $t \in I$,

$$\begin{aligned}\lambda_N\big(B_\rho(t,r)\big) &= \int_{\overline{X\left(B_\rho(t,r)\right)}} L(x, B_\rho(t,r))\,dx \\ &\le L^*(B_\rho(t,r)) \cdot \Big(\sup_{s,t\in B_\rho(t,r)} |X(s)-X(t)|\Big)^d,\end{aligned}$$

we can prove that the Hölder conditions in Theorem 3.1 are optimal.

4. Hausdorff measure of level sets

Let us consider the class $\mathscr{C}$ of functions $\varphi : [0, \delta_0] \to \mathbb{R}_+$ such that φ is nondecreasing, continuous, $\varphi(0) = 0$, and satisfies the doubling condition, i.e. there exists a finite constant $c_0 > 0$ such that $\varphi(2s)/\varphi(s) \le c_0$ for all $s \in (0, \delta_0/2)$.

Let $\varphi \in \mathscr{C}$ and ρ be a metric on $\mathbb{R}^N$. For any Borel set A in $\mathbb{R}^N$, the *Hausdorff measure* of A with respect to the function φ, in metric ρ is defined by

$$\mathcal{H}^{\varphi}_{\rho}(A) = \liminf_{\epsilon \to 0} \left\{ \sum_{n=1}^{\infty} \varphi(2r_n) : A \subseteq \bigcup_{n=1}^{\infty} B_\rho(t^n, r_n),\ r_n \le \epsilon \text{ for all } n \right\}.$$

Theorem 4.1 [Khoshnevisan, Lee, and X. (2022)]

Suppose X satisfies Conditions (4) and (C-2) on I and $d < Q$. Then there is a constant $C > 0$ such that for any $x \in \mathbb{R}^d$,

$$CL(x, I) \le \mathcal{H}_\rho^{\varphi_3}(X^{-1}(x) \cap I) \quad \text{a.s.}$$

In particular, if $H_1 = \cdots = H_N = H$, then

$$CL(x, I) \le \mathcal{H}^{\varphi_3}(X^{-1}(x) \cap I) \quad \text{a.s.}$$

where $\varphi_3(r) = r^{N-Hd}(\log\log(1/r))^{Hd/N}$.

5. System of stochastic heat equations

As an example, we consider the following system of stochastic heat equations:

$$\begin{cases} \dfrac{\partial}{\partial t} u_j(t,x) = \Delta u_j(t,x) + \dot{W}_j(t,x), & t \geq 0, x \in \mathbb{R}^N, \\ u_j(0,x) = 0 & j = 1, \ldots, d, \end{cases} \tag{11}$$

where $\dot{W} = (\dot{W}_1, \ldots, \dot{W}_d)$ is a d-dimensional Gaussian noise.

We assume that $\dot{W}_1, \ldots, \dot{W}_d$ are i.i.d. and $\dot{W}_j(t,x)$ is either (i) white in time and colored in space with covariance

$$\mathbb{E}[\dot{W}_j(t,x)\dot{W}_j(s,y)] = \delta_0(t-s)|x-y|^{-\beta}$$

for $N \geq 1$ and $0 < \beta < 2 \wedge N$, or (ii) the space-time white noise for $N = 1$ (take $\beta = 1$ in this case).

The solution of (13) is the Gaussian random field $u = \{u(t,x) : t \geq 0, x \in \mathbb{R}^N\}$ with i.i.d. components $u_1, \ldots, u_d$, given by

$$u_j(t,x) = \int_0^t \int_{\mathbb{R}^N} G(t-s, x-y) W_j(ds\, dy),$$

where G is the fundamental solution of the heat equation:

$$G(t,x) = \frac{1}{(4\pi t)^{N/2}} \exp\Big(-\frac{|x|^2}{4t}\Big) \mathbf{1}_{\{t>0\}}.$$

Recall from Dalang, Khoshnevisan and Nualart (2007) that for any $0 < a < b < \infty$, there exist positive finite constants C_1, C_2 such that

$$C_1\rho((t,x),(s,y)) \le \left(\mathbb{E}|u(t,x) - u(s,y)|^2\right)^{1/2} \le C_2\rho((t,x),(s,y)) \tag{12}$$

for all $(t,x),(s,y) \in [a,b] \times [-b,b]^N$, where

$$\rho((t,x),(s,y)) = |t-s|^{\frac{2-\beta}{4}} + |x-y|^{\frac{2-\beta}{2}}. \tag{13}$$

