Inverse Problems for Wave Propagation in 2 and 3 Dimensions

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Overview

- Motivation and model problems
- Inverse source problem
- Increasing stability
- Ongoing and future work

Source scattering problem

Source scattering problems are concerned with the relationship between radiating sources and wave fields.

- Inverse problem: To determine the radiating source which produces the measured wave field.
- Direct problem: To determine the wave field from the given source and the differential equation governing the wave motion.

Wave equation for source scattering

- The Helmholtz equation (acoustic wave):

$$(\Delta + k^2)u = f \quad in \quad \mathbb{R}^d.$$

- Attenuated Helmholtz equation:

$$(\Delta + k^2 + b)u = f \quad in \quad \mathbb{R}^d,$$

  where b is the attenuation factor.

- The Navier equation for elastic wave :

$$(\sigma + k^2)\mathbf{u} = \mathbf{f} \quad \text{in} \quad \mathbb{R}^d,$$

  where $\sigma = (\mu\Delta + (\mu + \lambda)\nabla \cdot \nabla)$, where μ, λ are Lame constants.

- Supported plate in a homogeneous medium:

$$(\Delta^2 - k^4)u = f \quad in \quad B_R \subset \mathbb{R}^2$$

Frame Title

- Helmholtz equation for two layered medium

$$u'' + (k^2(x) + i\alpha k(x))u = f$$

where the wave number k defines as follows

$$k(x) = \begin{cases} k_p & \text{if } x > 0 \\ k_n & \text{if } x < 0, \end{cases}$$

Application

- Antenna synthesis
- Tomography (PAT)
- Medical imaging (MEG, EEG, ENG)
- Geology
- Neuroscience (brain imaging)
- Material science and mechanical structure

Statement of the problem (I)

The scattering problem with the source term $-f_1 - ikf_0$,

$$(\Delta + k^2)u = -f_1 - ikf_0 \quad \text{in} \quad \mathbb{R}^2,$$

$$\lim r^{1/2}(\partial_r u - iku) = 0 \quad \text{as} \quad r = |x| \to +\infty,$$

where $supp f_0, supp f_1 \subset \Omega$, $\partial\Omega \in C^2$ and $\Gamma \subset \partial\Omega$ with outer unit normal ν.

The direct problem: Given f_1, f_0, to determine the random wave field u.
The inverse problem: To recover f_1, f_0 from given boundary data for $0 < k < K$.

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The direct problem: Given f_1, f_0, to determine the random wave field u.
The inverse problem: To recover f_1, f_0 from given boundary data for $0 < k < K$.
Our goals are uniqueness and stability of f_0, f_1 from the Cauchy data.

$$u = u_0,\ \partial_\nu u = u_1 \,\text{on}\, \Gamma, \text{ when} \quad K_* < k < K.$$

Known integral representation

$$u(x,k) = \frac{-i}{4}\int_{\Omega} H_0^{(1)}(k|x-y|)(f_1(y) + ikf_0(y))dy,$$

where $H_0^1(z) = \frac{1}{\pi i}\int_{1+i\infty}^{1} e^{izs}(s^2-1)^{-1/2}ds$, for $Rez > 0$, is the Hankel function of the first kind.

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It is also can be defined as

$$H_0^{(1)}(z) = J_0(z) + iY_0(z),$$

The Hankel function of fist kind

- Recurrence formula for the Hankel functions of first kind:

$$z\frac{dH_\nu^{(1)}(z)}{dz} + \nu H_\nu^{(1)}(z) = zH_{\nu-1}^{(1)}(z),$$

$$\frac{d}{dz}(z^\nu H_\nu^{(1)}(z)) = z^\nu H_{\nu-1}^{(1)}(z), \quad \nu = 1, 2, \cdots .$$

- Bounds for the Hankel function:

$$|H_0^{(1)}(z)| \leq \frac{e^{|\mathfrak{Im} z|}}{(\mathfrak{Re} z)^{1/2}}, \quad |\overline{H_0^{(1)}(z)}| \leq \frac{e^{|\mathfrak{Im} z|}}{(\mathfrak{Re} z)^{1/2}}.$$

Goals

(1)Uniqueness of Source Functions f_1, f_0

Theorem

Let u be a solution to the scattering problem with $f_0 \in H^1(\Omega), f_1 \in L^2(\Omega)$. If the Cauchy data $u_0 = u_1 = 0$ on Γ when $k \in (K_*, K)$, then $f_0 = f_1 = 0$ in Ω.

