Intermittency for hyperbolic Anderson equations with time-independent Gaussian noise: Stratonovich regime

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Our model

The model in the talk is the hyperbolic Anderson Model (HAM)

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + \dot{W}(x)u(t,x) \\ \\ u(0,x) = 1 \ \text{ and } \ \dfrac{\partial u}{\partial t}(0,x) = 0 \quad x \in \mathbb{R}^d \end{cases}$$

where $\{\dot{W}(x);\ x \in \mathbb{R}^d\}$ is a mean-zero generalized stationary Gaussian field such that

$$\mathrm{Cov}\left(\dot{\mathrm{W}}(\mathrm{x}), \dot{\mathrm{W}}(\mathrm{y})\right) = \gamma(\mathrm{x}-\mathrm{y}) \quad \mathrm{x}, \mathrm{y} \in \mathbb{R}^{\mathrm{d}}$$

with $\gamma(\cdot) \geq 0$. In this talk, $\mathrm{d} = 1, 2, 3$.

Set-up of our model

Mathematically, the hyperbolic Anderson equation is defined by following mild equation

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(dy) ds$$

where the stochastic integral on the right hand side is defined in the sense of Stratanovich, i.e., a proper limit of

$$\int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) \dot{W}_\epsilon(x) ds \quad (\text{as } \epsilon \to 0^+)$$

Mathematical set-up

and $G(t,x)$ is the fundamental solution defined by the deterministic wave equation

$$\begin{cases} \dfrac{\partial^2 G}{\partial t^2}(t,x) = \Delta G(t,x) \\ G(0,x) = 0 \text{ and } \dfrac{\partial G}{\partial t}(0,x) = \delta_0(x) \quad x \in \mathbb{R}^d \end{cases}$$

Chaos expansion

Iterating the mild equation infinite times we formally have

$$u(t,x) = \sum_{n=0}^{\infty} S_n(t,x)$$

with $I_0(t,x) = 1$ and the recurrent relation

$$S_{n+1}(t,x) = \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) S_n(s,y) W(dy) ds$$

Chaos expansion

Iterating this relation we have

$$
\begin{aligned}
&S_n(t,x)\\
&= \int_{(\mathbb{R}^d)^n} \left[\int_{[0,t]^n_<} d\mathbf{r} G(t-r_n, y_n - x) \cdots G(r_2 - r_1, y_2 - y_1) \right]\\
&\times W(dx_1) \cdots W(dx_n)\\
&= \int_{(\mathbb{R}^d)^n} \left[\int_{[0,t]^n_<} d\mathbf{s} \left(\prod_{k=1}^n G(s_k - s_{k-1}, x_k - x_{k-1}) \right) \right] W(dx_1) \cdots W(dx_n)
\end{aligned}
$$

where the conventions $x_0 = x$ and $s_0 = 0$ are adopted and the second equality follows from the substitutions $s_k = t - r_{n-k+1}$ and $x_k = y_{n-k+1} - x$ $(k = 1, \cdots, n)$.

Set-up of our model

Essentially, the expansion

$$u(t,x) = \sum_{n=0}^{\infty} S_n(t,x)$$

is a stochastic version of what is called Feynman-Kac formula and is formulated by Dalang, Mueller and Tribe (2008).

We recently proved that this expansion $\mathcal{L}^2$-converges, and solves the hyperbolic Anerson equation under the Dalng's condition

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < \infty$$

Set-up of our model

where $\mu(\mathrm{d}\xi)$ is the spectral measure of the covariance function $\gamma(\cdot)$ determined by the relation

$$\gamma(\mathrm{x}) = \int_{\mathbb{R}^\mathrm{d}} \mathrm{e}^{\mathrm{i}\xi\cdot\mathrm{x}}\mu(\mathrm{dx}) \quad \mathrm{x} \in \mathbb{R}^\mathrm{d}$$

Prior to our progress, Balan (2022+) had reached the same conclusion under a more restrictive condition

$$\int_{\mathbb{R}^\mathrm{d}} \left(\frac{1}{1+|\xi|^2}\right)^{1/2} \mu(\mathrm{d}\xi) < \infty$$

In this talk, our attention is on the intermittency of the system, i.e., the asymptotic behavior of the integer moments

$$\mathbb{E}\, u^p(t, x) \quad \text{or} \quad \mathbb{E}\, |u(t, x)|^p$$

as $t \to \infty$ or $p \to \infty$. In the remaining of the talk, we assume

$$\gamma(cx) = c^{-\alpha}\gamma(x) \qquad c > 0, \quad x \in \mathbb{R}^d$$

for some $\alpha > 0$. In this case, Dalang's condition requests $0 < \alpha < 2$.

