# Asymptotics of the density of parabolic Anderson random fields

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# Outline

1. Main equation and motivation
2. Noise structure
3. Definition of Solution
4. Moment bounds and right tails
5. Tail probability density
6. Malliavin calculus

# 1. Main equation and motivation

Cauchy problem (multiplicative noise)

$$\frac{\partial}{\partial t}u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}, \quad t > 0, x \in \mathbb{R}^d,$$

where $u(0,x) = u_0(x)$, $\quad \Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian,

$\dot{W}$ = Gaussian noise.

The product $u(t,x)\dot{W}$ is understood in the sense of Skorohod.

# 2. Noise structure

$$\mathbb{E}\left[\dot{W}(r,y)\dot{W}(s,z)\right] \quad = \quad \gamma(s-r)\Lambda(y-z)\,.$$

In the above expression we assume that the noise in spatial variable is homogeneous. We shall further assume that

$$\Lambda(x) = \int_{\mathbb{R}^d} e^{-\iota x\xi}\mu(d\xi)\,, \quad \text{where} \quad \iota = \sqrt{-1}\,.$$

## Hypothesis (on $\gamma$)

*There exist constants $c_0, C_0$ and $0 \le \beta < 1$, such that*

$$c_0|t|^{-\alpha_0} \le \gamma(t) \le C_0|t|^{-\alpha_0}.$$

## Hypothesis (on $\Lambda$)

*$\Lambda$ satisfies one of the following conditions.*

(i) *There exist positive constants $c_1, C_1$ and $0 < \kappa < 2$ such that*

$$\begin{cases} d \geq 2\,, \\ c_1 |x|^{-\alpha} \leq \Lambda(x) \leq C_1 |x|^{-\alpha}\,. \end{cases}$$

(ii) *There exist positive constants $c_1, C_1$ and $\kappa_i$ such that*

$$\begin{cases} 0 < \alpha_i < 1\,, \qquad \sum_{i=1}^{d} \alpha_i < 2\,, \\ c_1 \prod_{i=1}^{d} |x_i|^{-\alpha_i} \leq \Lambda(x) \leq C_1 \prod_{i=1}^{d} |x_i|^{-\alpha_i}. \end{cases}$$

(iii)

$$\begin{cases} d = 1\,, \\ \Lambda(x) = \delta(x) \quad (\textit{Dirac delta function}). \end{cases}$$

# 3. Definitions of the solution

**Definition**

$u(t,x)$ is called mild solution to

$$\frac{\partial}{\partial t} u(t,x) = \frac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}, \quad t > 0, x \in \mathbb{R}^d,$$

if

$$u(t,x) = p_t u(0,x) + \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) u(s,y) W(ds,dx)\,.$$

$p_t(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right)$ and the integral is Skorohod

# 4. Moment bounds and right tails

We assume the following

$$\alpha = \alpha \text{ ( Case (i))}, \qquad \sum_{i=1}^{d} \alpha_i \text{ ( Case (ii))}, \qquad 1 \text{ ( Case (iii))}.$$

Then we have for all $t \geq 0$, $x \in \mathbb{R}^d$, $k \geq 2$,

$$\exp\left(Ct^{\frac{4-2\alpha_0-\alpha}{2-\alpha}} k^{\frac{4-\alpha}{2-\alpha}}\right) \leq \mathbb{E}\left[u_{t,x}^k\right] \leq \exp\left(C' t^{\frac{4-2\alpha_0-\alpha}{2-\alpha}} k^{\frac{4-\alpha}{2-\alpha}}\right)$$

where $C, C'$ are constants independent of $t$ and $k$.

If $\Lambda(x) = \delta_0(x)$ and $\dot{W}$ is time independent, then,

$$\exp\left(Ct^3 k^3\right) \leq \mathbb{E}\left[u_{t,x}^k\right] \leq \exp\left(C' t^3 k^3\right) ,$$

where $C, C' > 0$ are constants independent of $t$ and $k$.

Hu, Y.; Huang, J.; Nualart, D. and Tindel, S.

Stochastic heat equations with general multiplicative Gaussian noises: Hölder continuity and intermittency.

Electron. J. Probab. 20 (2015), no. 55, 50 pp.