(2) Inverse source problem

Theorem

Let $\| f_0 \|^2_{(4)} (\Omega) + \| f_1 \|^2_{(3)} (\Omega) \leq M$, $1 \leq M$, *and* $\delta < |x-y|, x \in \partial\Omega, y \in suppf_0 \cup suppf_1$ *for some positive* δ. *Then there exist a constant* $C = C(\Omega, \delta)$ *such that*

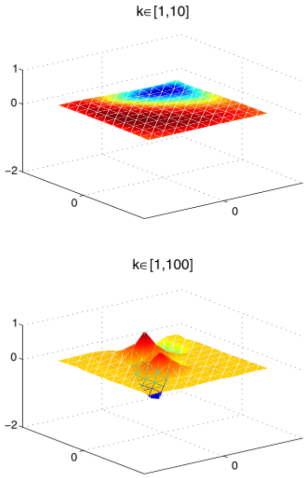
$$\| f_1 \|^2_{(0)} (\Omega) + \| f_0 \|^2_{(1)} (\Omega) \leq C\Big(\epsilon^2 + \frac{M^2}{1 + K^{\frac{2}{3}} |E|^{\frac{1}{4}}}\Big),$$

for all $u \in H^2(\Omega)$ *solving (1.1), (1.2) with* $1 < K$. *Here*

$$\epsilon^2 = \int_0^K \Big(\omega^2 \| u(,\omega) \|^2_{(0)} (\partial\Omega) + \| \nabla u(,\omega) \|^2_{(0)} (\partial\Omega)\Big) d\omega,$$

$$0 < E = -\ln \epsilon.$$

Numerical result



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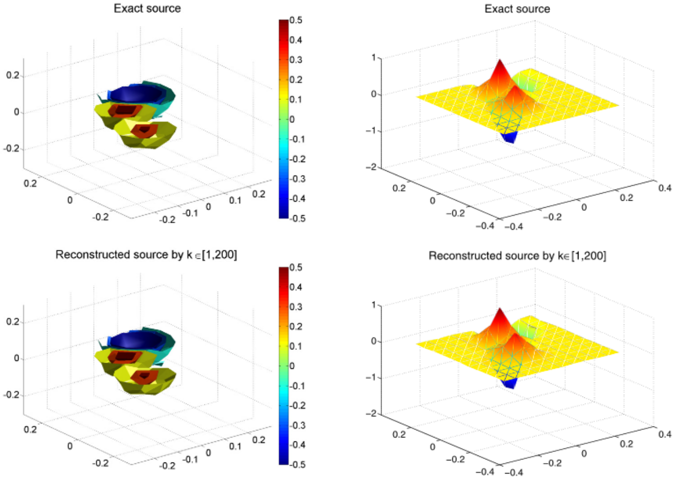

Sketch of proof

Step 1: denoting

$$\int_{-\infty}^{+\infty} \omega^2 \parallel u(,\omega) \parallel^2_{(0)} (\partial\Omega) d\omega = I_1(k) + \int_k^{\infty} \omega^2 \parallel u(,\omega) \parallel^2_{(0)} (\partial\Omega) d\omega,$$

&

$$\int_{-\infty}^{+\infty} \parallel \nabla u(,\omega) \parallel^2_{(0)} (\partial\Omega) d\omega = I_2(k) + \int_k^{\infty} \parallel \nabla u(,\omega) \parallel^2_{(0)} (\partial\Omega) d\omega.$$

where

$$I_1(k) = 2\int_0^k \omega^2 \int_{\partial\Omega} \Big(\int_\Omega \frac{-i}{4}(f_1(y) + i\omega f_0(y)) H_0^{(1)}(\omega|x-y|)dy\Big)$$

$$\Big(\int_\Omega \frac{i}{4}(f_1(y) - i\omega f_0(y)) \overline{H_0^{(1)}}(\omega|x-y|)dy\Big) d\Gamma(x) d\omega,$$

$$\&$$

$$I_2(k) = 2\int_0^k \int_{\partial\Omega} \Big(\int_{\partial\Omega} \frac{-i}{4}(f_1(y) + i\omega f_0(y)) \nabla_x H_0^{(1)}(\omega|x-y|)dy\Big)$$

$$\Big(\int_{\partial\Omega} \frac{i}{4}(f_1(y) - i\omega f_0(y)) \nabla_x \overline{H_0^{(1)}}(\omega|x-y|)dy\Big) d\Gamma(x) d\omega.$$

