Heat kernel estimates for Dirichlet forms decaying at the boundary

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Gaussian Random Fields, Fractals, SPDEs, and Extremes
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Reference

This talk is based on the following papers:

[KSV1] P. Kim, R. Song and Z. Vondracek: On potential theory of Markov processes with jump kernels decaying at the boundary, arXiv:1910.10961, to appear in Potential Analysis.

[KSV2] P. Kim, R. Song and Z. Vondracek: Sharp two-sided Green function estimates for Dirichlet forms degenerate at the boundary, arXiv:2011.00234, to appear in J. European Math.Soc.

[KSV3] P. Kim, R. Song and Z. Vondracek: Potential theory of Dirichlet forms degenerate at the boundary: the case of no killing potential, arXiv:2110.11653

[CKSV] S. Cho, P. Kim, R. Song and Z. Vondracek: Heat kernel estimates for Dirichlet forms degenerate at the boundary.

Outline

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In the last few decades, lots of progress has been made in the study of potential theoretic properties for various types of jump processes in open subsets D (or their closures) of $\mathbb{R}^d$. These include killed isotropic α-stable processes, more general killed symmetric Lévy processes, their reflected and censored versions.

In these studies, the jump kernel $J^D(x, y)$ of the process in the open set D (or its closure $\overline{D}$) is either the restriction of the jump kernel of the original process in $\mathbb{R}^d$ or comparable to such a kernel and it does not tend to zero as x or y tends to the boundary of D.

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Subordinate killed Brownian motions and subordinate killed Lévy processes form another important class of Markov processes. In case of a stable subordinator, the generator of the subordinate killed Brownian motion is the spectral fractional Laplacian. The spectral fractional Laplacian and, more generally, fractional powers of Dirichlet elliptic differential operators in domains have been studied by quite a few people in the PDE community.

In contrast with killed Lévy processes and censored processes, the jump kernel of a subordinate killed Lévy process in an open subset $D \subset \mathbb{R}^d$ tends to zero near the boundary of D. In this sense, the Dirichlet forms of subordinate killed Lévy processes are degenerate near the boundary. Partial differential equations degenerate at the boundary have been studied a lot in the PDE literature.

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In two recent papers, Kim-S.-Vondracek (**TAMS**, 2019) and Kim-S.-Vondracek (**Pot. Anal.**, 2019), we studied the potential theory of those processes. There were some unexpected results.

These processes are natural and important, but its structure is too rigid for applications. In some sense we are just dealing with particular processes. Is there is a general theory behind all these?

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Let $\mathbb{R}^d_+ = \{x = (\widetilde{x}, x_d) : \ x_d > 0\}$, $j(|x-y|) = |x-y|^{-\alpha-d}$, $0 < \alpha < 2$. Let $\mathcal{B}(x, y)$ be a function on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ satisfying the following assumptions:

(A1) $\mathcal{B}(x, y) = \mathcal{B}(y, x)$ for all $x, y \in \mathbb{R}^d_+$.

(A2) If $\alpha \geq 1$, then there exist $\theta > \alpha - 1$ and $C_1 > 0$ such that

$$|\mathcal{B}(x, x) - \mathcal{B}(x, y)| \leq C_1 \left(\frac{|x-y|}{x_d \wedge y_d} \right)^{\theta}.$$

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(A3) There exist $C_2 \geq 1$ and parameters $\beta_1, \beta_2, \beta_3 \geq 0$, with $\beta_1 > 0$ if $\beta_3 > 0$, and $\beta_2 > 0$ if $\beta_4 > 0$, such that

$$C_2^{-1} \widetilde{B}(x,y) \leq \mathcal{B}(x,y) \leq C_2 \widetilde{B}(x,y)\,, \qquad x, y \in \mathbb{R}^d_+\,,$$

where $\widetilde{B}(x,y)$ is defined to be

$$\left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^{\beta_1} \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{\beta_2} \left[\log\left(1 + \frac{(x_d \vee y_d) \wedge |x-y|}{x_d \wedge y_d \wedge |x-y|}\right)\right]^{\beta_3} \times$$
$$\times \left[\log\left(1 + \frac{|x-y|}{(x_d \vee y_d) \wedge |x-y|}\right)\right]^{\beta_4}.$$

(A4) For all $x, y \in \mathbb{R}^d_+$ and $a > 0$, $\mathcal{B}(ax, ay) = \mathcal{B}(x, y)$. In case $d \geq 2$, for all $x, y \in \mathbb{R}^d_+$ and $\widetilde{z} \in \mathbb{R}^{d-1}$, $\mathcal{B}(x + (\widetilde{z}, 0), y + (\widetilde{z}, 0)) = \mathcal{B}(x, y)$.