## Theorem (Hu-Le 2022)

*Assume that the initial condition $u_0$ is uniformly bounded from above and from below by two positive constants. Then, there are positive constants $a_0, b_0$, $c_j$, $\tilde{c}_j$, $j = 1, 2, 3$ (independent of $t$ and $a$) such that*

$$\tilde{c}_1 \exp\left(-\tilde{c}_2 t^{-\frac{4-2\alpha_0-\alpha}{2}} (\log(\tilde{c}_3 a))^{\frac{4-\alpha}{2}}\right) \le P(u(t,x) \ge a)$$
$$\le c_1 \exp\left(-c_2 t^{-\frac{4-2\alpha_0-\alpha}{2}} (\log(c_3 a))^{\frac{4-\alpha}{2}}\right)$$

*for all $a \ge a_0 e^{b_0 t^{\beta}}$, where $\beta = \frac{4-2\alpha_0-\alpha}{2-\alpha}$.*

*Proof* Chebyshev inequality for upper bound and Paley-Zygmund inequality,

$$P\left(Z_{\varepsilon,\delta}(t,x) \ge \frac{1}{2}\mathbb{E}Z_{\varepsilon,\delta}(t,x)\right) \ge \frac{|\mathbb{E}Z_{\varepsilon,\delta}(t,x)|^2}{4\mathbb{E}|Z_{\varepsilon,\delta}(t,x)|^2}$$

for lower bound.

# Left tail

## Theorem (Hu-Le 2022)

*Let $t > 0$ and $x \in \mathbb{R}^d$ be fixed. there are positive constants $C$, $c_i$, $i = 1, \dots, 5$ such that for every $t > 0, x \in \mathbb{R}^d$ and for every $0 < r < \frac{1}{2}\exp\{-c_4 e^{c_5 t^\beta}\}$,*

$$P\left(\frac{u(t,x)}{p_t * u_0(x)} \le r\right) \le C \exp\left\{-\left[c_1 \exp\left(-c_3 t^\beta\right) |\log(2r)| + c_2\sqrt{1+t^\beta}\right]^2\right\}.$$

In the case when $\dot{W}$ is a space time white noise with spatial dimension one (that is $d = \alpha_0 = \alpha = \beta = 1$), the above theorem yields

$$P\left(\frac{u(t,x)}{p_t * u_0(x)} \le r\right) \le C \exp\left\{-\left(c_1 \exp\left(-c_3 t\right) |\log(2r)| + c_2\sqrt{1+t}\right)^2\right\},$$

for sufficiently small $r > 0$.

This is consistent with the result of Moreno, Flores, Gregorio R. AP 42 (2014), 1635-1643. in which $u_0$ is the Dirac mass. However, in this one dimensional space time white noise case if the initial condition is the Dirac delta mass at 0, then a recent work of Corwin and Ghosal (Lower tail of the KPZ equation, Duke Math. J. 169 (2020), 1329-1395) improves the above bound as follow:

$$P(u(t,0) \leq r) \simeq \exp\left\{-c_1 t^{-1/2}(c_2 + |\log r|)^{5/2}\right\}$$

for sufficiently small $r > 0$ and for all $t \leq T_0$.

Amir, G.; Corwin, I. and Quastel, J.

Probability distribution of the free energy of the continuum directed random polymer in $1+1$ dimensions.

Comm. Pure Appl. Math. 64 (2011), no. 4, 466-537.

$p(t,x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$.

$$F(t,x) = \log\left(\frac{u(t,x)}{p(t,x)}\right)$$

$$F_T(s) = P\left(F(T,x) + \frac{T}{4!} \leq s\right).$$

$$
\begin{aligned}
\mathrm{Ai}(x) &= \frac{1}{\pi}\int_0^\infty \cos\left(\frac{1}{3}t^3 + xt\right) dt \\
K_\sigma(x,y) &= \int_{-\infty}^{\infty} \sigma(t)\,\mathrm{Ai}(x+t)\,\mathrm{Ai}(y+t) dt \\
F_T(s) &= \int_{\tilde{C}} \frac{d\tilde{\mu}}{\tilde{\mu}} e^{-\tilde{\mu}} \det\left(I - K_{\sigma_{T,\tilde{\mu}}}\right)_{L^2(K_T^{-1}a,\ \infty)} ,
\end{aligned}
$$

where

$$
\begin{aligned}
\sigma_{T,\tilde{\mu}} &= \frac{\tilde{\mu}}{\tilde{\mu} - e^{-K_T t}} \\
a &= a(s) = s - \log\sqrt{2\pi T} \\
K_t &= 2^{-1/3} T^{1/3} \\
\tilde{C} &= \left\{e^{i\theta}\right\}_{\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}} \cup \{x + \pm i\}_{x>0} .
\end{aligned}
$$

# 5. Tail probability density

For any $(t,x)$, $u(t,x)$ is a random variable.

