Necessary and sufficient conditions to solve parabolic Anderson model with rough noise

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Joint work with Yaozhong Hu and Shuhui Liu

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Some background

In this work, we are interested in the following stochastic heat equation on $\mathbb{R}^d$ $(d \geq 1)$, also known as parabolic Anderson model (PAM):

$$\begin{cases} \dfrac{\partial}{\partial t} u(t,x) = \dfrac{1}{2}\Delta u(t,x) + u(t,x)\dot{W}(t,x)\,, \ \ t > 0\,; \\ u(0,x) = u_0(x) = 1\,, \ \ x \in \mathbb{R}^d\,. \end{cases} \tag{1}$$

Some interpretations of PAM:

- diffusion mechanism: $\Delta = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian in $\mathbb{R}^d$;
- random potential: $\dot{W}(t,x) = \frac{\partial^{1+d}}{\partial t \partial x_1 \cdots \partial x_d} W(t,x)$ where $W(t,x)$ is a centered Gaussian field;
- $u(t,x)$ describes a heat flow through a field of random sources;
- $u(t,x)$ describes the population density with random birth and death rates.

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Covariance functions

The covariance function of $\dot{W}(t,x)$ is given formally by

$$\mathrm{Cov}(\dot{W}(t,x),\dot{W}(s,y)) = \gamma_0(s-t)\gamma(x-y)\,. \tag{2}$$

- Space time white, i.e., $d=1$, $\gamma_0(t)=\delta_0(t)$ and $\gamma(x)=\delta_0(x)$;
- Riesz potential, i.e., $\gamma_0(t)=c_{\alpha_0}|t|^{-\alpha_0}$ and $\gamma(x)=c_{\alpha,d}|x|^{-\alpha}$ where $\alpha_0\in(0,1)$ and $\alpha\in(0,d)$.

Remark: In this two cases, $\gamma_0(t)$ and $\gamma(x)$ are locally integrable.

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Remark: In this two cases, $\gamma_0(t)$ and $\gamma(x)$ are locally integrable.

- Fractional, i.e.,

$$\gamma_0(t) = c_{H_0}|t|^{2H_0-2}, \quad \gamma(x) = \prod_{j=1}^{d} \gamma_j(x_j) = \prod_{j=1}^{d} c_{H_j,d}|x_j|^{2H_j-2}, \tag{3}$$

with the Hurst parameters $(H_0, H_1, \cdots, H_d)$ satisfying $H_0 \in (\frac{1}{2}, 1]$ and $H_j \in (0, 1]$ $\forall j = 1, \cdots, d$.

Remark: In this case, $\gamma_0(t)$ is integrable, $\gamma_j(x_j)$ is locally integrable if $H \in (1/2, 1]$ (regular) but is NOT locally integrable if $H \in (0, 1/2)$ (**rough**).

Wiener chaos expansion

We consider the mild solution to (1):

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)u(s,y)W(ds,dy) \tag{4}$$

where $G_t(x) = \frac{1}{\sqrt{2\pi t}} \exp(-\frac{|x|^2}{2t})$ is the heat kernel.

Iterating the mild solution, we have the **Wiener chaos expansion**:

$$u(t,x) = \sum_{n=0}^{\infty} u_n(t,x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} I_n(f_n(\cdot,t,x)) \,. \tag{5}$$

Here $I_n(\cdot)$ denotes the multiple stochastic integrals

$$I_n(f_n(\cdot,t,x)) = \int_{\mathbb{R}_+^n \times \mathbb{R}^{nd}} f_n(t_1,x_1,\cdots,t_n,x_n,t,x) \prod_{j=1}^n W(dt_j,dx_j)\,,$$

and

$$\begin{aligned} f_n(\cdot,t,x) &:= f_n(t_1,x_1,\cdots,t_n,x_n,t,x) \qquad (6)\\ &= \sum_{\sigma\in S_n} G_{t-t_{\sigma(n)}}(x-x_{\sigma(n)}) G_{t_{\sigma(n)}-t_{\sigma(n-1)}}(x_{\sigma(n)}-x_{\sigma(n-1)})\cdots \\ &\quad \times G_{t_{\sigma(2)}-t_{\sigma(1)}}(x_{\sigma(2)}-x_{\sigma(1)})\mathbf{1}_{\{0<t_{\sigma(1)}<\cdots<t_{\sigma(n)}<t\}} \end{aligned}$$

is the symmetrization of the product of heat kernel $G_t(x)$.