Lemma

Let $f_1 \in H^1(\Omega), f_0 \in H^1(\Omega)$ *and* $suppf_0, suppf_1 \subset \Omega$. *Then*

$$|I_1(k)| \leq \frac{\pi}{2}|\partial\Omega| d\Big(\frac{1}{3}|k|^3 \parallel f_1 \parallel^2_{(0)} (\Omega) + \frac{1}{5}|k|^5 \parallel f_0 \parallel^2_{(0)} (\Omega)\Big)\frac{e^{2d|k_2|}}{k_1},$$

$$|I_2(k)| \leq \frac{\pi}{2}|\partial\Omega| d\Big(\frac{1}{3}|k| \parallel f_1 \parallel^2_{(1)} (\Omega) + \frac{1}{5}|k|^3 \parallel f_0 \parallel^2_{(1)} (\Omega)\Big)\frac{e^{2d|k_2|}}{k_1},$$

where $|\partial\Omega|$ *is the length of* $\partial\Omega$, $d = sup|x-y|$ *over* $x, y \in \Omega$.

$$\epsilon^2 = \int_0^K \left(\omega^2 \parallel u(,\omega) \parallel^2_{(0)} (\partial\Omega) + \parallel \nabla u(,\omega) \parallel^2_{(0)} (\partial\Omega)\right) d\omega$$

$$|I_1(k)e^{-2(d+1)k}| \leq \epsilon^2 \quad on \quad [0, K],$$

Using harmonic measure on sector $S = \{k \in \mathbb{C} : |\arg k| < \frac{\pi}{4}\}$ or $|k_2| \leq k_1$ for any $k \in S$. So we obtain the following band for $K < k < \infty$.

$$|I_1(k)e^{-2(d+1)k}| \leq C\epsilon^{2\mu(k)} M^2,$$

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Lemma

Let $\mu(k)$ *be the harmonic measure of* $[0,K]$ *in* $S \backslash [0,K]$. *Then*

$$\begin{cases} \frac{1}{2} \leq \mu(k) & \textit{if} \quad 0 < k < 2^{\frac{1}{4}} K, \\ \frac{1}{\pi}\left((\frac{k}{K})^4 - 1\right)^{-1/2} \leq \mu(k) & \textit{if} \quad 2^{\frac{1}{4}} K < k. \end{cases}$$

Step 2: link between low and high frequencies

$$(\partial_t^2 - \Delta)U = 0 \text{ in } \mathbb{R}^2 \times (0, \infty),$$

$$U(,0) = -f_0, \quad \partial_t U(,0) = f_1 \text{ on } \mathbb{R}^2,$$

where U has the following well-known definition:

$$U(x,t) = \frac{1}{2\pi} \iint_{|x-y|<t} \frac{f_1(y)}{\sqrt{t^2 - r^2}} dy + \partial_t \Big(\frac{1}{2\pi} \iint_{|x-y|<t} \frac{-f_0(y)}{\sqrt{t^2 - r^2}} dy \Big),$$

where $r = |x - y|$.

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where $r = |x - y|$.
We define $U(x,t) = 0$ when $t < 0$ and claim:

$$u(x,k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty U(x,t) e^{ikt} dt.$$

Lemma

Let function u be a solution to the forward problem, with radiation condition (1.2) with $f_1 \in H^3(\Omega)$ *and* $f_0 \in H^4(\Omega)$, $suppf_1, suppf_0 \subset \Omega$. *Then*

$$\int_k^\infty \omega^2 \parallel u(,\omega) \parallel_{(0)}^2 (\partial\Omega) d\omega + \int_k^\infty \parallel \nabla u(,\omega) \parallel_{(0)}^2 (\partial\Omega) d\omega \leq$$

$$C(\delta) k^{-1} \Big(\parallel f_0 \parallel_{(4)}^2 (\Omega) + \parallel f_1 \parallel_{(3)}^2 (\Omega) \Big),$$

where $\delta = inf|y - x|$ *over* $x \in \partial\Omega$ *and* $y \in suppf_1 \cup suppf_0$.

