A physical model with cyclical long-range dependence

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Cyclical long-range dependence (LRD)

Continuous time: Stationary process $Y = \{Y_t\}_{t \in \mathbb{R}}$. Assume $\mathbb{E} Y_t = 0$ for simplicity.

Spectrum: For some frequency $\nu_0 > 0$,

$$\boxed{S(\nu) \simeq C_S (\nu_0 - \nu)^{-2\delta}, \text{ as } \nu \uparrow \nu_0,}$$

with $\delta \in (0, 1/2)$ and constant $C_S > 0$.[1]

ACVF: For $R(h) = \mathbb{E} Y_{t+h} Y_t = \int_0^\infty \cos(h\nu) S(\nu) d\nu$,

$$\boxed{R(h) \simeq C_R \cos(\nu_0 h + \phi_R) h^{2\delta - 1}, \text{ as } h \to \infty.}$$

Notes: $\int_0^\infty |R(h)| dh = \infty$, but $\left| \int_0^\infty R(h) dh \right| < \infty$. Traditional LRD can be thought as above with $\nu_0 = 0$.

[1] More generally, divergence can occur on both sides of the frequency ν_0, and more than one fixed frequency can be considered. Other names used as well.

For small fixed $\epsilon > 0$,

$$\int_{\nu_0-\epsilon}^{\nu_0} e^{ih\nu} S(\nu) d\nu \simeq C_S \int_{\nu_0-\epsilon}^{\nu_0} e^{ih\nu} (\nu_0 - \nu)^{-2\delta} d\nu = C_S e^{ih\nu_0} \int_0^{\epsilon} e^{-ihz} z^{-2\delta} dz$$

$$= C_S e^{ih\nu_0} h^{2\delta-1} \int_0^{\epsilon h} e^{-ix} x^{-2\delta} dx \simeq C_S a_\delta e^{i(h\nu_0+\phi_\delta)} h^{2\delta-1},$$

as $h \to \infty$. Thus, we expect that

$$R(h) \simeq C_R \cos(\nu_0 h + \phi_R) h^{2\delta-1},$$

as $h \to \infty$.

Squared process: Under Gaussianity, $R_2(h) = 2(R(h))^2$, so that

$$R_2(h) \simeq 2C_R^2 \cos^2(\nu_0 h + \phi_R) h^{4\delta - 2}$$

or

$$\boxed{R_2(h) \simeq C_R^2 h^{4\delta-2} + C_R^2 \cos(2\nu_0 h + 2\phi_R) h^{4\delta-2}.}$$

Notes:

$$\int_0^\infty R_2(h)dh \left\{ \begin{array}{c} < \infty \\ = \infty \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \delta \in (0, \frac{1}{4}) \\ \delta \in [\frac{1}{4}, \frac{1}{2}) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{c} \text{SRD} \\ \text{LRD} \end{array} \right\}.$$

For $\delta \in (\frac{1}{4}, \frac{1}{2})$, expect

$$\boxed{S_2(\nu) \simeq C_{1,S} \nu^{-4\delta+1}, \text{ as } \nu \downarrow 0, \quad \simeq C_{2,S}(2\nu_0 - \nu)^{-4\delta+1}, \text{ as } \nu \uparrow 2\nu_0,}$$

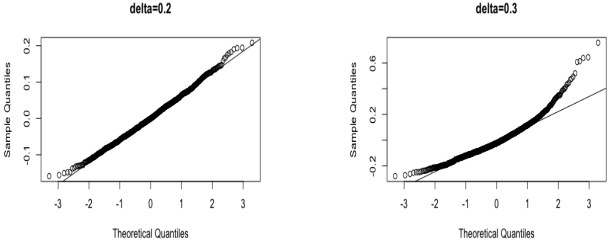
When $\delta = \frac{1}{4}$, still expect logarithmic divergence. (This cannot be seen just considering the spectra.)

Cyclical LRD

(Non)Central limit theorems:

$$\frac{1}{T}\int_0^T Y_t^2 dt \simeq \left\{ \begin{array}{l} \text{Normal}, \ \ \delta \in (0, \frac{1}{4}) \\ \text{Non-Normal}, \ \ \delta \in (\frac{1}{4}, \frac{1}{2}) \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} \text{SRD} \\ \text{LRD} \end{array} \right\}.$$

Normal QQ-plots for averages of squared LRD series, $\delta = 0.2$ and 0.3:



The boundary case $\delta = \frac{1}{4}$ seems unresolved.

This is important in e.g. setting confidence intervals for the variance.