The solvability of (1) is reduced to two problems:

(i) The first one is to evaluate $\mathbb{E}\left[u_n^2(t,x)\right]$.

(ii) The second one is to show the convergence of

$$\mathbb{E}\left[u^2(t,x)\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[u_n^2(t,x)\right] .$$

Some Known results

- $\gamma_0(\cdot) = \delta_0(\cdot)$: Dalang (Electron. J. Probab. 1999) obtianed

$$\text{Dalang's condition } \Leftrightarrow \begin{cases} \alpha < 2 & \text{Riesz kernel noise} \\ \sum_{j=1}^{d} \alpha_j < 2 & \text{regular fractional noise} \end{cases} \tag{7}$$

is **necessary and sufficient** for the existence of the mild solution to (1).

- $\gamma_0(t) = c_{\alpha_0}|t|^{-\alpha_0}$: Hu-Huang-Nualart-Tindel (Electron. J. Probab. 2015) obtianed the Dalang's condition (7) is **sufficient** for the solvability problem of (1);
- $\gamma_0(t) = c_{\alpha_0}|t|^{-\alpha_0}$ and $\gamma(x) = c_{\alpha,d}|x|^{-\alpha}$: Balan-Conus (Stat. Probab. Lett. 2014) proved that the Dalang's condition (7) is also **neccesary**.

- Consider hyperbolic Anderson model (HAM) (i.e., we replace $\frac{\partial}{\partial t}$ by $\frac{\partial^2}{\partial t^2}$ in (1)).
  In the case of $\gamma_0(t) = c_{\alpha_0}|t|^{-\alpha_0}$, $\gamma(x) = c_{\alpha,d}|x|^{-\alpha}$ or $\gamma(x) = \prod_{j=1}^d c_{\alpha_j}|x|^{-\alpha_j}$, Chen-Deya-Song-Tindel (2021+) showed that

$$\alpha_0 + \alpha < 3 \quad \text{or} \quad \alpha_0 + \sum_{j=1}^{d} \alpha_j < 3$$

  is the **necessary and sufficient** condition for HAM to admit a unique solution.

When the noise is rough?

- Chen (AIHP 2019, 2020) obtained some sufficient conditions on the parameters $(H_0, H_1, \cdots, H_d)$;
- Chen-Hu (2021+) obtained some sharpened conditions. In particular, when $d = 1$ they showed that

$$\begin{cases} H + H_0 > \dfrac{3}{4} & \text{is a sufficient condition} \quad (8) \\ H + 2H_0 > \dfrac{5}{4} & \text{is a necessary condition} \quad (9) \end{cases}$$

  for (1) to be solvable.
- There are some results on the $L^2(\Omega)$ boundedness of n-th chaos.

Main results

In this talk, we focus on the rough noise and one dimensional case. Namely,

$$\boldsymbol{d = 1\,, H_0 \times H \in (1/2, 1) \times (0, 1/2)}$$

with $H = H_1$.

Results on n-th chaos

Theorem (Hu-Liu-Wang 2022)

If $d = 1$, $H_0 \geq \frac{1}{2}$ and $0 < H < \frac{1}{2}$, then the necessary and sufficient condition so that $\mathbb{E}[u_n(t,x)^2] < +\infty$ for $n > 1$ is

$$H + 2H_0 > \frac{5}{4}\,. \tag{10}$$

Remark: when $d > 1$ and general fractional noise, we refer the result Theorem 4.1 in our preprint.