In the next two sections, we always assume that

$$d > (\alpha + \beta_1 + \beta_2) \wedge 2, \quad p \in ((\alpha - 1)_+, \alpha + \beta_1) \quad \text{and}$$

$J(x, y) = |x - y|^{-d-\alpha}\mathcal{B}(x, y)$ on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$ with $\mathcal{B}$ satisfying **(A1)** – **(A4)**.

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Let $\mathbf{e}_d = (\tilde{0}, 1)$. To every parameter $p \in ((\alpha - 1)_+, \alpha + \beta_1)$, we associate a constant $C(p) = C(\alpha, p, \mathcal{B}) \in (0, \infty)$ defined as

$$C(p) = \int_{\mathbb{R}^{d-1}} \frac{1}{(|\widetilde{u}|^2 + 1)^{(d+\alpha)/2}} \int_0^1 \frac{(s^p - 1)(1 - s^{\alpha - p - 1})}{(1 - s)^{1+\alpha}} \mathcal{B}\big((1 - s)\widetilde{u}, 1), s\mathbf{e}_d\big)\, ds d\widetilde{u}\,,$$

In case $d = 1$, $C(p)$ is defined as

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Note that $\lim_{p \downarrow (\alpha - 1)_+} C(p) = 0$, $\lim_{p \uparrow \alpha + \beta_1} C(p) = \infty$ and that the function $p \mapsto C(p)$ is strictly increasing.

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Let $\kappa(x) = C(p)x_d^{-\alpha}$ on $\mathbb{R}^d_+$.

Define

$$\mathcal{E}(u,v) := \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))J(x,y)\,dy\,dx + \int_{\mathbb{R}^d_+} u(x)v(x)\kappa(x)dx.$$

Let $\mathcal{F}$ be the closure of $C_c^\infty(\mathbb{R}^d_+)$ under $\mathcal{E}_1(u,u) = \mathcal{E}(u,u) + (u,u)$. Then $(\mathcal{E},\mathcal{F})$ is a Dirichlet form, degenerate at the boundary due to **(A3)**.

Let $((Y_t)_{t\geq 0}, (\mathbb{P}_x)_{x\in\mathbb{R}^d_+})$ be the associated Hunt process with lifetime ζ. We add a cemetery point ∂ to the state space $\mathbb{R}^d_+$ and define $Y_t = \partial$ for $t \geq \zeta$. The process Y is transient.

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In [KSV1], we proved that the Harnack inequality and Carleson's estimate hold for the non-negative harmonic functions of Y. For $d \geq 2$ and $\widetilde{w} \in \mathbb{R}^{d-1}$, we define $D_{\widetilde{w}}(a, b)$ to be $\{x = (\widetilde{x}, x_d) \in \mathbb{R}^d_+ : |\widetilde{x} - \widetilde{w}| < a, x_d < b\}$. When $d = 1$, $D_{\widetilde{w}}(a, b)$ stands for the interval $(0, b)$.

Theorem 1(Boundary Harnack principle), [KSV1, KSV2]

Suppose $p \in ((\alpha - 1)_+, \alpha + (\beta_1 \wedge \beta_2))$. Then there exists $C \geq 1$ such that for all $r > 0$, $\widetilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in $\mathbb{R}^d_+$ which is harmonic in $D_{\widetilde{w}}(2r, 2r)$ with respect to Y and vanishes continuously on $B((\widetilde{w}, 0), 2r) \cap \partial\mathbb{R}^d_+$, we have

$$\frac{f(x)}{x_d^p} \leq C_3 \frac{f(y)}{y_d^p}, \quad x, y \in D_{\widetilde{w}}(r/2, r/2).$$

Theorem 2, [KSV1, KSV2]

If $\alpha + \beta_2 \leq p < \alpha + \beta_1$, then the boundary Harnack principle is not valid for Y.

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If $\alpha + \beta_2 \leq p < \alpha + \beta_1$, then the boundary Harnack principle is not valid for Y.

A function $G : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to [0, \infty]$ is called the Green function for Y if G is the density of the occupation time:

$$\mathbb{E}_x \int_0^\zeta f(Y_t)dt = \int_{\mathbb{R}^d_+} G(x, y)f(y)dy, \quad x \in \mathbb{R}^d_+.$$

Theorem 3, [KSV2]

The process Y admits a Green function $G : \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to [0, \infty]$ such that $G(x, \cdot)$ is continuous in $\mathbb{R}^d_+ \setminus \{x\}$ and regular harmonic with respect to Y in $\mathbb{R}^d_+ \setminus B(x, \epsilon)$ for any $\epsilon > 0$. Moreover,