Is $u(t,x)$ continuous type?

Can one find a pdf $\rho(t,x;y)$ such that

$$P(u(t,x)\in A)=\int_A \rho(t,x;y)dy\,,\quad \forall \text{ Borel set } A?$$

What is the form of $\rho(t,x;y)$?

What $\rho(t,x;y)$ looks like When $y\to\infty$?

If $u_0(x)\geq 0$, then $u(t,x)\geq 0$ almost surely. This means $\rho(t,x;y)=0$ when $y<0$. $\quad \lim_{y\to 0+}\rho(t,x;y)=0$?

How fast does $\rho(t,x;y)$ converge to 0 when $y\to 0+$?

# Case 1: Large $y$ (right tail)

## Theorem (Hu-Le 2022)

*Suppose $|u_0| \le c < \infty$. Then, the law of $u(t,x)$ has a density $\rho(t,x;y)$ with respect to the Lebesgue measure. Moreover, there are positive constants $c_1, c_2, c_3, \tilde{c}_1, \tilde{c}_3$ and $\tilde{c}_3 > 0$ such that*

$$\tilde{c}_1 t^{-\frac{4-2\alpha_0-\alpha}{2}} \exp\left\{-\tilde{c}_2 t^{-\frac{4-2\alpha_0-\alpha}{2}} (\log(\tilde{c}_3 y))^{\frac{4-\alpha}{2}}\right\}$$
$$\le \rho(t,x;y) \le c_1 t^{-\frac{4-2\alpha_0-\alpha}{4}} \exp\left\{-c_2 t^{-\frac{4-2\alpha_0-\alpha}{2}} (\log(c_3 y))^{\frac{4-\alpha}{2}}\right\}$$

*for every $t > 0$ and for $y$ sufficiently large.*

# Case 2: Small $y$ (left tail)

## Theorem (Hu-Le 2022)

*Assume that the initial condition $u_0$ is uniformly bounded from above and from below by two positive constants. For fixed $T > 0$, there are positive constants $C, a_0, b_0$ and $c_1(T), c_2(T)$ such that for every $t \in [0, T]$, $x \in \mathbb{R}^d$ and $0 < y < a_0 e^{-b_0 t^\beta}$*

$$\rho(t, x; y) \leq C t^{-\frac{4-2\alpha_0-\alpha}{4}} \exp\left\{ -\left(-c_1(T) \log y - c_2(T)\right)^2 \right\} .$$

*In particular, when the noise $\dot{W}$ is one dimensional space-time white, we have*

$$\rho(t, x; y) \leq C t^{-\frac{1}{4}} \exp\left\{ -\left(-c_1 \log y - c_2\right)^2 \right\} .$$

The tool to use is Malliavin calculus

D. Nualart

The Malliavin calculus and related topics (Second edition).

Springer-Verlag, Berlin, 2006.

Hu, Y.

Analysis on Gaussian spaces.

World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.

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ANALYSIS ON GAUSSIAN SPACES
by Yaozhong Hu
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ANALYSIS ON GAUSSIAN SPACES
Yaozhong Hu
World Scientific
Analysis of functions on the finite dimensional Euclidean space with respect to the Lebesgue measure is fundamental in mathematics. The extension to infinite dimension is a great challenge due to the lack of Lebesgue measure on infinite dimensional space. Instead the most popular measure used in infinite dimensional space is the Gaussian measure, which has been unified under the terminology of "abstract Wiener space".
Out of the large amount of work on this topic, this book presents some fundamental results plus recent progress. We shall present some results on the Gaussian space itself such as the Brunn-Minkowski inequality, Small ball estimates, large tail estimates. The majority part of this book is devoted to the analysis of nonlinear functions on the Gaussian space. Derivative, Sobolev spaces are introduced, while the famous Poincaré inequality, logarithmic inequality, hypercontractive inequality, Meyer's inequality, Littlewood-Paley-Stein-Meyer theory are given in details.
This book includes some basic material that cannot be found elsewhere that the author believes should be an integral part of the subject. For example, the book includes some interesting and important inequalities, the Littlewood-Paley-Stein-Meyer theory, and the Hörmander theorem. The book also includes some recent progress achieved by the author and collaborators on density convergence, numerical solutions, local times.
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Graduate students and researchers in probability and stochastic processes and functional analysis.
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# 6. Malliavin Calculus

Newton calculus, Itô calculus

Malliavin calulus

$\Omega = C_0([0, T], \mathbb{R})$ = The set of all continuous functions $\omega$ starting at 0 ($\omega(0) = 0$).