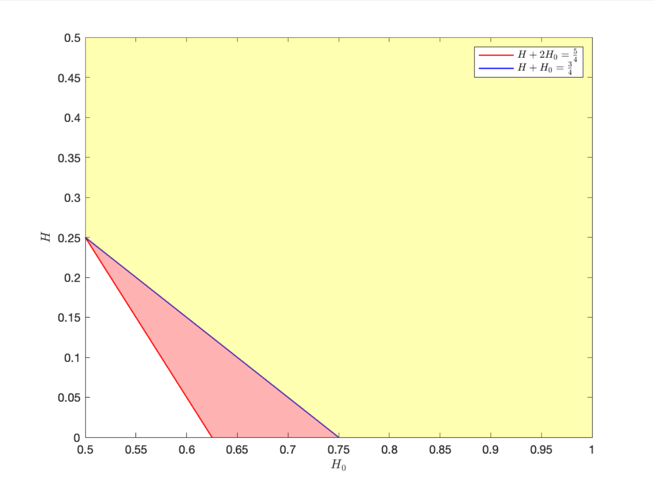


Figure: BL region (red) and HLS region (yellow)

Result on the chaos expansion

Theorem (Hu-Liu-Wang 2022)

Let $u(t,x)$ be the solution candidate to (1). Suppose $d=1$, $H_0>\frac{1}{2}$ and $H=H_1<\frac{1}{2}$. If $(H_0,H)\in\mathcal{A}_1\cup\mathcal{A}_2$, where

$$\mathcal{A}_1=\{(H_0,H)\in(1/2,1)\times(1/20,1/4):\ 2H_0+H>5/4\}\ ; \tag{11}$$

$$\mathcal{A}_2=\{(H_0,H)\in(1/2,1)\times(0,1/20):\ 4H_0+12H>3\}\ , \tag{12}$$

then

$$\mathbb{E}[|u(t,x)|^2]=\sum_{n=0}^{\infty}\mathbb{E}[|u_n(t,x)|^2]<+\infty. \tag{13}$$

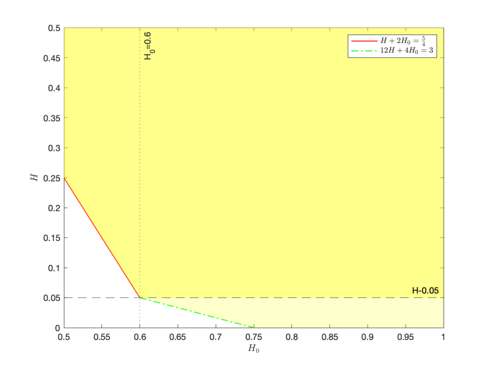
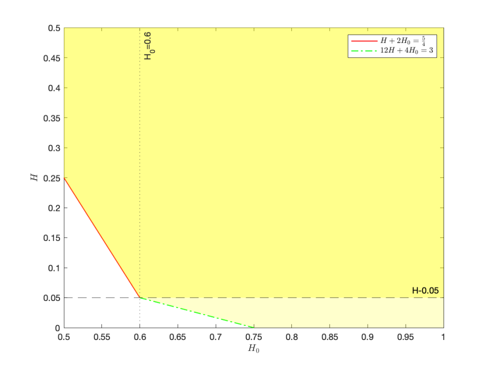


Figure: The regions $\mathcal{A}_1$ and $\mathcal{A}_2$

It is not hard to see that if $H > 0.05$ (or $H_0 < 0.6$), then the condition

$$H + 2H_0 > \frac{5}{4}$$

is **both necessary and sufficient**.



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Hardy-Littlewood-Sobolev inequality

Lemma (Hardy-Littlewood-Sobolev inequality)

For any $\varphi \in L^{1/H_0}(\mathbb{R}^n)$, *it holds*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \varphi(\vec{\mathbf{r}})\varphi(\vec{\mathbf{s}}) \prod_{i=1}^{n} |s_i - r_i|^{2H_0-2} d\vec{\mathbf{r}} d\vec{\mathbf{s}} \le C_{H_0}^n \left(\int_{\mathbb{R}^n} |\varphi(\vec{\mathbf{r}})|^{1/H_0} d\vec{\mathbf{r}}\right)^{2H_0}, \tag{14}$$

where $C_{H_0} > 0$.