Let $n \geq 1$, $(t^1, x^1), \ldots, (t^n, x^n) \in \mathbb{R}_+ \times \mathbb{R}^N$ and $a_1, \ldots, a_n \in \mathbb{R}$. Let

$$g(s,y) = \sum_{j=1}^{n} a_j G(t^j - s, x^j - y) \mathbf{1}_{[0,t^j]}.$$

Then by Plancherel's theorem, we have

$$\mathbb{E}\left[\left(\sum_{j=1}^{n} a_j u_1(t^j, x^j)\right)^2\right] = C \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^N} |\mathscr{F} g(\tau, \xi)|^2 \, |\xi|^{\beta - N} d\xi, \tag{14}$$

where $\mathscr{F} g$ denotes the Fourier transform of g, that is,

$$\mathscr{F} g(\tau, \xi) = \int_{\mathbb{R}} \int_{\mathbb{R}^N} e^{-i\tau s - i\langle \xi, y\rangle} g(t, x) \, ds \, dy.$$

One can directly verify that

$$\mathscr{F}\left(G(t - \cdot, x - \cdot)\mathbf{1}_{[0,t]}\right)(\tau, \xi) = e^{-i\langle \xi, x\rangle} \frac{e^{-i\tau t} - e^{-t|\xi|^2}}{|\xi|^2 - i\tau}. \tag{15}$$

The following proposition shows that u satisfies the strong LND property (C-2).

Proposition 5.1 [Khoshnevisan, Lee, and X. (2022)]

For any $0 < a < b < \infty$, there exists a constant $C > 0$ such that for all integers $n \geq 1$ and all (t, x), $(t^1, x^1), \ldots, (t^n, x^n) \in [a, b] \times [-b, b]^N$,

$$\begin{aligned} &\mathrm{Var}(u_1(t, x) | u_1(t^1, x^1), \ldots, u_1(t^n, x^n)) \\ &\geq C \min_{1 \leq i \leq n} \rho((t, x), (t^i, x^i))^2. \end{aligned} \tag{16}$$

Consequently, Theorems 3.1 and 4.1 are applicable to the solution $u = \{u(t, x) : t \geq 0, x \in \mathbb{R}^N\}$.

Proof. Since u is Gaussian, we have

$$\begin{aligned}&\mathrm{Var}(u_1(t,x)|u_1(t^1,x^1),\ldots,u_1(t^n,x^n))\\&=\inf_{a_1,\ldots,a_n\in\mathbb{R}}\mathbb{E}\left[\left(u_1(t,x)-\sum_{j=1}^n a_j u_1(t^j,x^j)\right)^2\right].\end{aligned}$$

Let $r=\min_{1\le j\le n}(|t-t^j|^{1/2}\vee|x-x^j|)$. It suffices to show that there exists a positive constant C such that

$$\mathbb{E}\left[\left(u_1(t,x)-\sum_{j=1}^n a_j u_1(t^j,x^j)\right)^2\right]\ge Cr^{2-\beta},$$

for all $n\ge 1$, $(t,x),(t^1,x^1),\ldots,(t^n,x^n)\in[a,a']\times[-b,b]^N$, and all $a_1,\ldots,a_n\in\mathbb{R}$.

From (16) and (17), we have

$$\mathbb{E}\left[\left(u_1(t,x) - \sum_{j=1}^n a_j u_1(t^j, x^j)\right)^2\right] = C\int_{\mathbb{R}} d\tau \int_{\mathbb{R}^N} d\xi$$

$$\left|e^{-i\langle \xi, x\rangle}(e^{-i\tau t} - e^{-t|\xi|^2}) - \sum_{j=1}^n a_j e^{-i\langle \xi, x^j\rangle}(e^{-i\tau t^j} - e^{-t^j|\xi|^2})\right|^2 \frac{|\xi|^{\beta-N}}{|\xi|^4 + |\tau|^2}. \tag{17}$$

Let M be such that $|t-s|^{1/2} \vee |x-y| \le M$ for all $(t,x),(s,y) \in [a,a'] \times [-b,b]^N$. Let $h = \min\{a/M^2, 1\}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ and $\psi : \mathbb{R}^N \to \mathbb{R}$ be nonnegative smooth test functions that vanish outside the interval $(-h,h)$ and the unit ball respectively and satisfy $\varphi(0) = \psi(0) = 1$. Let $\varphi_r(\tau) = r^{-2}\varphi(r^{-2}\tau)$ and $\phi_r(\xi) = r^{-N}\psi(r^{-1}\xi)$.