Main theorem

Theorem (Chen-Hu)

Assume that $0 < \alpha < 2$. *Then*

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}\, u^p(t,x) = \frac{3-\alpha}{2} p^{\frac{4-\alpha}{3-\alpha}} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

for any $p = 1, 2, \cdots$, *and*

$$\lim_{p\to\infty} p^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}\, |u(t,x)|^p = \frac{3-\alpha}{2} t^{\frac{4-\alpha}{3-\alpha}} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

for any $t > 0$. *where*

$$\mathcal{M} = \sup_{g\in\mathcal{F}_d} \left\{ \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \gamma(x-y) g^2(x) g^2(y) dxdy \right)^{1/2} - \int_{\mathbb{R}^d} |\nabla g(x)|^2 dx \right\}$$

Corollary

Corollary. When $\dot{W}(x)$ $(x \in \mathbb{R})$ is an 1-dimensional white noise (i.e., $\gamma(\cdot) = \delta_0(\cdot)$),

$$\lim_{t\to\infty} t^{-3/2} \log \mathbb{E}\, u^p(t,x) = \frac{1}{2}\sqrt[4]{\frac{3}{4}}p^{3/2} \qquad p = 1, 2, \cdots .$$

$$\lim_{p\to\infty} p^{-3/2} \log \mathbb{E}\, |u(t,x)|^p = \frac{1}{2}\sqrt[4]{\frac{3}{4}}t^{3/2} \qquad \forall t > 0$$

Remark.

In recent work by Balan, R., Chen, L. and Chen, X. (2022), the same p-limit and a slighly different t-limit

$$\lim_{t\to\infty} t^{-\frac{4-\alpha}{3-\alpha}} \log \mathbb{E}\,|u(t,x)|^p = \frac{3-\alpha}{2} p(p-1)^{\frac{1}{3-\alpha}} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

are obtained in Skorokhod regime, under the condition $0 < \alpha < 3$.

Chaos expansion

We only prove the large-t part. First, under our intial codition $u(t,x)$ is stationary in x. So we make $x = 0$ in our proof. From

$$u(t,0) = \sum_{n=0}^{\infty} S_n(t,0)$$

we have

$$\mathbb{E}\, u^p(t,0) = \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_p=n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(t,0)$$

$$= \sum_{n=0}^{\infty} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(t,0) = \sum_{n=0}^{\infty} t^{\frac{4-\alpha}{2}n} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(1,0)$$

where the last step follows from scaling.

Series decomposition of $\mathbb{E}\, u^p(t,x)$

Assume that we can prove

$$\lim_{n\to\infty}\frac{1}{n}\log(n!)^{3-\alpha}\left(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^{p}S_{l_j}(1,0)\right)$$

$$=\log\left(\frac{1}{2}\right)^{3-\alpha}p^{4-\alpha}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{4-\alpha}$$

Then the proof is completed by the computation

$$\lim_{t\to\infty}t^{-\frac{4-\alpha}{3-\alpha}}\log\sum_{n=0}^{\infty}t^{(4-\alpha)n}\left(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^{p}S_{l_j}(1,0)\right)$$

$$=\lim_{t\to\infty}t^{-\frac{4-\alpha}{3-\alpha}}\log\sum_{n=0}^{\infty}\frac{t^{(4-\alpha)n}}{(n!)^{3-\alpha}}\left(\left(\frac{1}{2}\right)^{3-\alpha}p^{4-\alpha}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{4-\alpha}\right)^{n}$$

$$=\frac{3-\alpha}{2}p^{\frac{4-\alpha}{3-\alpha}}\left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{\frac{4-\alpha}{3-\alpha}}$$

Reduction to high moment asymptotics

where the last step follows from the elementary fact that

$$\lim_{t\to\infty} t^{-1/\gamma} \log \sum_{n=0}^{\infty} \frac{\theta^n t^n}{(n!)^\gamma} = \gamma\theta^{1/\gamma} \quad (\theta, \gamma > 0)$$

with $\gamma = 3 - \alpha$ and t being replaced by $t^{4-\alpha}$.

In summary, the proof of our theorem is reduced to the proof of

$$\lim_{n\to\infty} \frac{1}{n} \log(n!)^{3-\alpha} \Big(\sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(1,0) \Big)$$

$$= \log \left(\frac{1}{2}\right)^{3-\alpha} p^{4-\alpha} \left(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\right)^{4-\alpha}$$

Laplacian moment representation

The following moment representation plays a fundamental role in our result:

Theorem (Representation of Stratonovich moment)

For any $\lambda > 0$, and $n = 0, 1, 2, \cdots$,

$$\int_0^\infty e^{-\lambda t} S_n(t,0)dt$$

$$= \frac{1}{n!}\frac{\lambda}{2}\Big(\frac{1}{2}\Big)^n \int_0^\infty \exp\Big\{-\frac{\lambda^2}{2}t\Big\}\mathbb{E}_0\Big[\int_0^t \dot{W}(B(s))ds\Big]^n dt \quad a.s.$$

where $B(s)$ is a d-dimensional Brownian motion independent of $\dot{W}$ with $B(0) = 0$, and "$\mathbb{E}_0$" is the expectation with respect to the Brownian motion.