Hölder-Young-Brascamp-Lieb inequality

Definition (Brascamp-Lieb datum)

We say $(c_1, L_1), \cdots, (c_m, L_m)$ to be a **Brascamp-Lieb datum** *on $\mathbb{R}^n$. This is, each c_i is a positive number and each L_j is a surjective linear mapping from $\mathbb{R}^n$ onto $\mathbb{R}^{n_i}$ for some $n_i \in \mathbb{N}$.*

For each Brascamp-Lieb datum, we can consider the m-linear Brascamp-Lieb inequality

$$\int_{\mathbb{R}^n} \prod_{j=1}^{m} f_j(L_j x)^{c_j} dx \leq C_{\mathrm{BL}} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j(x) dx \right)^{c_j} \tag{15}$$

where $f_j : \mathbb{R}^{n_j} \to \mathbb{R}_+$ are non-negative measurable functions and C_{BL} is the best constant for which the inequality holds.

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where $f_j : \mathbb{R}^{n_j} \to \mathbb{R}_+$ are non-negative measurable functions and C_{BL} is the best constant for which the inequality holds.

Theorem (Brascamp-Lieb, Adv. Math. 1976)

For a Brascamp-Lieb datum as above, assume that there exists a positive definite matrix A such that

$$A^{-1} = \sum_{j=1}^{m} c_j L_j^*(L_j A L_j^*)^{-1} L_j\,. \tag{16}$$

Then the **Brascamp-Lieb constant** *associated with the datum is*

$$C_{BL} = \left(\frac{det(A)}{\prod_{j=1}^{m} det(L_j A L_j^*)^{c_j}} \right)^{\frac{1}{2}} \tag{17}$$

and the equality in (15) *is archived for the Gaussian functions*

$$f_j(x) = \exp\left(-\frac{1}{2}\langle (L_j A L_j^*)^{-1} x, x\rangle\right), 1 \le j \le m\,.$$

BCCT inequality

Theorem (Bennett-Carbery-Christ-Tao, Geom. Funct. Anal. 2008)

Let $(c_1, L_1), \cdots, (c_m, L_m)$ *to be a* **Brascamp-Lieb datum**. *Then the Brascamp-Lieb constant* C_{BL} *is* **finite** *if and only if the following* **dimension conditions** *hold*

$$n = \sum_{j=1}^{m} c_j n_j\,; \tag{18}$$

$$\dim(V) \le \sum_{j=1}^{m} c_j \dim(L_j(V)) \quad \textit{for all subspaces } V \subseteq \mathbb{R}^n\,. \tag{19}$$

We need to consider the local version of (15)

$$\int_{|x|\leq 1} \prod_{j=1}^{m} f_j(L_j x)^{c_j} dx \leq C_{\mathrm{BL}} \prod_{j=1}^{m} \left(\int_{\mathbb{R}^{n_j}} f_j(x) dx \right)^{c_j} . \qquad (20)$$

Corollary (Bennett-Carbery-Christ-Tao, Geom. Funct. Anal. 2008)

A necessary and sufficient condition for (20) holds with $0 < \cancel{K}_{BL} < \infty$ for all nonnegative measurable functions f_j is that every subspace $V \subseteq \mathbb{R}^n$ satisfies the dimension condition

$$codim_{\mathbb{R}^n}(V) \geq \sum_j c_j codim_{\mathbb{R}^{n_j}}(L_j(V)) . \qquad (21)$$

