Level of noise and long time behavior of space-time fractional SPDEs in bounded domains

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Outline

Recently, Mijena and N. [2015], Foondun, Mijena, and N. [2016], and Foondun [2021] considered the following time fractional stochastic heat equation on the interval $(0, L)$ with Dirichlet boundary condition:

$$\begin{cases} \partial_t^\beta u_t(x) = \frac{1}{2}\partial_{xx} u_t(x) + I_t^{1-\beta}[\lambda\sigma(u_t(x))\dot{W}(t,x)], \quad 0 < x < L\,, \ t > 0 \\ u_t(0) = u_t(L) = 0 \quad \text{for} \quad t > 0, \end{cases} \tag{1.1}$$

where

- ∂_t^β is the Caputo fractional time derivative which first appeared in Caputo [1967] and is defined for $0 < \beta < 1$ by

$$\partial_t^\beta u_t(x) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{\partial u_r(x)}{\partial r} \frac{\mathrm{d}r}{(t-r)^\beta},$$

  and for $\gamma > 0$, I_t^β is the fractional order integral defined by

$$I_t^{\gamma} f(t) := \frac{1}{\Gamma(\gamma)} \int_0^t (t-\tau)^{\gamma-1} f(\tau) \mathrm{d}\tau.$$

We know that for $\beta \in (0,1)$, and $g \in L^{\infty}(\mathbb{R}_+)$ or $g \in \boldsymbol{C}(\mathbb{R}_+)$
$\partial_t^{\beta} I_t^{\beta} g(t) = g(t)$.

- $u_0 : B \to \mathbb{R}_+$ is a non-random measurable and bounded function that has support with positive measure inside B;
- $\dot{W}$ denotes a space-time white noise and $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function satisfying $l_\sigma |\mathrm{x}| \leq |\sigma(\mathrm{x})| \leq L_\sigma |\mathrm{x}|$ where l_σ and L_σ are positive constants;
- λ is a positive parameter called *the level of the noise*.

- If the partial derivative in time ∂_t in the classical heat equation $\partial_t u = \Delta u$ is substituted with fractional derivatives ∂_t^β for $0 < \beta < 1$, the processes explains the sticking and trapping behavior of particle,
- while if the Laplacian Δ is replaced with fractional power $-(-\Delta)^{\alpha/2}$ for $0 < \alpha < 2$, it describes long particle jumps.

In Foondun and Nualart [2015] and Foondun, Guerngar, and N. [2017], the authors looked at the behavior of the solution to equation (1.1) for small λ and large λ when $\beta = 1$. They showed that

- For λ large enough, the second moment of the solution u_t grows exponentially fast; while
- for small λ, the second moment of the solution u_t eventually decays exponentially.

However, Foondun [2021] has shown this phase transition is no longer valid and a more complicated situation occurs if the usual derivative is replaced by a fractional derivative in time when $0 < \beta < 1$.

From a practical point of view, the results in Foondun [2021] are relevant because fractional time derivatives are often used for modeling of various systems with memory.
Therefore, it is very important to understand that the use of such derivatives can cause significant changes in the qualitative properties of the solution.

Goal

Our main recent results investigate the long time behavior of the solution to the equation (2.1) with respect to λ. Our work extend the main results in Foondun [2021] to fractional Laplacian case in space dimensions $d = 1, 2, 3$.

Stable processes

- Let X_t denote a symmetric stable process of index $\alpha \in (0,2)$ in $\mathbb{R}^d$ and B be a regular bounded open subset of $\mathbb{R}^d$.
- Let X_t^B denote the symmetric stable process killed upon exiting B.
- The Dirichlet fractional heat kernel (the transition density of X_t^B) $p_B(t, \mathrm{x}, \mathrm{y})$ on B has spectral decomposition

$$p_B(t, \mathrm{x}, \mathrm{y}) := \sum_{n=1}^{\infty} e^{-\mu_n t}\varphi_n(\mathrm{x})\varphi_n(\mathrm{y}), \tag{1.2}$$

for all $\mathrm{x}, \mathrm{y} \in B$, $t > 0$, where $\{\varphi_n\}_{n\geq 0}$ is an orthonormal basis of $L^2(B)$, and $0 < \mu_1 < \mu_2 \leq \mu_3 \leq \cdots$ is a sequence of positive real numbers such that, for every $n \geq 1$, $P_t^B\varphi_n(x) = e^{-\mu_n t}\varphi_n(x)$, and $\varphi_n(x) = 0$ for $x \in B^C$.