(1) If $p \in ((\alpha - 1)_+, \alpha + \frac{1}{2}[\beta_1 + (\beta_1 \wedge \beta_2)])$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$G(x, y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d}{|x-y|} \wedge 1\right)^p \left(\frac{y_d}{|x-y|} \wedge 1\right)^p. \qquad (1)$$

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Theorem 3 (cont), [KSV2]

(2) If $p = \alpha + \frac{\beta_1+\beta_2}{2}$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

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(3) If $p \in (\alpha + \frac{\beta_1+\beta_2}{2}, \alpha+\beta_1)$, then on $\mathbb{R}^d_+ \times \mathbb{R}^d_+$,

$$G(x,y) \asymp \frac{1}{|x-y|^{d-\alpha}} \left(\frac{x_d \wedge y_d}{|x-y|} \wedge 1\right)^p \left(\frac{x_d \vee y_d}{|x-y|} \wedge 1\right)^{2\alpha-p+\beta_1+\beta_2} \times$$
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Outline

In proving the results of the previous section, the strict positivity of the killing function was used in an essential way in several places. This includes the proof of finite lifetime, Carleson estimate, and decay of the Green function at the boundary.

What happens if the killing function is identically zero?

In [KSV3], we studied this case. Throughout this section, we assume the killing function is identically zero. It is easy to show that when $\alpha \in (0, 1]$, the process Y will not approach $\partial\mathbb{R}^d_+$ at the end of its lifetime, so there is no "boundary theory". So in this section, we also assume $\alpha \in (1, 2)$.

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Theorem 4 (Hardy inequality), [KSV3]

Then there exists $C = C(\alpha) \in (0, \infty)$ such that for all $u \in \mathcal{F}$,

$$\mathcal{E}(u,u) \geq C \int_{\mathbb{R}^d_+} \frac{u(x)^2}{x_d^\alpha} dx.$$

As a consequence we can get that the lifetime of Y is finite with probability 1. Furthermore

(a) For all $x \in \mathbb{R}^d_+$, $\mathbb{P}_x(Y_{\zeta-} \in \partial\mathbb{R}^d_+) = 1$;

(b) There exists a constant $n_0 \geq 2$ such that for all $x \in \mathbb{R}^d_+$, $\mathbb{P}_x\left(\tau_{B(x,n_0 x_d)} = \zeta\right) \geq 1/2$.

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(a) implies that Y is transient. (b) plays an important role in the proof of Carleson's estimate.

Combining these with some results from [KSV1, KSV2], we can prove the following results.

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Theorem 5 (Boundary Harnack principle), KSV3

There exists $C \geq 1$ such that for all $r > 0$, $\widetilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in $\mathbb{R}^d_+$ which is harmonic in $D_{\widetilde{w}}(2r, 2r)$ with respect to Y and vanishes continuously on $B((\widetilde{w}, 0), 2r) \cap \partial\mathbb{R}^d_+$, we have

$$\frac{f(x)}{x_d^{\alpha-1}} \leq C \frac{f(y)}{y_d^{\alpha-1}}, \quad x, y \in D_{\widetilde{w}}(r/2, r/2).$$

Theorem 6, [KSV3]

Then there exists $C > 1$ such that for all $x, y \in \mathbb{R}^d_+$,

$$C^{-1}\left(\frac{x_d}{|x-y|} \wedge 1\right)^{\alpha-1}\left(\frac{y_d}{|x-y|} \wedge 1\right)^{\alpha-1}\frac{1}{|x-y|^{d-\alpha}} \leq G(x,y)$$

$$\leq C\left(\frac{x_d}{|x-y|} \wedge 1\right)^{\alpha-1}\left(\frac{y_d}{|x-y|} \wedge 1\right)^{\alpha-1}\frac{1}{|x-y|^{d-\alpha}}.$$

Theorem 5 (Boundary Harnack principle), KSV3

There exists $C \geq 1$ such that for all $r > 0$, $\widetilde{w} \in \mathbb{R}^{d-1}$, and any non-negative function f in $\mathbb{R}^d_+$ which is harmonic in $D_{\widetilde{w}}(2r, 2r)$ with respect to Y and vanishes continuously on $B((\widetilde{w}, 0), 2r) \cap \partial\mathbb{R}^d_+$, we have

$$\frac{f(x)}{x_d^{\alpha-1}} \leq C \frac{f(y)}{y_d^{\alpha-1}}, \quad x, y \in D_{\widetilde{w}}(r/2, r/2).$$

Theorem 6, [KSV3]

Then there exists $C > 1$ such that for all $x, y \in \mathbb{R}^d_+$,

$$C^{-1} \left(\frac{x_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \left(\frac{y_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \frac{1}{|x-y|^{d-\alpha}} \leq G(x,y)$$
$$\leq \quad C \left(\frac{x_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \left(\frac{y_d}{|x-y|} \wedge 1 \right)^{\alpha-1} \frac{1}{|x-y|^{d-\alpha}} .$$

Outline

Now we have Green functions estimates. Can get get sharp two-sided heat kernel estimates also? Fix $\kappa \in [0, \infty)$. That is, we are dealing with the case either with or without critical killing. Let $p(t, x, y)$ be the heat kernel of Y. We will not need the assumption $d > (\alpha + \beta_1 + \beta_2) \wedge 2$.