Mathematical set-up

This relation largely related to he generalized function $G(t,x)$. Its spatial Fourier transform that takes form to all $d \geq 1$:

$$\widehat{G}(t,\xi) = \frac{\sin(|\xi|t)}{|\xi|}.$$

uniform for all $d \geq 1$. In the dimensions $d = 1, 2, 3$, $G(t,x)$ can be expressed explicitly as

$$G(t,x) = \begin{cases} \frac{1}{2} 1_{\{|x| \leq t\}} & d = 1 \\ \frac{1}{2\pi} \frac{1_{\{|x| \leq t\}}}{\sqrt{t^2 - |x|^2}} & d = 2 \\ \frac{1}{4\pi t} \sigma_t(dx) & d = 3 \end{cases}$$

Mathematical set-up

where $\sigma_t(dx)$ is the surface measure on $\{x \in \mathbb{R}^3;\ |x| = t\}$.

The reason that we limit our discussion to $d = 1, 2, 3$ because these are only cases where $G(t, x) \geq 0$.

The reason behind is a simple fact that

$$\int_0^\infty e^{-\lambda t} G(t, x) dt = \frac{1}{2} \int_0^\infty e^{-\lambda^2 t/2} p(t, x) dt \quad x \in \mathbb{R}^d$$

for any $\lambda > 0$, where $p(t, x)$ is the density of $B(t)$:

$$p(t, x) = \frac{1}{(2\pi t)^{d/2}} \exp\left\{ - \frac{|x|^2}{2t} \right\} \quad (t, x) \in \mathbb{R}^+ \times \mathbb{R}^d$$

Laplacian moment representation

Indeed, the both sides has the same Fourier transform

$$\int_{\mathbb{R}^d} e^{i\xi\cdot x}\left[\int_0^\infty e^{-\lambda t} G(t,x)dt\right]dx$$
$$=\int_0^\infty e^{-\lambda t}\frac{\sin|\xi|t}{|\xi|}dt=\frac{1}{\lambda^2+|\xi|^2}$$
$$=\frac{1}{2}\int_0^\infty e^{-\lambda^2 t/2}\exp\left\{-\frac{1}{2}|\xi|^2 t\right\}dt$$
$$=\int_{\mathbb{R}^d} e^{i\xi\cdot x}\left[\frac{1}{2}\int_0^\infty e^{-\lambda^2 t/2}p(t,x)dt\right]dx$$

for every $\xi\in\mathbb{R}^d$.

Laplacian moment representation

Therefore,

$$\int_0^\infty e^{-\lambda t} S_n(t,0)dt$$

$$= \int_0^\infty dt e^{-\lambda t} \int_{(\mathbb{R}^d)^n} d\mathbf{x} \int_{[0,t]^n_<} d\mathbf{s} \left(\prod_{k=1}^n G(s_k - s_{k-1}, x_k - x_{k-1}) \right)$$

$$\times \left(\prod_{k=1}^n \dot{W}(x_k) \right)$$

$$= \lambda^{-1} \int_{(\mathbb{R}^d)^n} d\mathbf{x} \left(\prod_{k=1}^n \int_0^\infty e^{-\lambda t} G(t, x_k - x_{k-1} dt \right) \left(\prod_{k=1}^n \dot{W}(x_k) \right)$$

Laplacian moment representation

$$
\begin{aligned}
&= \lambda^{-1}\left(\frac{1}{2}\right)^n \int_{(\mathbb{R}^d)^n} d\mathbf{x}\left(\prod_{k=1}^n \int_0^\infty e^{-\lambda^2 t/2} p(t, x_k - x_{k-1}) dt\right) \\
&\times \left(\prod_{k=1}^n \dot{W}(x_k)\right) \\
&= \frac{\lambda}{2}\left(\frac{1}{2}\right)^n \int_0^\infty dt \exp\left\{-\frac{\lambda^2}{2} t\right\} \int_{[0,t]^n_<} d\mathbf{s} \\
&\times \int_{(\mathbb{R}^d)^n} d\mathbf{x}\left(\prod_{k=1}^n p(s_k - s_{k-1}, x_k - x_{k-1})\right)\left(\prod_{k=1}^n \dot{W}(x_k)\right)
\end{aligned}
$$

Laplacian moment representation

Given $(s_1, \cdots, s_n) \in [0, t]^n_<$, the random vector $\big(B(s_1), \cdots, B(s_n)\big)$ has the joint density

$$f_{s_1,\cdots,s_n}(x_1, \cdots, x_n) \triangleq \prod_{k=1}^{n} p(s_k - s_{k-1}, x_k - x_{k-1})$$