Stable subordinator

- Let $D = \{D_r, r \geq 0\}$ denote a β-stable subordinator and E_t be the inverse of a stable subordinator of index $\beta \in (0, 1)$.
- The process $X^B_{E_t}$ is a time-changed symmetric α-stable process X^B_t killed upon exiting B, and since $\beta \in (0, 1)$, $X^B_{E_t}$ moves slower than X^B_t.
- The density of the time-changed process $X^B_{E_t}$ is given by $G^{(\beta)}_B(t, x)$.

By conditioning, we have

$$G^{(\beta)}_B(t, x) = \int_0^\infty p_B(s, x) f_t(s) ds, \tag{1.3}$$

where

$$f_t(s) = t\beta^{-1}s^{-1-1/\beta}g_\beta(ts^{-1/\beta}). \tag{1.4}$$

where $g_\beta(.)$ (Cf. Meerschaert and Straka [2013]) is the density function of D_1 and is infinitely differentiable on the entire real line, with $g_\beta(u) = 0$ for $u \leq 0$; see Meerschaert and Scheffer [2004] for more information about for properties of the inverse stable subordinator E_t.

The function $u_t(\mathrm{x}) := \mathbb{E}^{\mathrm{x}}[u_0(X^B_{E_t})]$ solves the space-time fractional equation

$$\begin{cases} \partial_t^\beta u_t(\mathrm{x}) = -(-\Delta)^{\alpha/2}u_t(\mathrm{x}) \ \text{ for } \ \mathrm{x} \in B \ \text{ and } \ t > 0 \\ u_t(\mathrm{x}) = 0 \quad \text{for } \ \mathrm{x} \notin B, \end{cases} \tag{1.5}$$

with initial condition u_0 (Chen, Meerschaert, and N. [2012]).

Thus, we get the following representation of $u_t(x)$

$$
\begin{aligned}
u_t(\mathrm{x}) &:= \mathbb{E}^{\mathrm{x}}[u_0(X^B_{E_t})] \\
&= \int_B \int_0^\infty \sum_{n=1}^\infty e^{-\mu_n s} \varphi_n(\mathrm{x}) \varphi_n(\mathrm{y}) f_t(s)\, u_0(\mathrm{y})\, ds d\mathrm{y} \\
&= \int_B \sum_{n=1}^\infty E_\beta(-\mu_n t^\beta) \varphi_n(\mathrm{x}) \varphi_n(\mathrm{y}) u_0(\mathrm{y}) d\mathrm{y} \\
&= \int_B G_B^{(\beta)}(t, \mathrm{x}, \mathrm{y}) u_0(\mathrm{y})\, d\mathrm{y},
\end{aligned}
$$

where

$$
G_B^{(\beta)}(t, \mathrm{x}, \mathrm{y}) := \sum_{n=1}^\infty E_\beta(-\mu_n t^\beta) \varphi_n(\mathrm{x}) \varphi_n(\mathrm{y}) \tag{1.6}
$$