Lemma

Let U be a solution to (1.10) with $f_1 \in L^2(\Omega), f_0 \in H^1(\Omega)$ with $suppf_0, suppf_1 \subset \Omega$ and $T = 2diam\Omega + 2$.
Then there is C such that

$$\| f_0 \|^2_{(1)} (\Omega) + \| f_1 \|^2_{(0)} (\Omega) \leq C\Big(\| \partial_t U \|^2_{(0)} (\partial\Omega \times (0, T)) +$$

$$+ \| \nabla U \|^2_{(0)} (\partial\Omega \times (0, T))\Big).$$

By the Parseval's identity;

$$\int_{k<|\omega|} \omega^2 \| u(,\omega) \|^2_{(0)} (\partial\Omega) d\omega \leq k^{-2} \int_{k<|\omega|} \omega^4 \| u(,\omega) \|^2_{(0)} (\partial\Omega) d\omega$$

$$\leq k^{-2} \int_R \omega^4 \| u(,\omega) \|^2_{(0)} (\partial\Omega) d\omega = k^{-2} \int_R \| \partial_t^2 U(,t) \|^2_{(0)} (\partial\Omega) dt.$$

Statement of the problem (II)

Attenuated Helmholtz equation;

$$(\Delta + k^2 + ikb)u = -f_1 - bf_0 + ikf_0 \quad in\ \mathbb{R}^2,\ 0 < k,$$

$$u = u_0, \partial_\nu u = u_1 \ on\ \partial\Omega\ with\ 0 < k < K,$$

where ν is the outer unit normal to $\partial\Omega$.

Using the following change of variable

$$u^*(x,\kappa) := u(x,k), \quad \kappa := k\sqrt{1+\frac{b}{k}i},$$

then the equation (22) becomes

$$\Delta u^* + \kappa^2 u^* = -f_1 - bf_0 + ikf_0,$$

with the solution

$$u^*(x,\kappa) = \frac{i}{4}\int_\Omega H_0^{(1)}(\kappa|x-y|)\big(f_1(y) + bf_0(y) - ikf_0(y)\big)\,dy.$$

Main Result

Theorem

There exists a generic constant C depending on the domain Ω such that

$$\| f_0 \|^2_{(1)} (\Omega) + \| f_1 \|^2_{(0)} (\Omega) \leq Ce^{Cb^2} \Big(\epsilon^2 + \frac{(b^2+1)M_3^2}{1 + K^{\frac{2}{3}} |E|^{\frac{1}{4}} + b} \Big),$$

for all $u \in H^2(\Omega)$ solving (1.1), with $1 < K$. Where

$$\epsilon^2 = \int_0^K \big(\omega^2 \| u(,\omega) \|^2_{(0)} (\partial\Omega) + \| \nabla u(,\omega) \|^2_{(0)} (\partial\Omega) \big) d\omega,$$

$E = -ln\epsilon$ *and* $M_3 = \max \big\{ \| f_0 \|_{(4)} (\Omega) + \| f_1 \|_{(3)} (\Omega), 1 \big\}$

Using hyperbolic initial boundary value problem for higher frequencies;

$$\partial_t^2 U - \Delta U + b\partial_t U = 0 \ on \ \Omega \times (0, \infty),$$

$$U(x,0) = f_0, \ \partial_\nu U(x,0) = f_1 \ on \ \Omega.$$

with the fundamental solution:

$$U(x,t) = \partial_t\Big(\frac{e^{\frac{-bt}{2}}}{2\pi}\iint_{|x-y|<t}\frac{\cosh(\frac{b}{2}\sqrt{t^2-r^2})f_0(y)}{\sqrt{t^2-r^2}}dy\Big)$$

$$+\frac{e^{\frac{-bt}{2}}}{2\pi}\iint_{|x-y|<t}\frac{\cosh(\frac{b}{2}\sqrt{t^2-r^2})f_1(y)}{\sqrt{t^2-r^2}}dy,$$

with $r = |x-y|$.

Observability Bound for the Damped Wave Equation

Theorem

Let the observation time $4(D+1) < T < 5(D+1)$. *Then there exists a generic constant* C *depending on the domain* Ω *such that*

$$\| f_0 \|_{(1)}^2 (\Omega) + \| f_1 \|_{(0)}^2 (\Omega) \leq C$$

$$e^{Cb^2} \Big(\| \partial_t U \|_{(0)}^2 (\partial\Omega \times (0, T)) + \| \nabla U \|_{(0)}^2 (\partial\Omega \times (0, T)) \Big),$$

for all $U \in H^2(\Omega \times (0, \infty))$.