So we have

$$\int_{(\mathbb{R}^d)^n} d\mathbf{x} \left(\prod_{k=1}^{n} p(s_k - s_{k-1}, x_k - x_{k-1}) \right) \left(\prod_{k=1}^{n} \dot{W}(x_k) \right)$$
$$= \mathbb{E}_0 \prod_{k=1}^{n} \dot{W}\big(B(s_k)\big)$$

Laplacian moment representation

Finally,

$$
\begin{aligned}
&\int_0^\infty e^{-\lambda t} S_n(t,0)dt \\
&= \frac{\lambda}{2}\Big(\frac{1}{2}\Big)^n \int_0^\infty dt \exp\Big\{-\frac{\lambda^2}{2}t\Big\} \int_{[0,t]_<^n} d\mathbf{s}\, \mathbb{E}_0 \prod_{k=1}^n \dot{W}\big(B(s_k)\big) \\
&= \frac{1}{n!}\frac{\lambda}{2}\Big(\frac{1}{2}\Big)^n \int_0^\infty \exp\Big\{-\frac{\lambda^2}{2}t\Big\} \mathbb{E}_0 \Big[\int_0^t \dot{W}\big(B(s)\big)ds\Big]^n dt
\end{aligned}
$$

□

Laplacian moment representation

Corollary (Laplacian moment representation)

Given $\lambda_1, \cdots, \lambda_p > 0$,

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p \lambda_j t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^p S_{l_j}(t_j, 0)$$

$$= \Big(\frac{1}{2}\Big)^{3n} \frac{1}{n!} \Big(\prod_{j=1}^p \frac{\lambda_j}{2}\Big) \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\frac{1}{2}\sum_{j=1}^p \lambda_j^2 t_j\Big\}$$

$$\times \mathbb{E}_0 \Big[\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s) - B_k(r)\big) ds dr\Big]^n \quad n = 0, 1, 2, \cdots$$

where $B_1(t), \cdots, B_p(t)$ *are independent* d*-dimensional Brownian motions starting at 0.*

Laplacian moment representation

Proof.

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p \lambda_j t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \prod_{j=1}^p S_{l_j}(t_j,0)$$

$$= \sum_{l_1+\cdots+l_p=2n} \prod_{j=1}^p \int_0^\infty e^{-\lambda_j t} S_{l_j}(t_j,0)dt$$

$$= \sum_{l_1+\cdots+l_p=2n} \bigg(\prod_{j=1}^p \frac{\lambda_j}{2}\Big(\frac{1}{2}\Big)^{l_j}\bigg) \prod_{j=1}^p \frac{1}{l_j!} \int_0^\infty dt e^{-\lambda^2 t/2} \mathbb{E}_0 \bigg[\int_0^t \dot{W}(B(s))\,ds\bigg]^{l_j}$$

$$= \Big(\frac{1}{2}\Big)^{2n} \Big(\prod_{j=1}^p \frac{\lambda_j}{2}\Big) \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\frac{1}{2}\sum_{j=1}^p \lambda_j^2 t_j\Big\}$$

$$\times \frac{1}{(2n)!} \mathbb{E}_0 \bigg[\sum_{j=1}^p \int_0^{t_j} \dot{W}(B_j(s))\,ds\bigg]^{2n}$$

Laplacian moment representation

The remaining of the proof relies on the fact that conditioning on the Brownian motions,

$$\sum_{j=1}^{p}\int_0^{t_j}\dot{W}\big(B_j(s)\big)ds$$

is normal with zero mean and the variance

$$\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr$$

Laplacian moment representation

Consequently,

$$\mathbb{E}\left[\sum_{j=1}^{p}\int_0^{t_j}\dot{W}\big(B_j(s)\big)ds\right]^{2n}$$

$$=\frac{(2n)!}{2^n n!}\left[\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\right]^n$$

Together with the computation by far, this completes the proof. □

Laplacian moment asymptotics

We now start the proof of the main theorem. The first step is to show

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$
$$=\log 2\mathcal{M}^{\frac{4-\alpha}{2}}$$

Taking $\lambda_1=\cdots=\lambda_p=1$ in Laplacian moment representation, it is equivalent to

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}$$
$$\times\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n=\log 2^4\mathcal{M}^{\frac{4-\alpha}{2}}$$