Mittag-Leffler function

Note that $E_\beta(x) = \sum_{k=1}^{\infty} \frac{x^k}{\Gamma(1+\beta k)}$ is the Mittag-Leffler function and has the following property,

$$\frac{1}{1+\Gamma(1-\beta)x} \leq E_\beta(-x) \leq \frac{1}{1+\Gamma(1+\beta)^{-1}x} \quad \text{for } x > 0. \quad (1.7)$$

and

$$E_\beta(-\mu_n t^\beta) = \int_0^\infty e^{-\mu_n s} f_t(s)\, ds = \mathbb{E}(e^{-\mu_n E_t})$$

is the eigenfunctions of the Caputo time fractional derivative:

$$\partial_t^\beta E_\beta(-\mu_n t^\beta) = -\mu_n E_\beta(-\mu_n t^\beta).$$

We consider the following stochastic heat equation on a regular bounded domain B in $\mathbb{R}^d$, $d \geq 1$ with Dirichlet boundary condition:

$$\begin{cases} \partial_t^\beta u_t(x) = -(-\Delta)^{\frac{\alpha}{2}} u_t(x) + I_t^{1-\beta}[\lambda \sigma(u_t(x)) \dot{W}(t,x)], & x \in B, \quad t > 0 \\ u_t(x) = 0 \quad \text{for} \quad x \notin B \ \text{and} \ t > 0, \end{cases} \tag{2.1}$$

where

- the operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $0 < \alpha \leq 2$, is the L^2-generator of a symmetric α-stable process X_t^B killed when exiting B;
- u_0, $\dot{W}$, σ and λ are as mention in the previous slides.

Walsh mild solution

Following Walsh [1986], we look at the mild solution of the equation (2.1) satisfying the following integral equation.

$$u_t(x) = (\mathcal{G}_B^{(\beta)} u_0)_t(\mathrm{x}) + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s, \mathrm{x}, \mathrm{y}) \sigma(u_s(\mathrm{y})) W(\mathrm{d}s, \mathrm{d}\mathrm{y}), \quad (2.2)$$

where $G_B^{(\beta)}(t, \mathrm{x}, \mathrm{y})$ denotes the heat kernel of the space-time fractional diffusion equation with Dirichlet boundary conditions in (1.5), and

$$(\mathcal{G}_B^{(\beta)} u_0)_t(x) := \int_B G_B^{(\beta)}(t, \mathrm{x}, \mathrm{y}) u_0(\mathrm{y})\, \mathrm{d}\mathrm{y}. \quad (2.3)$$

Existence of unique random-field solution

Theorem 1 (Foondun, Mijena and N. [2016])

If $d < (2 \wedge \beta^{-1})\alpha$, equation (2.1) has a unique random-field solution u_t satisfying

$$\sup_{x \in B} \mathbb{E}|u_t(x)|^2 \leq c_1 e^{c_2 t \lambda^{\frac{2\alpha}{\alpha - \beta d}}}$$

for all $t > 0$.

Theorem 2 (Mijena, N. and Negash, 2022+)

Suppose that $d < (2 \wedge \beta^{-1})\alpha$. *Let* u_t *denote the unique solution to* (2.2).

- *Then no matter what* λ *is, the second moment of* u_t *cannot decay exponentially fast.*
- *In fact, if we further assume that* $\beta \in (0, \frac{1}{2}]$, *then as* t *gets large,* $\sup_{x\in B} \mathbb{E}|u_t(x)|^2$ *grows exponentially fast for any* λ.

Remark 3

If $d = 1$, $\beta = 1$, *and* $\alpha = 2$, *our result above gives Theorem 1.1 of Foondun and Nualart [2015] which says the second moment of the solution having exponential growth in time when the level of the noise* λ *is large (as explained in frame 6).*

Since the Mittag-Leffler function $E_\beta(-t^\beta)$ behaves as a stretched exponential for $t \to 0$;

$$E_\beta(-t^\beta) \backsimeq 1 - \frac{t^\beta}{\Gamma(\beta+1)} \backsimeq e^{-t^\beta/\Gamma(\beta+1)}, \text{ for } 0 < t \leq 1,$$

and as a polynomial decay for $t \to \infty$,

$$E_\beta(-t^\beta) \backsimeq \frac{\sin(\beta\pi)}{\pi} \frac{\Gamma(\beta)}{t^\beta}, \text{ for } t \geq 1,$$

the polynomial decay behaviour of $E_\beta(-t^\beta)$ illustrates the need for the sharp condition of $\beta \in (0, \frac{1}{2}]$ in Theorem 2.