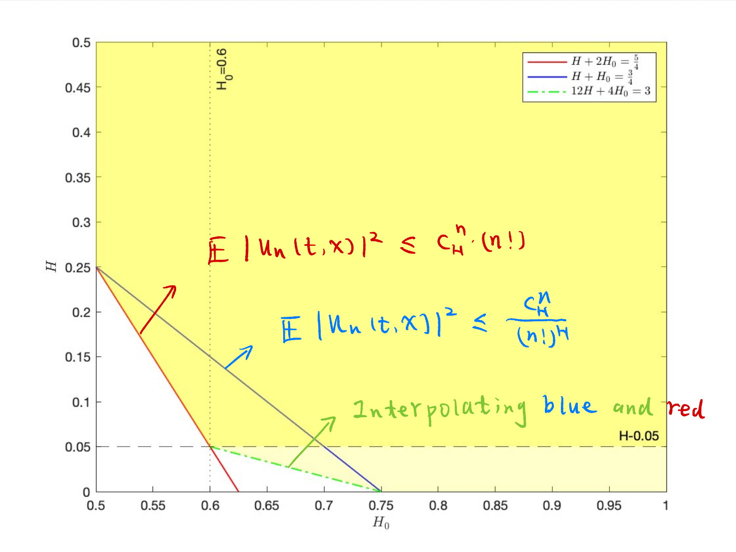
Second moment bound of n-th chaos

We are trying to use the HLS inequality and HYBL inequality to estimate the integral on the simplex

$$\mathbb{E}[|u_n(t,x)|^2] \lesssim \frac{t^{n(H+2H_0-1)}}{n!} \sum_{\sigma,\rho} \sum_{\vec{\rho}\in\mathcal{D}_n} \int_{\mathbb{T}_1(\vec{\mathbf{s}}_\sigma)\times\mathbb{T}_1(\vec{\mathbf{r}}_\rho)} \prod_{i=1}^{n} |s_{\sigma(i+1)} - s_{\sigma(i)}|^{-\rho_i} \times \prod_{i=1}^{n-1} |r_{\rho(i+1)} - r_{\rho(i)}|^{-\rho_i} \prod_{i=1}^{n} \gamma_0(s_i - r_i) d\vec{\mathbf{s}} d\vec{\mathbf{r}}, \tag{22}$$

with $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_n) \in \mathcal{D}_n$ is given by $\rho_i = (\frac{3}{4} - H) - \frac{\alpha_i}{2}(\frac{1}{2} - H)$ and

$$\alpha_i \in \{0, 1, 2\}, \ \boldsymbol{\alpha_i + \alpha_{i+1} \geq 1}, \ \sum_{i=1}^{n} \alpha_i = n. \tag{23}$$


$H+2H_0=\frac{5}{4}$
$H+H_0=\frac{3}{4}$
$12H+4H_0=3$
H_0=0.6
H-0.05
H
H_0
$\mathbb{E}|u_n(t,x)|^2 \leq C_H^n \cdot (n!)$
$\mathbb{E}|u_n(t,x)|^2 \leq \frac{C_H^n}{(n!)^H}$
Interpolating blue and red

Some key steps

First idea (HLS inequality): We can bound (22) by

$$\frac{C_{H_0}^n}{n!}\sum_{\vec{\rho}\in\mathcal{D}_n}\left(\int_{[0,1]^n}\left[\sum_{\sigma}\prod_{i=1}^{n}|s_{\sigma(i+1)}-s_{\sigma(i)}|^{-\rho_i}\mathbf{1}_{\mathbf{s}_\sigma}\right]^{\frac{1}{H_0}}d\mathbf{s}\right)^{2H_0}$$

$$\leq C_{H_0}^n(n!)^{2H_0-1}\sum_{\vec{\rho}\in\mathcal{D}_n}\left(\int_{\mathbb{T}_1(\vec{\mathbf{s}})}\prod_{i=1}^{n}|s_{i+1}-s_i|^{-\frac{\rho_i}{H_0}}\right)^{2H_0}$$

$$\leq\frac{C_{H_0}^n(n!)^{2H_0-1}}{\Gamma(n[1-(\frac{1}{2}-\frac{H}{2})/H_0])^{2H_0}}\quad\text{since }\sum_{i=1}^{n}\rho_i=(\frac{1}{2}-\frac{H}{2})n$$