Upper bound of Laplacian moment

By Parseval's indentity

$$\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr=\int_{\mathbb{R}^d}\mu(d\xi)\bigg|\sum_{j=1}^{p}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\bigg|^2$$

$$=(t_1+\cdots+t_p)^2\int_{\mathbb{R}^d}\mu(d\xi)\bigg|\sum_{j=1}^{p}\frac{t_j}{t_1+\cdots+t_p}\frac{1}{t_j}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\bigg|^2$$

$$\le(t_1+\cdots+t_p)\sum_{j=1}^{p}t_j\int_{\mathbb{R}^d}\mu(d\xi)\bigg|\frac{1}{t_j}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\bigg|^2$$

$$=(t_1+\cdots+t_p)\sum_{j=1}^{p}\frac{1}{t_j}\int_0^{t_j}\int_0^{t_j}\gamma\big(B_j(s)-B_j(r)\big)dsdr$$

$$\overset{d}{=}(t_1+\cdots+t_p)\sum_{j=1}^{p}t_j^{\frac{2-\alpha}{2}}\int_0^1\int_0^1\gamma\big(B_j(s)-B_j(r)\big)dsdr$$

Upper bound of Laplacian moment

where the inequality follows from Jensen and the last step from Brownian scaling and homogenity of $\gamma(\cdot)$.

So we have

$$\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n$$

$$\le (t_1+\cdots+t_p)^n\mathbb{E}_0\Big[\sum_{j=1}^p t_j^{\frac{2-\alpha}{2}}\int_0^1\int_0^1\gamma\big(B_j(s)-B_j(r)\big)dsdr\Big]^n$$

$$=(t_1+\cdots+t_p)^n\sum_{l_1+\cdots+l_p=n}\frac{n!}{l_1!\cdots l_p!}$$

$$\times\prod_{j=1}^p t_j^{\frac{2-\alpha}{2}l_j}\mathbb{E}_0\Big[\int_0^1\int_0^1\gamma\big(B(s)-B(r)\big)dsdr\Big]^{l_j}$$

Upper bound of Laplacian moment

and therefore

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}\mathbb{E}_0\Big[\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s)-B_k(r)\big)\,dsdr\Big]^n$$

$$\leq n! \sum_{l_1+\cdots+l_p=n} \frac{1}{l_1!\cdots l_p!}\Big\{\prod_{j=1}^p \mathbb{E}_0\Big[\int_0^1\int_0^1 \gamma\big(B(s)-B(r)\big)\,dsdr\Big]^{l_j}\Big\}$$

$$\times \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p (t_1+\cdots+t_p)^n \exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}\prod_{j=1}^p t_j^{\frac{2-\alpha}{2}l_j}$$

$$= n! \sum_{l_1+\cdots+l_p=n} \frac{1}{l_1!\cdots l_p!}\Big\{\prod_{j=1}^p \mathbb{E}_0\Big[\int_0^1\int_0^1 \gamma\big(B(s)-B(r)\big)\,dsdr\Big]^{l_j}\Big\}$$

$$\times 2^p 2^{\frac{4-\alpha}{2}n}\Big(\prod_{j=1}^p \Gamma\Big(1+\frac{2-\alpha}{2}l_j\Big)\Big)\Gamma\Big(p+\frac{2-\alpha}{2}n\Big)^{-1}\Gamma\Big(p+\frac{4-\alpha}{2}n\Big)$$

Upper bound of Laplacian moment

From the large deviation for self-intersection local time

$$\lim_{n\to\infty}\frac{1}{n}\log(n!)^{-\alpha/2}\mathbb{E}_0\left[\int_0^1\int_0^1\gamma(B_s-B_r)dsdr\right]^n=\log 2^\alpha\left(\frac{4\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2}}$$

That means: We are allowed to do the replacement

$$\mathbb{E}_0\left[\int_0^1\int_0^1\gamma(B(s)-B(r))dsdr\right]^{l_j}\approx(l_j!)^{\alpha/2}\left(2^\alpha\left(\frac{4\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2}}\right)^{l_j}$$

in our computation

Upper bound of Laplacian moment

Using Stirling formula

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\left\{-\frac{1}{2}\sum_{j=1}^p t_j\right\} \mathbb{E}_0\left[\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s) - B_k(r)\big)\, ds dr\right]^n$$

$$\preceq n! 2^{\frac{4-\alpha}{2}n} \left(\frac{2-\alpha}{2}\right)^n \left(2^\alpha \left(\frac{4\mathcal{M}}{4-\alpha}\right)^{\frac{4-\alpha}{2}}\right)^n \Gamma\left(p + \frac{2-\alpha}{2} n\right)^{-1}$$

$$\times \Gamma\left(p + \frac{4-\alpha}{2} n\right) \sum_{l_1+\cdots+l_p=n} 1$$

$$\approx (n!)^2 2^{4n} \mathcal{M}^{\frac{4-\alpha}{2}n} \begin{pmatrix} n+p-1 \\ p-1 \end{pmatrix} \approx (n!)^2 2^{4n} \mathcal{M}^{\frac{4-\alpha}{2}n}$$