The following two theorems demonstrate that the condition $\beta \in (0, \frac{1}{2}]$ is not only a technical limitation of the proof of Theorem 2.

Theorem 4 (Mijena, N. and Negash, 2022+)

In Theorem 2, if $\beta \in (\frac{1}{2}, 1)$, then there exist a strictly positive real number λ_u such that for all $\lambda > \lambda_u$, $\sup_{x \in B} \mathbb{E}|u_t(x)|^2$ grows exponentially fast as time gets large.

Theorem 5 (Mijena, N. and Negash, 2022+)

In Theorem 2, suppose that $\beta \in (\frac{1}{2}, 1)$. Suppose also that either

- *$d < \alpha/2\beta$ (this implies $(d = 1) < \alpha/2\beta$) or*
- *$\{\varphi_n\}_{n\geq 1}$ are uniformly bounded by a constant $C(B)$,*

then there exist a strictly positive real number λ_l such that for $\lambda < \lambda_l$ the quantity $\sup_{t>0} \sup_{x \in B} \mathbb{E}|u_t(x)|^2$ is finite.

Foondun's interpretation of these three results

- The above three results show that for any fixed β, the solution to the stochastic heat equation (2.1) behaves very differently to that of the case when $\beta = 1$ as described in frame 6.
- When the $\beta \in (0, \frac{1}{2}]$, the process $X^B_{E_t}$ do not reach the boundary quickly enough which allows the non-linear term to kick in.
- When $\beta \in (\frac{1}{2}, 1)$, this process proceeds quickly enough to the boundary so that the non-linear term does not have time to generate the exponential growth.

[**Sketch of the Proof of Theorem 5**] We establish the required result using Ito-Walsh isometry and the global Lipschitz assumption on σ. The second moment of the mild formulation given by (2.2) becomes:

$$\begin{aligned} \mathrm{E}|u_t(x)|^2 &= |(\mathcal{G}_B^{(\beta)} u)_t(\mathrm{x})|^2 \\ &+ \lambda^2 \int_B \int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})^2 \mathrm{E}|\sigma(u_s(\mathrm{y}))|^2 \mathrm{d}s\,\mathrm{dy} \\ &\le |(\mathcal{G}_B^{(\beta)} u)_t(\mathrm{x})|^2 \\ &+ \lambda^2 L_\sigma^2 \int_B \int_0^t G_B^{(\beta)}(t-s,\,\mathrm{x},\,\mathrm{y})^2 \mathrm{E}|u_s(\mathrm{y})|^2 \mathrm{d}s\,\mathrm{dy} \\ &:= J_1 + J_2. \end{aligned} \tag{2.4}$$

We observe J_1 is bounded by the square of the same constant as initial condition, since $\int_B G_B^{(\beta)}(t, x)\,\mathrm{d}x \leq \int_{\mathbb{R}^d} G^{(\beta)}(t, x)\,\mathrm{d}x = 1$ and the initial datum is assumed to be bounded above by a constant.
It remains to bound J_2. Since $\{\varphi_n\}_{n\geq 1}$ is an orthonormal sequence, for each fixed $t > 0$ we obtain

$$J_2 := \lambda^2 L_\sigma^2 \int_B \int_0^t G_B^{(\beta)}(t-s, \mathrm{x}, \mathrm{y})^2 \mathrm{E}|u_s(\mathrm{y})|^2 \mathrm{d}s\,\mathrm{d}\mathrm{y}$$
$$\leq a_t \lambda^2 L_\sigma^2 \int_0^t \sum_{n=1}^{\infty} E_\beta(-\mu_n (t-s)^\beta)^2 \varphi_n^2(\mathrm{x})\,\mathrm{d}s < \infty, \tag{2.5}$$

where $a_t = \sup_{0<s<t} \sup_{\mathrm{x}\in B} \mathrm{E}|u_s(\mathrm{x})|^2$.