Cyclical LRD

Origins: Hosking (1981), "Fractional differencing," *Biometrika*, last para:

Finally we mention two other processes involving fractional differencing which may prove useful in applications. The fractional equal-root integrated moving-average process is defined by $\nabla^q y_t = (1-\theta B)^q a_t$, $|q| < \frac{1}{2}$, $|\theta| < 1$; as a forecasting model it corresponds to 'fractional order multiple exponential smoothing'. The process $(1-2\phi B+B^2)^d y_t = a_t$, $|d| < \frac{1}{2}$, $|\phi| < 1$, exhibits both long-term persistence and quasiperiodic behaviour; its correlation function resembles a hyperbolically damped sine wave.

There were a number of follow-up papers looking at this phenomenon, with some applications. But it has largely stayed at the margins of LRD research. E.g. pp. 185-191 in Giraitis et al. (2012; 500+ pages); pp. 496-499 in Beran et al. (2013; 800+ pages); no mention in Pipiras and Taqqu (2017; 600+ pages); etc.

No known physical model leading to this phenomenon.

Wave elevation model

Longuet-Higgins model: Along x-direction and time t,

$$\zeta(x,t) = \sum_{n=1}^{N} a_n \cos(k_n(\cos\mu_0)x - w_n t + \phi_n)$$

with frequencies w_n, amplitudes $a_n = \sqrt{2S(w_n)\Delta w}$ for spectrum $S(w)$, heading μ_0, random phases ϕ_n, and wave numbers

$$k_n = w_n^2/g. \qquad \text{(Dispersion relation)}$$

At $x = 0$,

$$\boxed{\zeta_0(t) = \sum_{n=1}^{N} a_n \cos(-w_n t + \phi_n)}$$

ACVF:

$$R(h) = \sum_{n=1}^{N} \cos(hw_n)S(w_n)\Delta w \simeq \int_0^\infty \cos(hw)S(w)dw.$$

Wave elevation at non-zero speed

Non-zero forward speed: Setting $x = U_0 t$ for speed U_0, the model becomes

$$\boxed{\zeta_e(t) = \sum_{n=1}^{N} a_n \cos(-w_{e,n} t + \phi_n)}$$

for encounter frequencies

$$w_{e,n} = w_n - \frac{U_0}{g}(\cos \mu_0) w_n^2 = w_n - q w_n^2.$$

ACVF:

$$R(h) = \sum_{n=1}^{N} \cos(h w_{e,n}) S(w_n) \Delta w \simeq \int_0^\infty \cos(h w_e) S(w) dw.$$

Original and transformed spectra

Note: With $w_e = w - qw^2 = w - \frac{U_0}{g}\cos\mu_0 w^2$ and for $q > 0$ ($\mu_0 \in (-\frac{\pi}{2}, \frac{\pi}{2})$),

$$\int_0^\infty \cos(hw_e)S(w)dw = \int_0^\infty \cos(h\nu)\widetilde{S}(\nu)d\nu,$$

where

$$\widetilde{S}(\nu) = \frac{S(w_1(\nu)) + S(w_2(\nu))}{(1-4q\nu)^{1/2}} + \frac{S(w_3(\nu))}{(1+4q\nu)^{1/2}}$$

for $\nu \in (0, 1/4q)$, and

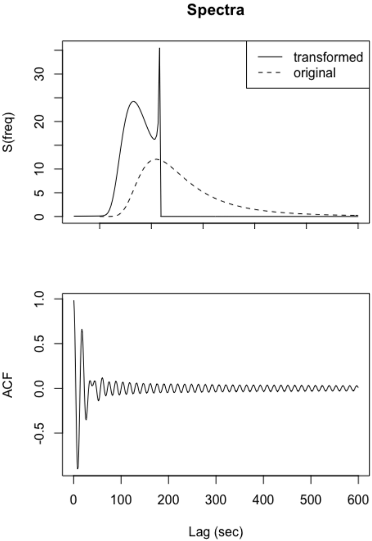
$$\widetilde{S}(\nu) = \frac{S(w_3(\nu))}{(1+4q\nu)^{1/2}}$$

for $\nu \in (1/4q, \infty)$. So cyclical LRD with

$$\boxed{\delta = \frac{1}{4}.}$$

(This is directly related to the dispersion relation.)