It is a Banach space with the sup norm $\|\omega\| = \sup_{0 \le t \le T} |\omega(t)|$.

$\mathcal{F}$ be the $\sigma$-algebra generated by the open sets

$P$ is the canonical Wiener measure on $(\Omega, \mathcal{F})$ such that $B_t : \Omega \to \mathbb{R}$ defined by $B_t(\omega) = \omega(t)$ is the standard Brownian motion.

A functional from $\Omega \to \mathbb{R}$ is called a Wiener functional.

## Example

1. $B_t$ 2. $\int_0^T |B_t|^p dt$
3. $\sup_{0 \le t \le T} |B_t|$
4. $I_{\{\sup_{0 \le t \le T} |B_t|\}}$

5. $\int_0^T f(t)dB_t$, where $f : [0, T] \to \mathbb{R}$ s.t. $\int_0^T f^2(t)dt < \infty$

6. multiple Itô-Wiener integral $I_n(f_n) = \int_{[0,T]^n} f_n(t_1, \cdots, t_n)dB_{t_1} \cdots dB_{t_n}$, where $f_n : [0, T]^n \to \mathbb{R}$ is symmetric and
$$\int_{[0,T]^n} f_n^2(t_1, \cdots, t_n)dt_1 \cdots dt_n < \infty.$$

7. $x_{t_0}$, $dx_t = b(x_t)dt + \sigma(x_t)dB_t$.

8. Functionals of the form $F = f(\int_0^T h_1(t)dB_t, ..., \int_0^T h_n(t)dB_t)$ is dense in $L^2(\Omega, \mathcal{F}, P)$,

where $f$ can be the sets of all polynomials, smooth functions of polynomial growth, smooth functions of compact supports

$h_1, h_2, \cdots, h_n, \cdots$ is ONB of $L^2([0, T])$

Itô-Wiener's chaos expansion theorem:

Any $F \in L^2(\Omega, \mathcal{F}, P)$ can be written as

$$F = \sum_{n=0}^{\infty} I_n(f_n)\,,$$

where

$$f_n \in L^2([0,T]^n) \quad \text{and} \quad I_n(f_n) = \int_{[0,T]^n} f_n(t_1, \cdots, t_n) dB_{t_1} \cdots dB_{t_n}\,.$$

Exercises: 1. Find the chaos expansion for $I_{\{\sup_{0\le t\le} |B_t| \le \varepsilon\}}$

2. Find the chaos expansion of $x_t$, where
$dx_t = b(x_t)dt + \sigma(x_t)dB_t\,, x_0 = x$.

Nonlinear functional analysis on a Banach space with a measure (infinite dimensional harmonic analysis)

Gaussian measure (Lebesgue measure does not exist in infinite dimensions)

Why Malliavin derivative?

$x_{t_0}$ , $dx_t = b(x_t)dt + \sigma(x_t)dB_t$.

$x_{t_0} : \Omega \to \mathbb{R}^d$ is not continuous.

Example: $\int_0^T \left(B_t^2 dB_t^1 - B_t^2 dB_t^1\right)$

Malliavin, P.

Stochastic calculus of variation and hypoelliptic operators. Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976), pp. 195-263, Wiley, New York-Chichester-Brisbane, 1978.

# Malliavin derivative

Let $(B_t; t \geq 0)$ be a standard Brownian motion.

Given $F = f(\int_0^T h_1(t)dB_t, ..., \int_0^T h_n(t)dB_t)$, where $h_1, h_2, \cdots, h_n, \cdots$ are continuous functions of $t$ and constitute an orthonormal basis of $L^2([0,T])$

$$D_t F = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\int_0^T h_1(t)dB_t, ..., \int_0^T h_n(t)dB_t)h_i(t).$$

The derivative operator $D$ is a closable and unbounded operator

$$\|F\|_{1,p}^p = E(|F|^p) + E\left(\int_0^T |D_t F|^2 \, dt\right)^{p/2}$$

Higher order derivatives

$\|F\|_{k,p}$

Spaces $\mathbb{D}_{1,p}$, $\mathbb{D}_{k,p}$

If $F = I_q(f_q)$, then

$$D_t F = \sum_{q=1}^{\infty} q I_{q-1}(f_q(\cdot, t)).$$

If $F = \sup_{0 \leq t \leq T} B_t$, then

$$D_t F = I_{[0,\theta_T]}(t)\,,$$

where $\theta_T$ is the unqiue maximum point of $B_t$ over $[0, T]$

chain rule, $D_t g(F) = g'(F) D_t F$

Malliavin calculus can be developed for general Gaussian processes, for Poisson processes, Lévy processes