Upper bound of Laplacian moment

In summary, we have established the upper bound

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}$$
$$\times\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n\le\log 2^4\mathcal{M}^{\frac{4-\alpha}{2}}$$

In the following we prove the lower bound

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}$$
$$\times\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n\ge\log 2^4\mathcal{M}^{\frac{4-\alpha}{2}}$$

Lower bound of Laplacian moment

By Cauchy-Schwartz inequality

$$\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr=\int_{\mathbb{R}^d}\mu(d\xi)\left|\sum_{j=1}^{p}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\right|^2$$

$$\geq\left[\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\left(\sum_{j=1}^{p}\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\right)\right]^2$$

$$=\left[\sum_{j=1}^{p}\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\left(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\right)\right]^2$$

for any non-negative $f(\xi)$ with $f(-\xi)=f(\xi)$ and

$$\int_{\mathbb{R}^d}|f(\xi)|^2\mu(d\xi)=1$$

Lower bound of Laplacian moment

Therefore

$$\mathbb{E}_0\left[\sum_{j,k=1}^{p}\int_0^{t_j}\int_0^{t_k}\gamma(B_j(s)-B_k(r))dsdr\right]^n$$

$$\geq \mathbb{E}_0\left[\sum_{j=1}^{p}\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\left(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\right)\right]^{2n}$$

$$=\sum_{l_1+\cdots+l_p=2n}\frac{(2n)!}{l_1!\cdots l_p!}\prod_{j=1}^{p}\mathbb{E}_0\left[\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\left(\int_0^{t_j}e^{i\xi\cdot B_j(s)}ds\right)\right]^{l_j}$$

$$=(2n)!\sum_{l_1+\cdots+l_p=2n}\prod_{j=1}^{p}\int_{(\mathbb{R}^d)^{l_j}}\mu(d\xi)\left(\prod_{k=1}^{l_j}f(\xi_k)\right)$$

$$\times\int_{[0,t_j]_<^{l_j}}d\mathbf{s}\prod_{k=1}^{l_j}\exp\left\{-\frac{s_k-s_{k-1}}{2}\left|\sum_{i=k}^{l_j}\xi_i\right|^2\right\}$$

Lower bound of Laplacian moment

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\left\{-\frac{1}{2}\sum_{j=1}^p t_j\right\}\mathbb{E}_0\left[\sum_{j,k=1}^p \int_0^{t_j}\int_0^{t_k} \gamma\big(B_j(s)-B_k(r)\big)\,ds\,dr\right]^n$$

$$\geq (2n)! \sum_{l_1+\cdots+l_p=2n} \prod_{j=1}^p \int_{(\mathbb{R}^d)^{l_j}} \mu(d\xi)\left(\prod_{k=1}^{l_j} f(\xi_k)\right)$$

$$\times \int_0^\infty dt e^{-t/2} \int_{[0,t]_<^{l_j}} d\mathbf{s} \prod_{k=1}^{l_j} \exp\left\{-\frac{s_k - s_{k-1}}{2}\left|\sum_{i=k}^{l_j}\xi_i\right|^2\right\}$$

$$= 2^{2n+1}(2n)! \sum_{l_1+\cdots+l_p=2n} \prod_{j=1}^p \int_{(\mathbb{R}^d)^{l_j}} \mu(d\xi) \prod_{k=1}^{l_j} f(\xi_k)\left(1+\left|\sum_{i=k}^{l_j}\xi_i\right|^2\right)^{-1}$$

Lower bound of Laplacian moment

The spectral method yields that

$$\lim_{n\to\infty}\frac{1}{n}\log\int_{(\mathbb{R}^d)^n}\mu(d\xi)\prod_{k=1}^n f(\xi_k)\Big(1+\Big|\sum_{i=k}^n\xi_i\Big|^2\Big)^{-1}$$

$$=\sup_{\|\varphi\|_2=1}\int_{\mathbb{R}^d}\mu(d\xi)f(\xi)\Big[\int_{\mathbb{R}^d}d\eta\frac{\varphi(\eta)\varphi(\eta+\xi)}{\sqrt{(1+|\eta|^2)(1+|\xi+\eta|^2)}}\Big]\triangleq\rho(f)$$

Consequently, we are allowed to do the replacement

$$\int_{(\mathbb{R}^d)^{l_j}}\mu(d\xi)\prod_{k=1}^{l_j} f(\xi_k)\Big(1+\Big|\sum_{i=k}^{l_j}\xi_i\Big|^2\Big)^{-1}\approx\rho(f)^{l_j}$$

Lower bound of Laplacian moment

Therefore,

$$\liminf_{n\to\infty}\frac{1}{n}\log\frac{1}{(n!)^2}\int_{(\mathbb{R}^+)^p}dt_1\cdots dt_p\exp\Big\{-\frac{1}{2}\sum_{j=1}^p t_j\Big\}$$
$$\times\,\mathbb{E}_0\Big[\sum_{j,k=1}^p\int_0^{t_j}\int_0^{t_k}\gamma\big(B_j(s)-B_k(r)\big)dsdr\Big]^n\geq\log 2^4\rho^2(f)$$