Frame Title

Let

$$E(t) := \int_\Omega (|\partial_t U(x,t)|^2 + |\nabla U(x,t)|^2 + |U(x,t)|^2)dx,$$

$$E_0(t) := \int_\Omega (|\partial_t U(x,t)|^2 + |\nabla U(x,t)|^2)dx,$$

and

$$F^2 := \frac{1}{2}\int_{\partial\Omega\times(0,T)} (|\partial_t U(x,t)|^2 + |\nabla U(x,t)|^2 + U^2(x,t))d\Gamma(x).$$

Let U be a solution of the forward problem (3.25) with $f_1 \in H^1(\Omega)$ and $f_0 \in H^2(\Omega)$, $suppf_0$, $suppf_1 \subset \Omega$. Let $0 \le t_1 \le t_2 \le T$ and $1 \le 2b$. Then

$$E(t_2) \le e^{4(t_2-t_1)^2}(2E(t_1) + F^2)$$

and

$$E(t_1) \le 2e^{(2b+4(t_2-t_1))(t_2-t_1)}(E(t_2) + F^2).$$

Lemma

Let u be a solution to the forward problem with $f_1 \in H^3(\Omega)$ *and* $f_0 \in H^4(\Omega)$ *with* $suppf_0, suppf_1 \subset \Omega$, *then*

$$\int_k^\infty \omega^2 \parallel u(,\omega) \parallel_{(0)}^2 (\partial\Omega) d\omega + \int_k^\infty \parallel \nabla u(,\omega) \parallel_{(0)}^2 (\partial\Omega) d\omega$$

$$\leq C(k+b)^{-1}\Big((1+b^2) \parallel f_0 \parallel_{(4)}^2 (\Omega) + \parallel f_1 \parallel_{(3)}^2 (\Omega)\Big),$$

where $C = C(\Omega, \delta)$, $\delta = inf|y-x|$, $x \in \partial\Omega$ *and* $y \in suppf_1 \cup suppf_0$.

Statement of problem (III), elasticity system

$$\sigma(\mathbf{u}) + \rho k^2 \mathbf{u} = -\mathbf{f}_1 - \mathrm{i}k\mathbf{f}_0 \quad \text{in} \quad \mathbb{R}^3,$$

where $\sigma = (\mu\Delta + (\mu + \lambda)\nabla \cdot \nabla)$, where μ, λ are Lame constants.

Goal: Recovering the external force from the Dirichlet data

$$\mathbf{u} = \mathbf{u}_0 \quad \text{on} \quad \Gamma.$$

By the Helmholtz decomposition, the displacement filed $\mathbf{u}$ can be written as

$$\mathbf{u} = \mathbf{u}_p + \mathbf{u}_s \quad \text{in} \quad \mathbb{R}^3 \backslash \overline{\Omega}$$

where $\mathbf{u}_p$ and the sheer part $\mathbf{u}_s$ which satisfy Sommerfeld radiation conditions

$$\lim_{r\to\infty} r(\partial_r \mathbf{u}_p - \mathrm{i}k_p \mathbf{u}_p) = 0, \quad \lim_{r\to\infty} r(\partial_r \mathbf{u}_s - \mathrm{i}k_s \mathbf{u}_s) = 0, \quad r = |\mathbf{x}|,$$

where k_p, k_s are the compressional and shear wavenumbers defined as follows

$$k_p = \frac{k}{(\lambda + 2\mu)^{1/2}} = c_p k, \quad k_s = \frac{k}{\mu^{1/2}} = c_s k.$$

$$\mathbf{u}(x,k) = \int_\Omega \mathbf{G}(x,y;k)\cdot(\mathbf{f}_1(y) + \mathrm{i}k\mathbf{f}_0(y))d(y),$$

where $\mathbf{G}(x,y;k)$ is Green's tensor with following representation

$$\mathbf{G}(x,y;k) = \frac{1}{4\pi\mu}\frac{e^{\mathrm{i}k_s|\mathbf{x}-\mathbf{y}|}}{|x-y|}\mathbf{I}_3 + \frac{1}{k^2}\nabla_x\nabla_x^\top\left(\frac{e^{\mathrm{i}k_s|x-y|}}{|x-y|} - \frac{e^{\mathrm{i}k_p|x-y|}}{|x-y|}\right)$$

where $\mathbf{I_3}$ is 3×3 identity matrix.