## $H = L^2([0, T])$

Denote by $\delta$ the adjoint operator of $D$ (*divergence* operator)

$$E(\delta(u)F) = E(\langle DF, u\rangle_H) \quad \text{for any } F \in \mathbb{D}_{1,2}.$$

Ornstein-Uhlenbeck operator

$$\delta DF = -LF.$$

Meyer's inequality

$$c_p\|F\|_{k,p} \leq \|(I+L)^{k/2}F\|_p \leq C_p\|F\|_{k,p}\,.$$

Interpolation inequality (Decreusefond-Hu-Üstünel)

$$\|(I+L)^{1/2}F\|_p \leq \frac{2}{\Gamma(1/2)}\|F\|_p^{1/2}\|(I+L)V\|_p^{1/2}.$$

Combined with Meyer's inequality

$$\|DF\|_p \leq C_p(\|F\|_p + \|F\|_p^{1/2}\|D^2F\|_p^{1/2})$$

# Key formula

### Lemma

*Let $F \in \mathbb{D}^{1,2}$ such that $\frac{DF}{\|DF\|_H^2} \in Dom(\delta)$. Then the law of $F$ has a continuous and bounded density given by*

$$p(x) = E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right].$$

*Proof*

$$\begin{aligned}
p(x) &= \int_{\mathbb{R}} \delta_x(y)p(y)dy = E(\delta_x(F)) \\
&= E\left(\frac{d}{dy}\mathbf{1}_{\{y\geq x\}}\Big|_{y=F}\right) \\
&= E\left[\langle D\left(\mathbf{1}_{\{F>x\}}\right), DF\rangle_H \frac{1}{\|DF\|_H^2}\right] \\
&= E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{DF}{\|DF\|_H^2}\right)\right].
\end{aligned}$$

Another formula

$$
\begin{aligned}
p(x) &= E\left(\frac{d}{dy}\mathbf{1}_{\{y\geq x\}}\Big|_{y=F}\right) \\
&= E\left[\left\langle D\left(\mathbf{1}_{\{F>x\}}\right), \xi\right\rangle_H \frac{1}{\langle DF, \xi\rangle_H}\right] \\
&= E\left[\mathbf{1}_{\{F>x\}}\delta\left(\frac{\xi}{\langle DF, \xi\rangle_H}\right)\right].
\end{aligned}
$$

For any smooth function of compact support $g$

$$
\begin{aligned}
& \int_{\mathbb{R}} g(x) E\left[\mathbf{1}_{\{F>x\}} \delta\left(\frac{u}{\langle DF, u\rangle_H}\right)\right] dx \\
= \quad & E\left[\int_{-\infty}^{F} g(x) dx \delta\left(\frac{u}{\langle DF, u\rangle_H}\right)\right] \\
= \quad & E\left[\langle D \int_{-\infty}^{F} g(x) dx, \frac{u}{\langle DF, u\rangle_H}\rangle_H\right] \\
= \quad & E\left[\langle g(F) DF, \frac{u}{\langle DF, u\rangle_H}\rangle_H\right] \\
= \quad & \mathbb{E}\left[g(F)\right]
\end{aligned}
$$

(big) problem of negative moments

## Lemma

*Let*

$$b(t) = \left(8 \sup_{x\in\mathbb{R}^d} \frac{\mathbb{E}u^2(t,x)}{|p_t * u_0(x)|^2}\right)^{-1} .$$

*For every $p > 0$ and every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, we have*

$$\mathbb{E}|u(t,x)|^{-p} \le 2^p e^{2p\sqrt{\lambda(t)\log\frac{2}{b(t)}}} \left(1 + 4\sqrt{\pi p^2 \lambda(t)} e^{p^2\lambda(t)}\right) |p_t * u_0(x)|^{-p} .$$

## Lemma

*For every $T > 0$ and $p > 0$, there exist positive constants $C, c_T$ such that*

$$\mathbb{E}\|Du(t,x)\|_{\mathcal{H}}^{-2p} \le C e^{c_T p^2} t^{-p(2-\alpha_0-\frac{\alpha}{2})} \quad \textit{for all } t,x \in [0,T]\times\mathbb{R}^d .$$