Finally, the desired lower bound follows from the relation

$$\sup_{\|f\|_2=1}\rho^2(f)=\sup_{\|\varphi\|_2=1}\int_{\mathbb{R}^d}\mu(d\xi)\Big[\int_{\mathbb{R}^d}d\eta\frac{\varphi(\eta)\varphi(\eta+\xi)}{\sqrt{(1+|\eta|^2)(1+|\xi+\eta|^2)}}\Big]^2$$
$$=\mathcal{M}^{\frac{4-\alpha}{2}}$$

Taking “Laplacian inverse”

In summary, we have reached the conclusion that

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$
$$=\log 2\mathcal{M}^{\frac{4-\alpha}{2}}$$

As the last step, we now prove that

$$\lim_{n\to\infty}\frac{1}{n}\log(n!)^{3-\alpha}\Big(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Big)$$
$$=\log\Big(\frac{1}{2}\Big)^{3-\alpha}p^{4-\alpha}\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}$$

Taking “Laplacian inverse”

We first prove the upper bound

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(t_j, 0)$$

$$\geq \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}\Big(\min_{1\le j\le p} t_j,\ 0\Big)$$

$$= \Big\{\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(1, 0)\Big\}$$

$$\times \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\le j\le p} t_j\Big)^{(4-\alpha)n}$$

Taking "Laplacian inverse"

By the fact for the i.i.d. $\exp(1)$-random variables $\tau_1, \cdots, \tau_p$, $\min_{1\le j\le p} \tau_j \sim \exp(p)$,

$$\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\le j\le p} t_j\Big)^{(4-\alpha)n}$$

$$= p\int_0^\infty e^{-pt} t^{(4-\alpha)n} dt = \Big(\frac{1}{p}\Big)^{(4-\alpha)n}\Gamma\Big(1+(4-\alpha)n\Big)$$

Using Stirling formula we obtain the desired upper bound

$$\limsup_{n\to\infty}\frac{1}{n}\log(n!)^{3-\alpha}\Big(\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Big)$$

$$\le \log\Big(\frac{1}{2}\Big)^{3-\alpha} p^{4-\alpha}\Big(\frac{2\mathcal{M}^{1/2}}{4-\alpha}\Big)^{4-\alpha}$$

Taking "Laplacian inverse"

The same argument can be adapted for the lower bound with some extra effort. Let $\delta > 0$ be fixed but small. When $(t_1, \cdots, t_p) \in \big(n(4-\alpha-\delta), n(4-\alpha+\delta)\big)^p$,

$$t_j \le (4-\alpha+\delta)n \le \frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k \quad j = 1, \cdots, p$$

So we have

$$\sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(t_j, 0) \le \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}\Big(\frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k, 0\Big)$$

$$= \Big(\frac{4-\alpha+\delta}{4-\alpha-\delta} \min_{1\le k\le p} t_k\Big)^{(4-\alpha)n} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^{p} S_{l_j}(1, 0)$$

Taking “Laplacian inverse”

Therefore,

$$\int_{(n(4-\alpha-\delta),n(4-\alpha+\delta))^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$\le \Big\{\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Big\}\Big(\frac{4-\alpha+\delta}{4-\alpha-\delta}\Big)^{(4-\alpha)n}$$

$$\times \int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\Big(\min_{1\le j\le p} t_j\Big)^{(4-\alpha)n}$$

$$= \Big\{\sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(1,0)\Big\}\Big(\frac{4-\alpha+\delta}{4-\alpha-\delta}\Big)^{(4-\alpha)n}$$

$$\times \Big(\frac{1}{p}\Big)^{(4-\alpha)n}\Gamma\Big(1+(4-\alpha)n\Big)$$

Concentration of exponential times

To complete the proof for the lower bound, therefore, all we need is to show

$$\int_{(n(4-\alpha-\delta),n(4-\alpha+\delta))^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$\sim \int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)\quad (n\to\infty)$$

for any small $\delta > 0$. This is proceded below.

Concentration of exponential times

First recall that

$$\lim_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\left\{-\sum_{j=1}^p t_j\right\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$=\log 2\mathcal{M}^{\frac{4-\alpha}{2}}$$

Working harder on the moment representation, we can prove that for any $\lambda_1,\cdots,\lambda_p>0$

$$\limsup_{n\to\infty}\frac{1}{n}\log\frac{1}{n!}\int_{(\mathbb{R}^+)^p} dt_1\cdots dt_p \exp\left\{-\sum_{j=1}^p \lambda_j t_j\right\}\sum_{l_1+\cdots+l_p=2n}\mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$\le \log 2\mathcal{M}^{\frac{4-\alpha}{2}}+\frac{4-\alpha}{2}\sum_{j=1}^p\frac{\lambda_j^{-2}\log\lambda_j^{-2}}{\lambda_1^{-2}+\cdots+\lambda_p^{-2}}$$

The correspondent lower bound is very likely, but we are not able to prove it.