Original and transformed spectra



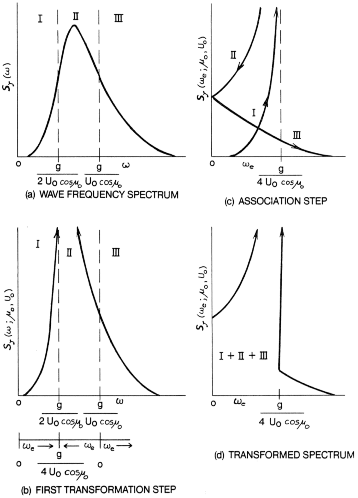
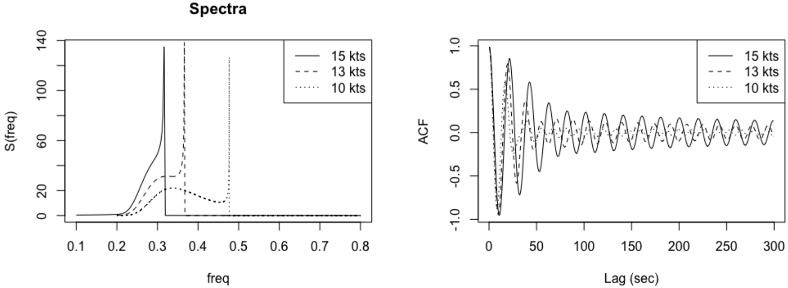


Fig. 68 Transformation of wave spectrum to encounter spectrum, following or quartering waves (long-crested)

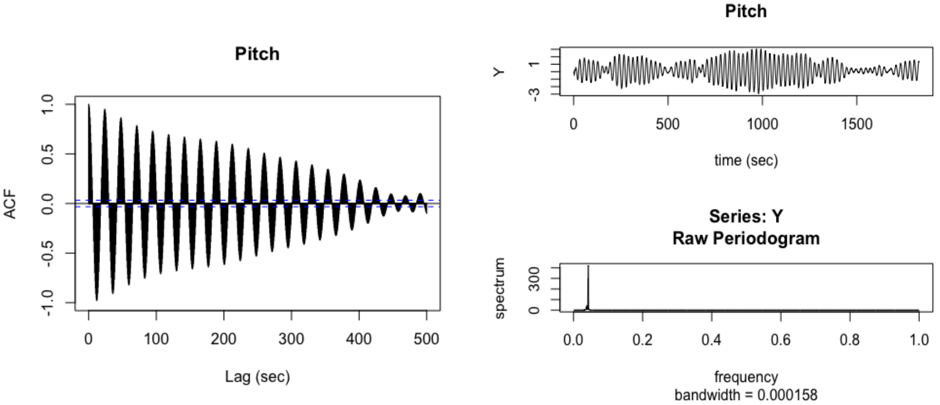
Transformed spectrum and ACF



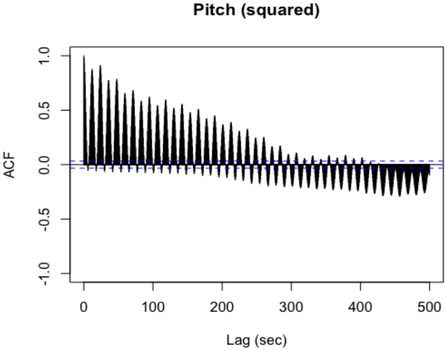
The decay will always be slow but magnitude of ACF coefficients at lags will depend on the underlying spectrum and speed.

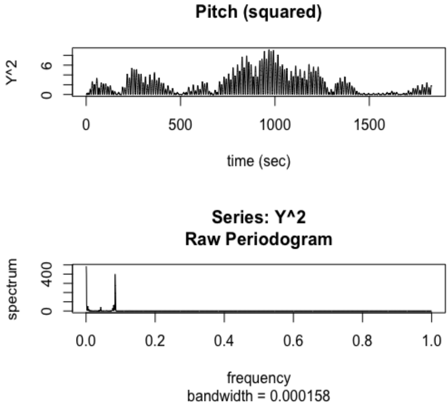
Bits of data

Ship motion: The plot presents one such ACF for the pitch motion from a 30-minute-long record. This is for the flared variant of the ONR Topsides Geometry Series, in sea state 6, the heading of 45°, and traveling at 25 kts.



For the same pitch motion process:



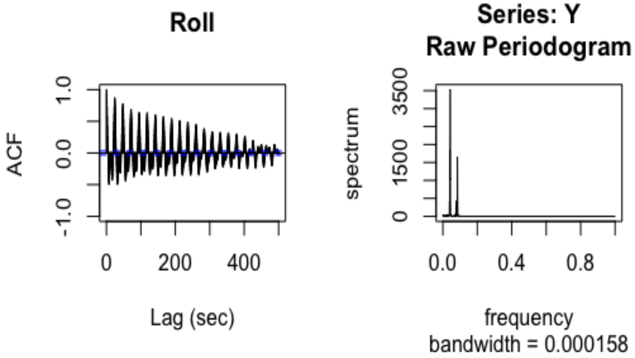


How to put confidence intervals, e.g., on the variance?

Conclusions

Key takeaway: Model for wave elevation and ship motions at non-zero speed characterized by (cyclical) LRD.

Question: What is going on with another motion, so-called roll?



Other physical models for cyclical LRD?