Consider the integral

$$I := \int_{\mathbb{R}} d\tau \int_{\mathbb{R}^N} \Big[e^{-i\langle \xi, x\rangle} \big(e^{-i\tau t} - e^{-t|\xi|^2}\big) - \sum_{j=1}^{n} a_j e^{-i\langle \xi, x^j\rangle} \big(e^{-i\tau t^j} - e^{-t^j|\xi|^2}\big)\Big]$$
$$\times\, e^{i\langle \xi, x\rangle} e^{i\tau t} \widehat{\varphi}_r(\tau) \widehat{\psi}_r(\xi)\, d\xi.$$

By the inverse Fourier transform, we have

$$I = (2\pi)^{1+N} \Big[\varphi_r(0)\psi_r(0) - \varphi_r(t)(p_t * \psi_r)(0)$$
$$- \sum_{j=1}^{n} a_j \Big(\varphi_r(t - t^j)\psi_r(x - x^j) - \varphi_r(t)(p_{t^j} * \psi_r)(x - x^j)\Big)\Big],$$

where $p_t(x) = G(t, x)$ is the heat kernel.

By the definition of r, $|t - t^j| \geq r^2$ or $|x - x^j| \geq r$ for every j, thus $\varphi_r(t - t^j)\psi_r(x - x^j) = 0$. Moreover, since $t/r^2 \geq a/M^2 \geq h$, we have $\varphi_r(t) = 0$ and hence

$$I = (2\pi)^{1+N} r^{-2-N}. \tag{18}$$

On the other hand, by the Cauchy-Schwarz inequality and (19),

$$I^2 \leq C\,\mathbb{E}\bigg[\Big(u_1(t,x) - \sum_{j=1}^n a_j u_1(t^j, x^j)\Big)^2\bigg] \times \int_{\mathbb{R}} \int_{\mathbb{R}^N} \big|\widehat{\varphi}_r(\tau)\widehat{\psi}_r(\xi)\big|^2 \big(|\xi|^4 + |\tau|^2\big)|\xi|^{N-\beta} d\tau\, d\xi.$$

Note that $\widehat{\varphi}_r(\tau) = \widehat{\varphi}(r^2\tau)$ and $\widehat{\psi}_r(\xi) = \widehat{\psi}(r\xi)$. Then

$$\begin{aligned}&\int_{\mathbb{R}}\int_{\mathbb{R}^N} \left|\widehat{\varphi}_r(\tau)\widehat{\psi}_r(\xi)\right|^2 \left(|\xi|^4 + |\tau|^2\right)|\xi|^{N-\beta} d\tau\, d\xi \\ &= r^{-6+\beta-2N}\int_{\mathbb{R}}\int_{\mathbb{R}^N} \left|\widehat{\varphi}(\tau)\widehat{\psi}(\xi)\right|^2 \left(|\xi|^4 + |\tau|^2\right)|\xi|^{N-\beta} d\tau\, d\xi.\end{aligned}$$

The last integral is finite since $\widehat{\varphi}$ and $\widehat{\psi}$ are rapidly decreasing functions. It follows that

$$I^2 \le C_0 r^{-6+\beta-2N}\,\mathbb{E}\left[\left(u_1(t,x) - \sum_{j=1}^n a_j u_1(t^j,x^j)\right)^2\right] \tag{19}$$

for some constant C_0. Combining (20) and (21), we get that

$$\mathbb{E}\left[\left(u_1(t,x) - \sum_{j=1}^n a_j u_1(t^j,x^j)\right)^2\right] \ge (2\pi)^{2+2N} C_0^{-1} r^{2-\beta}.$$

This finishes the proof.

Thank you!