Concentration of exponential times

Define the probability measures $\mu_n(A)$ on $(\mathbb{R}^+)^p$

$$\mu_n(A) = \frac{\int_A dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^p S_{l_j}(t_j, 0)}{\int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E} \prod_{j=1}^p S_{l_j}(t_j, 0)}$$

For any $(\theta_1, \cdots, \theta_p) \in \mathbb{R}^p$,

$$\limsup_{n\to\infty} \frac{1}{n} \log \int_{(\mathbb{R}^+)^p} \exp\Big\{\sum_{j=1}^p \theta_j t_j\Big\} \mu_n(dt_1 \cdots dt_p) \le \Lambda(\theta_1, \cdots, \theta_p)$$

Concentration of exponential times

where

$$\Lambda(\theta_1, \cdots, \theta_p) = \frac{4-\alpha}{2} \sum_{j=1}^{p} \frac{(1-\theta_j)^{-2} \log(1-\theta_j)^{-2}}{(1-\theta_1)^{-2} + \cdots + (1-\theta_p)^{-2}}$$

when $\theta_1, \cdots, \theta_p < 1$, and $\Lambda(\theta_1, \cdots, \theta_p) = +\infty$ if otherwise.

By the upper bound of Gärtner-Ellis theorem,

$$\limsup_{n\to\infty} \frac{1}{n} \log \mu_n(nF) \le - \inf_{(t_1, \cdots, t_p)\in F} \Lambda^*(t_1, \cdots, t_p)$$

for every close set $F \subset (\mathbb{R}^+)^p$, where

$$\Lambda^*(t_1, \cdots, t_p) = \sup_{\theta_1, \cdots, \theta_p} \left\{ \sum_{j=1}^{p} \theta_j t_j - \Lambda(\theta_1, \cdots, \theta_p) \right\}$$

Concentration of exponential times

It is not easy (perhaps) and unnecessary to find the close form of $\Lambda^*(\cdot)$. Clearly, $\Lambda^*(t_1, \cdots, t_p) \geq 0$. Further, we claim that $\Lambda^*(t_1, \cdots, t_p) > 0$ whenever $t_j \neq 4 - \alpha$ for any $1 \leq j \leq p$.

Indeed, assume $(t_1, \cdots, t_p) \in (\mathbb{R}^+)^p$ that makes $\Lambda^*(t_1, \cdots, t_p) = 0$. We must have

$$\sum_{j=1}^{p} \theta_j t_j \leq \frac{4-\alpha}{2} \sum_{j=1}^{p} \frac{(1-\theta_j)^{-2} \log(1-\theta_j)^{-2}}{(1-\theta_1)^{-2} + \cdots + (1-\theta_p)^{-2}}$$

for every $(\theta_1, \cdots \theta_p) \in (-\infty, 1)^p$. In particular, for given j, take $\theta_j = \theta$ and $\theta_k = 0$ for $k \neq j$:

$$\theta t_j \leq \frac{4-\alpha}{2} \log(1-\theta)^{-2} = (4-\alpha) \log(1-\theta)^{-1}$$

Concentration of exponential times

Thus,

$$t_j \le (4-\alpha)\frac{1}{\theta}\log(1-\theta)^{-1} \qquad \theta > 0$$

$$t_j \ge (4-\alpha)\frac{1}{\theta}\log(1-\theta)^{-1} \qquad \theta < 0$$

Letting $\theta \to 0^+$ and $\theta \to 0^-$ separately, we have $t_j = 4-\alpha$.

Write $G_\delta = (4-\alpha-\delta,\ 4-\alpha+\delta)^p$. We have, therefore,

$$\inf_{(t_1,\cdots,t_p)\in G_\delta^c} \Lambda^*(t_1,\cdots,t_p) > 0$$

Taking $F = G_\delta^c$ in the large deviation upper bound,

$$\limsup_{n\to\infty}\frac{1}{n}\log\mu_n(nG_\delta^c) < 0$$

Concentration of exponential times

In particular,

$$\int_{nG_\delta} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0)$$

$$\sim \int_{(\mathbb{R}^+)^p} dt_1 \cdots dt_p \exp\Big\{-\sum_{j=1}^p t_j\Big\} \sum_{l_1+\cdots+l_p=2n} \mathbb{E}\prod_{j=1}^p S_{l_j}(t_j,0) \quad (n\to\infty)$$

That is what we try to prove. □

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Thank you!