$$\rho\partial_t^2\mathbf{U} - \sigma(\mathbf{U}) = 0 \quad \text{in} \quad (0,) \times \mathbb{R}^3,$$

$$\mathbf{U}(,0) = \mathbf{f}_0, \quad \partial_t\mathbf{U}(,0) = \mathbf{f}_1 \quad on \quad \mathbb{R}^3.$$

$$\mathbf{u}(x,k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{U}(x,t)e^{ikt}dt$$

where

$$\mathbf{U}(x,t) = \mathbf{V}(-c_s^2\Delta\mathbf{f}_1 + (c_P^2 - c_s^2)\nabla div\mathbf{f}_1)(x,t) +$$

$$\partial_t\mathbf{V}(-c_s^2\Delta\mathbf{f}_0 + (c_P^2 - c_s^2)\nabla div\mathbf{f}_0)(x,t) + \partial_t^2\mathbf{V}(\mathbf{f}_1)(x,t) + \partial_t^3\mathbf{V}(\mathbf{f}_1)(x,t)$$

and

$$\mathbf{V}(\mathbf{F})(x,y) = \frac{1}{c_P^2 - c_s^2}(\int_{|x-y|\leq c_s t}\frac{c_p - c_s}{c_s}\mathbf{F}(y)dy +$$

$$\int_{c_s\leq|x-y|\leq c_p t}\frac{c_p t - |x-y|}{|x-y|}\mathbf{F}(y)dy).$$

Result for elasticity system

Theorem

There exist a constant $C = C(\Omega, \delta)$ *such that*

$$\| \mathbf{f}_1 \|^2_{(0)} (\Omega) + \| \mathbf{f}_0 \|^2_{(1)} (\Omega) \leq C\Big(\epsilon^2 + \frac{M^2}{1 + K^{\frac{4}{3}} |E_e|^{\frac{1}{2}}}\Big),$$

for all $\mathbf{u} \in H^2(\Omega)$ *solving (1.1), (1.2) with* $1 < K$. *Here*

$$\epsilon^2 = \int_0^K \Big(\omega^2 \| \mathbf{u}(,\omega) \|^2_{(0)} (\partial\Omega) + \| \mathbf{u}(,\omega) \|^2_{(1)} (\partial\Omega)\Big) d\omega,$$

$$0 < E_e = -\ln \epsilon, M = \| \mathbf{f}_1 \|^2_{(3)} (\Omega) + \| \mathbf{f}_0 \|^2_{(3)} (\Omega).$$

Significant of the results?

References

- Jin Cheng, Victor Isakov and Shuai Lu Increasing stability in the inverse source problem with many frequencies Journal of Differential Equations Volume 260, Issue 5, 5 March 2016.
- Victor Isakov and Shuai Lu: Increasing stability in the inverse source problem with attenuation and many frequencies, SIAM J.Appl. Math., 78(1), 1–18.
- Peijun L and Ganchua Yuan: Increasing stability for the inverse source scattering problem with multi frequencies, Inverse Problems and Imaging Volume 11, No. 4, 2017, 745–759.
- Peijun L and Ganchua Yuan, Stability on the inverse random source scattering problem for the one-dimensional Helmholtz equation, J. Math. Anal. Appl., 450 (2017), 872-887.

- Entekhabi and Victor Isakov , On increasing stability in the two dimensional inverse source scattering problem with many frequencies, Inverse Probl., 34 (2018), 055005.
- Entekhabi, Increasing stability in the two dimensional inverse source scattering problem with attenuation and many frequencies, Inverse Problems 34, 2018 115001.
- Entekhabi and Victor Isakov , On increasing stability in the inverse source acoustic and elastic scattering problems with many frequencies(2018).
- Entekhabi and A. Gunaratne, A logarithmic estimate for the inverse source scattering problem with attenuation in a two-layered medium, Journal of Inverse and Ill-posed Problems (published online ahead of print). doi: https://doi.org/10.1515/jiip-2019-0025.
- B.-Z. Guo and Z.-X. Zhang, Well-Posedness of Systems of Linear Elasticity with Dirichlet Boundary Control and Observation, SIAM J. Control Optimiz.48(2009), 2139–2167.

Ongoing and future work

- Partial data
- Higher order equation
- Random data
- Extending to the inverse problem in time domain

New Research: Trapped Mode in Two-Dimensional Waveguides with Inclusion

We seek a potential $\phi(x, y)$ satisfying

$$(\Delta + k^2)\phi \quad in \quad r < a, |y| < d, r = (x^2 + y^2)^{1/2}$$

$$\phi_y = 0, |y| = d, -\infty < x < \infty$$

$$\phi_r = 0, r = a,$$

$$\phi = 0, y = 0, |x| \geq a$$

$$\phi \to 0, |x| \to \infty, |y| \leq d$$

Thank You !