$$\leq\frac{C^n}{(n!)^H}.$$

The condition for us to apply the HLS inequality is

$$\frac{\rho_i}{H_0} < 1 \quad \Leftrightarrow \quad \frac{\frac{3}{4} - H}{H_0} < 1\,, \ \frac{\frac{1}{2} - \frac{H}{2}}{H_0} < 1\,, \ \frac{\frac{1}{2} - \frac{H}{2}}{H_0} < 1$$

$$\Leftrightarrow \quad H + H_0 > \frac{3}{4}\,.$$

Second idea (HYBL inequality): First by Cauchy Schwarz inequality and then Hölder inequality, (22) is bound by

$$\mathbb{E}[|u_n(t,x)|^2] \lesssim t^{n(H+2H_0-1)} \sum_{\vec{\rho}\in\mathcal{D}_n} \sum_{\sigma} \int_{\mathbb{T}_1(\vec{\mathbf{s}}_\sigma)\times\mathbb{T}_1(\vec{\mathbf{r}}_\sigma)} \prod_{i=1}^{n} |s_{\sigma(i+1)} - s_{\sigma(i)}|^{-\rho_i}$$

$$\times \prod_{i=1}^{n-1} |r_{\sigma(i+1)} - r_{\sigma(i)}|^{-\rho_i} \prod_{i=1}^{n} \gamma_0(s_i - r_i) d\vec{\mathbf{s}} d\vec{\mathbf{r}}$$

$$= n! \, t^{n(H+2H_0-1)} \sum_{\vec{\rho}\in\mathcal{D}_n} \int_{\mathbb{T}_1(\vec{\mathbf{s}})\times\mathbb{T}_1(\vec{\mathbf{r}})} \prod_{i=1}^{n} |s_{i+1} - s_i|^{-\rho_i}$$

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where $\boldsymbol{\rho} = (\rho_1, \cdots, \rho_n) \in \mathcal{D}_n$ is given by $\rho_i = (\frac{3}{4} - H) - \frac{\alpha_i}{2}(\frac{1}{2} - H)$.

Recall $\{\alpha_i : i = 1, \cdots, n\}$ are defined by (23). Then it is relatively easy to see the case $\alpha_i = 0$ implies that $H_0 + H > \frac{3}{4}$ must hold.

However, the fact $\alpha_i + \alpha_{i+1} \geq 1$ inspires the following trick.

$$\begin{cases} \textbf{Case 1:} \quad \alpha_i = 0\,, \alpha_{i+1} \neq 0\,, & \text{we use the HYBL inequality}\,; \\ \textbf{Case 2:} \quad \alpha_i \neq 0\,, & \text{we use the HLS inequality}\,. \end{cases} \tag{24}$$

The **Case 2** can be treated as before, so we skip the details.

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Case 1: When $\alpha_1 = 0$, $\alpha_2 \neq 0$, we integrate s_1, s_2, r_1, r_2 first. Namely, we write now

$$\mathcal{I}(\rho,\gamma) := \int_{\mathbb{T}_1(\vec{\mathbf{s}}_3)\times\mathbb{T}_1(\vec{\mathbf{r}}_3)} (1-s_n)^{-\rho_n}(1-r_n)^{-\rho_n}\prod_{i=3}^{n}\gamma_0(s_i-r_i) \quad (25)$$
$$\times\prod_{i=3}^{n-1}|s_{i+1}-s_i|^{-\rho_i}\prod_{i=3}^{n-1}|r_{i+1}-r_i|^{-\rho_i}\mathcal{I}_2(s_3,r_3)d\vec{\mathbf{s}}_3 d\vec{\mathbf{r}}_3\,,$$

where $\rho_1 = \frac{3}{4} - H$, $\rho_2 = \frac{3}{4} - H - \frac{\alpha_2}{2}(\frac{1}{2} - H)$, $\gamma = 2 - 2H_0$ and

$$\mathcal{I}_2(s_3,r_3) := \int_{\substack{0<s_1<s_2<s_3\\0<r_1<r_2<r_3}}\prod_{i=1}^{2}|s_i-r_i|^{-\gamma}|s_3-s_2|^{-\rho_2}|r_3-r_2|^{-\rho_2}$$
$$|s_2-s_1|^{-\rho_1}|r_2-r_1|^{-\rho_1}ds_1ds_2dr_1dr_2\,. \quad (26)$$