Theorem 7, (CKSV)

Suppose that **(A1)**-**(A4)** hold. Then the process Y have a heat kernel $p : (0, \infty) \times \mathbb{R}^d_+ \times \mathbb{R}^d_+ \to (0, \infty)$ which is jointly continuous. Moreover, the heat kernel p has the following estimates: For $x \in \mathbb{R}^d_+$ and $t > 0$, let $x^t := x + t^{1/\alpha}\mathbf{e}_d$.

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Theorem 7 (Cont)

(a) If $\beta_2 < \alpha + \beta_1$, Then $p(t,x,y)$ is comparable with

$$\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p \widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x^t,y^t) \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right)$$

$$\asymp \left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p \left(t^{-d/\alpha} \wedge \frac{t\widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x^t,y^t)}{|x^t-y^t|^{d+\alpha}}\right).$$

Theorem 7 (Cont)

(b) If $\beta_2 > \alpha + \beta_1$, Then $p(t,x,y)$ is comparable with

$$\left(1 \wedge \frac{x_d}{t^{1/\alpha}}\right)^p \left(1 \wedge \frac{y_d}{t^{1/\alpha}}\right)^p \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}\right) \Big[\widetilde{B}_{\beta_1,\beta_2,\beta_3,\beta_4}(x^t, y^t) + \left(1 \wedge \frac{t}{|x-y|^\alpha}\right) \widetilde{B}_{\beta_1,\beta_1,0,\beta_3}(x^t, y^t) \log^{\beta_3}\left(e + \frac{|x-y|}{x_d^t \wedge y_d^t \wedge |x-y|}\right)\Big].$$

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(c) If $\beta_2 = \alpha + \beta_1$, Then $p(t,x,y)$ is comparable with

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Recall

$$\mathcal{E}(u,v) = \frac{1}{2}\int_{\mathbb{R}^d_+}\int_{\mathbb{R}^d_+}(u(x)-u(y))(v(x)-v(y))J(x,y)\,dy\,dx + \int_{\mathbb{R}^d_+} u(x)v(x)\kappa(x)dx.$$

Let $\overline{\mathcal{F}}$ be the closure of $C_c^\infty(\overline{\mathbb{R}^d_+})$ under $\mathcal{E}_1$. Then $(\mathcal{E},\overline{\mathcal{F}})$ is a regular Dirichlet form on $L^2(\overline{\mathbb{R}^d_+})$. We denote the associated Hunt process by $\overline{Y}$ (reflected process)

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Integrating our heat kernel estimates, we can get sharp two-sided Green function estimates. In these way, we can get rid of the assumption $d > (\alpha + \beta_1 + \beta_2) \wedge 2$ from [KSV2]

Before proving two-sided heat kernel estimates, we first prove a Nash-type inequality. Using this inequality, one can show that that there exists $c > 0$ such that

$$p(t, x, y) \leq c \left(t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}} \right),$$

for all $t > 0$ and $x, y \in \mathbb{R}^d_+$.

For the two-sided heat kernel estimates, the upper bound is the more difficult one. The following consequence of the Lévy system formula is repeatedly used to obtain the upper bound.

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Lemma 8

Let V_1 and V_3 be open subsets of $\mathbb{R}^d_+$ with $\mathrm{dist}(V_1, V_3) > 0$. Set $V_2 := \mathbb{R}^d_+ \setminus (V_1 \cup V_3)$. For any $x \in V_1$, $y \in V_3$ and $t > 0$, it holds that

$$p(t,x,y) \le \mathbb{P}_x(\tau_{V_1} < t < \zeta) \sup_{s \le t,\, z \in V_2} p(s,z,y)$$
$$+ \mathrm{dist}(V_1, V_3)^{-d-\alpha} \int_0^t \int_{V_3} \int_{V_1} p^{V_1}(t-s,x,u)\mathcal{B}(u,w)p(s,y,w)\,du\,dw\,ds.$$

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for all $t > 0$ and $x, y \in \mathbb{R}^d_+$.

For $p < \alpha$, Lemma 9 is not too difficult to prove. To get rid of the condition $p < \alpha$, we have to use a bootstrap method (induction).

To get sharp upper bound estimates, we have use a bootstrap method again. The final form of our two-sided heat kernel estimates tells us quite some detailed analysis are involved.

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Thank you!