For any $\mathbb{H}$-valued random variable $F$ in $\mathbb{D}^{1,2}(\mathcal{H})$, it is well-known that

$$\begin{aligned}\rho(t,x;y) &= \mathbb{E}\left[I_{\{u(t,x)\geq y\}}\delta\left(\frac{F}{\langle Du(t,x),F\rangle_{\mathbb{H}}}\right)\right]\\ &= \mathbb{E}\left[I_{\{u(t,x)\geq y\}}\left(\frac{\delta(F)}{\langle Du(t,x),F\rangle_{\mathbb{H}}}+\frac{\langle D\langle Du(t,x),F\rangle_{\mathbb{H}},F\rangle_{\mathbb{H}}}{\langle Du(t,x),F\rangle^2_{\mathbb{H}}}\right)\right]\end{aligned}$$

With the choice

$$F(\tau,\xi)=\varphi(\tau,\xi,t,x)=1_{[0,t)}(\tau)p_{t-\tau}(x-\xi)u(\tau,\xi)$$

the above identity becomes

$$\rho(t,x;y)=\mathbb{E}\left[I_{\{u(t,x)\geq y\}}\left(\frac{u(t,x)-p_t*u_0(x)}{A(t,x)}+\frac{\langle D.A(t,x),\varphi(\cdot,t,x)\rangle_{\mathbb{H}}}{A(t,x)^2}\right)\right],$$

where

$$A(t,x)=\langle Du(t,x),\varphi(\cdot,t,x)\rangle_{\mathbb{H}}\,.$$

Since $\|Du\|_{\mathbb{H}}$ and $u$ have finite moments, we see that $A(t,x)$ also has finite moments of all orders.

$$A(t,x) = \int_{\mathbb{R}^{2d+2}} D_{\tau,\xi} u(t,x) 1_{[0,t]}(\tau') p_{t-\tau'}(x-\xi') u(\tau',\xi') \gamma_0(\tau-\tau') \gamma(\xi-\xi') d\xi d\xi' d\tau d\tau' .$$

Its Malliavin derivative is

$$D_{\lambda,\eta} A(t,x) = \int_{\mathbb{R}^{2d+2}} D^2_{\tau,\xi,\lambda,\eta} u(t,x) 1_{[0,t]}(\tau') p_{t-\tau'}(x-\xi') u(\tau',\xi') \gamma_0(\tau-\tau') \gamma(\xi-\xi') d\xi d\xi' d\tau d\tau' + \int_{\mathbb{R}^{2d+2}} D_{\tau,\xi} u(t,x) 1_{[0,t]}(\tau') p_{t-\tau'}(x-\xi') D_{\lambda,\eta} u(\tau,\xi') \gamma_0(\tau-\tau') \gamma(\xi-\xi') d\xi d\xi' d\tau d\tau' .$$

Thus, $D_{\lambda,\eta} u(t,x) \geq 0$ and $D^2_{\tau,\xi,\lambda,\eta} u(t,x) \geq 0$. This implies that $A(t,x) \geq 0$ and $D_{\lambda,\eta} A(t,x) \geq 0$. Clearly, we have $F \geq 0$ and hence

$$\langle D_{\cdot} A(t,x), \varphi(\cdot,t,x) \rangle_{\mathbb{H}} \geq 0 .$$

As a consequence,

$$\rho(t, x; y) \geq \mathbb{E}\left[ I_{\{u(t,x) \geq y\}} \frac{u(t,x) - p_t * u_0(x)}{A(t,x)} \right] .$$

When $y > p_t * u_0(x) + 1$, we have $u(t,x) - p_t * u_0(x) > 1$ on the event $\{u(t,x) > y\}$. This means

$$\rho(t, x; y) \geq \mathbb{E}\left[ \frac{I_{\{u(t,x) \geq y\}}}{A(t,x)} \right] .$$

Applying Hölder inequality,

$$\left(\mathbb{E}\left[ I_{\{u(t,x) \geq y\}} \right]\right)^2 \leq \mathbb{E}\left( \frac{I_{\{u(t,x) \geq y\}}}{A(t,x)} \right) \left(\mathbb{E} A(t,x)\right) .$$

Thus

$$\rho(t, x; y) \geq \frac{\left(P(u(t,x) \geq y)\right)^2}{\mathbb{E} A(t,x)} .$$

**THANKS**