We shall employ **HYBL inequality** to show that

$$\sup_{0\le s_3, r_3\le 1} \mathcal{I}_2(s_3, r_3) \le C_{\mathrm{BL}} < +\infty \,.$$

We can find $f_j(\cdot)$ and establish the BL datum $(c_j = \frac{1}{p_j}, L_j)$ according to the form of $\mathcal{I}_2(s_3, r_3)$. For example,

$$f_1(x) = |x|^{-\rho_1} \mathbf{1}_{\{0<x<r_3\}} \,, \; L_1(\mathbf{s}, \mathbf{r}) = r_2 - r_1 \;\; c_1 = \frac{1}{p_1} > \rho_1 \,.$$

Dimension conditions

Key condition: We select $V = span\{(1,1,1,1)\}$. Then

$$co\dim_{\mathbb{R}}(L_1(V)) = 1\,, co\dim_{\mathbb{R}}(L_3(V)) = 0\,, co\dim_{\mathbb{R}}(L_5(V)) = 1\,,$$
$$co\dim_{\mathbb{R}}(L_2(V)) = 1\,, co\dim_{\mathbb{R}}(L_4(V)) = 0\,, co\dim_{\mathbb{R}}(L_6(V)) = 1\,.$$

Then, in this case the dimension condition (21) is

$$codim_{\mathbb{R}^n}(V) \geq \sum_j c_j codim_{\mathbb{R}^{n_j}}(L_j(V)) \quad \Leftrightarrow \quad \begin{aligned} 3 &\geq c_1 + c_2 + c_5 + c_6 \\ &> \rho_1 + \rho_1 + \gamma + \gamma \\ &= (\frac{3}{2} - 2H) + 4 - 4H_0 \end{aligned}$$

$$\Leftrightarrow \quad H + 2H_0 > \frac{5}{4}\,. \qquad (27)$$

Remark: The condition $H + 2H_0 > \frac{5}{4}$ is also proven to be necessary, i.e., $\mathbb{E}[|u_n(t,x)|^2] = \infty$ if $H + 2H_0 \leq \frac{5}{4}$.

Boundedness of Wiener chaos expansion

We obtain that

$$\mathbb{E}[|u_n(t,x)|^2] \leq \begin{cases} \frac{C^n}{(n!)^H} t^{n(H+2H_0-1)} & \text{if } H+H_0 > \frac{3}{4}\,, \text{ using HLS}\,; \\ C^n n! t^{n(H+2H_0-1)} & \text{if } H+2H_0 > \frac{5}{4}\,, \text{ using HYBL}\,. \end{cases}$$

It is then natural to interpolate these two inequalities and obtain

$$\mathbb{E}[|u(t,x)|^2] = \sum_{n=1}^{\infty} \mathbb{E}[|u_n(t,x)|^2] < +\infty. \tag{28}$$

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$$\mathbb{E}[|u(t,x)|^2] = \sum_{n=1}^{\infty} \mathbb{E}[|u_n(t,x)|^2] < +\infty. \tag{28}$$

Sketch of proof: Recall that

$$\mathbb{E}[|u_n(t,x)|^2] \le C^n n! \sum_{\boldsymbol{\alpha}} \int_{\mathbb{R}_+^{2n}} \int_{\mathbb{R}^n} \overbrace{\frac{1}{n!}\sum_{\sigma} \widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{r}},\eta) \cdot \frac{1}{n!}\sum_{\varsigma} \overline{\widehat{f}_{n,\varsigma}^{(t,x)}(\vec{\mathbf{s}},\eta)}}^{\frac{1}{p}+\frac{1}{q}=1}$$

$$\times \prod_{j=1}^{n} \underbrace{|\eta_j|^{(1-2H)\alpha_j}}_{k+k'=1} \prod_{i=1}^{n} \underbrace{|s_i - r_i|^{-\gamma}}_{m+m'=1} d\eta d\vec{\mathbf{s}} d\vec{\mathbf{r}}\,.$$

Applying Hölder inequality, we get

$$\mathbb{E}[|u_n(t,x)|^2] \le C^n n! \sum_{\boldsymbol{\alpha}} \left(\int_{\mathbb{R}_+^{2n}} \Psi_{n,\boldsymbol{\tau}}(\vec{\mathbf{r}},\vec{\mathbf{s}}) \prod_{i=1}^n |s_i - r_i|^{-pm\gamma} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \right)^{\frac{1}{p}}$$

$$\times \left(\int_{\mathbb{R}_+^{2n}} \Psi_{n,\boldsymbol{\tau}'}(\vec{\mathbf{r}},\vec{\mathbf{s}}) \prod_{i=1}^n |s_i - r_i|^{-qm'\gamma} d\vec{\mathbf{s}} d\vec{\mathbf{r}} \right)^{\frac{1}{q}}$$

$$=: C^n n! \sum_{\boldsymbol{\alpha}} \underbrace{\mathcal{J}_1(p,m,k)}_{HLS \lesssim \frac{1}{n!}} \times \underbrace{\mathcal{J}_2(q,m',k')}_{HYBL \sim C^n},$$

where $\tau_j = kp(1-2H)\alpha_j$, $\tau_j' = k'q(1-2H)\alpha_j$ and

$$\Psi_{n,\boldsymbol{\tau}}(\vec{\mathbf{r}},\vec{\mathbf{s}}) := \int_{\mathbb{R}^n} \frac{1}{n!} \sum_{\sigma} \widehat{f}_{n,\sigma}^{(t,x)}(\vec{\mathbf{r}},\boldsymbol{\eta}) \cdot \frac{1}{n!} \sum_{\varsigma} \overline{\widehat{f}_{n,\varsigma}^{(t,x)}(\vec{\mathbf{s}},\boldsymbol{\eta})} \prod_{j=1}^n |\eta_j|^{\tau_j} d\boldsymbol{\eta}.$$

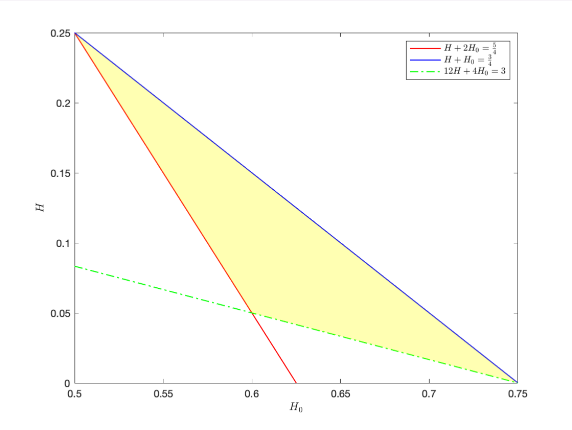
As a result,

$$\mathbb{E}[u(t,x)^2] \le \sum_{n=0}^{\infty} n! \sum_{\pi} \mathcal{J}_1 \times \mathcal{J}_2$$
$$\le \sum_{n=0}^{\infty} C^n (n!)^{1-\frac{2}{q}+[\frac{1}{2q}+\frac{k'}{2}(1-2H)]}.$$

To guarantee $\mathbb{E}[|u(t,x)|^2] < +\infty$, we must have

$$1 < p < \frac{5}{2k(1-2H)+8m(1-H_0)} \wedge \frac{1}{2m(1-H_0)}; \tag{29}$$
$$1 < q < \frac{3}{2k'(1-2H)+4m'(1-H_0)} \wedge \frac{1}{2m'(1-H_0)}; \tag{30}$$
$$1 < q < \frac{3}{k'(1-2H)+2}. \tag{31}$$

By selecting parameters p (q), m (m'), and k (k') properly, we obtain the result in Theorem 2.2.



Thanks for your attention!