Uniform bounds on the eigenfunctions

Example 6

Let X_t denote a Brownian motion in $\mathbb{R}^2$ and X_t^B denote the Brownian motion killed upon exiting the rectangular domain $B := [0,\ L_1] \times [0,\ L_2]$. The eigenvalues of the Dirichlet Laplacian are $\mu_{m,n} = (\frac{m\pi}{L_1})^2 + (\frac{n\pi}{L_2})^2$ and $\varphi_{m,n}(x,y) := \varphi_m(x)\varphi_n(y) = \frac{2}{\sqrt{L_1 L_2}} \sin(\frac{m\pi x}{L_1}) \sin(\frac{n\pi y}{L_2})$ are the corresponding eigenfunction so that $|\varphi_{m,n}(x.y)| \leq \frac{2}{\sqrt{L_1 L_2}}$ for all $(x,y) \in B$. A similar result is valid for the Brownian motion in higher dimensional rectangular boxes.

The other class of equations that we consider in this talk is equation with space colored noise:

$$\begin{cases} \partial_t^\beta u_t(\mathrm{x}) = -(-\Delta)^{\frac{\alpha}{2}} u_t(\mathrm{x}) + I_t^{1-\beta}[\lambda\sigma(u_t(\mathrm{x}))\dot{F}(t,\mathrm{x})] \ \ \mathrm{x} \in B, \quad t > 0 \\ u_t(\mathrm{x}) = 0 \quad \text{for} \quad \mathrm{x} \notin B \ \text{ and } \ t > 0, \end{cases} \tag{3.1}$$

and the initial condition $u_0 : B \to \mathbb{R}_+$ is a non-random measurable and bounded function that has support with positive measure inside B. The operator $-(-\Delta)^{\frac{\alpha}{2}}$, where $0 < \alpha \leq 2$, is the L^2-generator of a symmetric α-stable process X_t^B killed when exiting B.

The function $\sigma : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function satisfying $l_\sigma|x| \le |\sigma(x)| \le L_\sigma|x|$ where l_σ and L_σ are positive constants. The positive parameter λ is called *the level of the noise*.
The noise $\dot{F}(t, x)$ is white in time and colored in space satisfying

$$Cov(\dot{F}(t, x), \dot{F}(s, y)) = \delta_0(t-s)f(x, y),$$

where $0 < f(x, y) \le g(x-y)$ and g is a locally integrable function on $\mathbb{R}^d$ with possible singularity at 0 satisfying Dalang condition

$$\int_{\mathbb{R}^d} \frac{\hat{g}(\xi)}{1+|\xi|^\alpha} \mathrm{d}\xi < \infty, \tag{3.2}$$

where $\hat{g}$ denotes the Fourier transform of g.

We will need the following non degeneracy condition on the spatial correlation of the noise.

Assumption 7

Assume there exists some positive number K_f such that

$$\inf_{x,\, y \in B} f(x,\, y) \geq K_f .$$

This assumption is very mild as it is shown by the following examples.

Example 8

For the following list of examples Dalang's condition–Assumption 3.2– is satisfied.

- Riesz Kernel:

$$f(x,y) = \frac{1}{|x-y|^{\gamma}} \text{ with } \gamma < d \wedge \alpha.$$

- The Exponential-type kernel: $f(x,y) = exp[-(x \cdot y)]$.
- The Ornstein-Uhlenbeck-type kernels: $f(x,\ y) = exp[-|x-y|^{\delta}]$ with $\delta \in (0,\ 2]$.
- Poisson Kernels:

$$f(x,y) = \left(\frac{1}{|x-y|^2+1} \right)^{\frac{d+1}{2}}.$$

- Cauchy Kernels:

$$f(x,y) = \sum_{j=1}^{d} \left(\frac{1}{1+(x_j-y_j)^2} \right).$$

Following Walsh [1986], u_t is a mild solution to (3.1) if

$$u_t(x) = (\mathcal{G}_B^{(\beta)} u_0)_t(\mathrm{x}) + \lambda \int_B \int_0^t G_B^{(\beta)}(t-s,\, \mathrm{x},\, \mathrm{y}) \sigma(u_s(\mathrm{y})) F(\mathrm{d}s\,, \mathrm{dy}), \tag{3.3}$$

where $G_B^{(\beta)}(t, \mathrm{x}, \mathrm{y})$ denotes the heat kernel of the space-time fractional diffusion equation with Dirichlet boundary conditions in (3.1), and

$$(\mathcal{G}_B^{(\beta)} u_0)_t(x) := \int_B G_B^{(\beta)}(t,\, \mathrm{x},\, \mathrm{y}) u_0(\mathrm{y})\, \mathrm{dy}.$$

Exponentially growing second moments

Theorem 9 (Mijena, N., Negash, 2022+)

Suppose that the Dalang condition (3.2) holds. Let u_t denote the unique solution to (2.2). Then no matter what λ is, the second moment of u_t cannot decay exponentially fast. In fact, if we further assume that $\beta \in (0, \frac{1}{2}]$ and Assumption 7 holds, then as t gets large, $\sup_{x \in B} \mathrm{E}|u_t(x)|^2$ grows exponentially fast for any λ.

Theorem 10 (Mijena, N., Negash, 2022+)

In Theorem 9, if $\beta \in (\frac{1}{2}, 1)$ and Assumption 7 hold, then there exist a strictly positive real number λ_u such that for all $\lambda > \lambda_u$, $\sup_{x \in B} \mathrm{E}|u_t(x)|^2$ grows exponentially fast as time gets large.

Bounded second moments

Theorem 11 (Mijena, N., Negash, 2022+)

In Theorem 9, suppose that $\beta \in (\frac{1}{2}, 1)$, $d < \frac{\alpha}{2\beta}$, and $\int_{B\times B} f(x, y)\,\mathrm{d}x\,\mathrm{d}y < \infty$. If $\{\varphi_n\}_{n\geq 1}$ are uniformly bounded by a constant $C(B)$, then there exist a strictly positive real number λ_l such that for $\lambda < \lambda_l$ the quantity $\sup_{t>0}\sup_{x\in B}\mathrm{E}|u_t(x)|^2$ is finite.

Corollary 12

In Theorem 11, if $d = 1$ and $\alpha = 2$ then the conclusion of theorem follows.

Corollary 13

In Theorem 11, if $d = 1$ and f is Riesz Kernel function, then the conclusion of theorem follows.

Continuity of solutions in the fractional parameter β

The following continuity theorem of the unique solution $u_t^{\beta}(x)$ of equation (14) with respect to the parameter β is Theorem 4.3(b) of Dang et al. [2018].

Theorem 14 (Dang, N. and Nguyen [2018])

Let $u_t^{(\gamma)}$ and $u_t^{(\beta)}$ denote the solution to the following equation for parameters $\gamma, \beta \in (0, 1)$ with $\gamma \to \beta$. The initial condition u_0 is the same for both equations.

$$\begin{aligned} \frac{\partial^{\beta}}{\partial t^{\beta}} u_t(x) &= \Delta u_t(x),\ x \in B,\ t > 0, \\ u_t(x) &= 0,\ x \in \partial D,\ t > 0, \\ u(0, x) &= f(x),\ x \in B. \end{aligned} \tag{4.1}$$

Then, we have

$$\lim_{\gamma \to \beta} ||u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)||_H^2 = 0.$$

Continuity of the solution $u_t^\beta(x)$

where H is the Hilbert space of all functions with the bounded norm induced by

$$||f||_H = \sqrt{\sum_{k=1}^{\infty} \mu_k^2 |\langle f, \varphi_k \rangle|^2}.$$

Remark 15

Meerschaert et al. [2009] established the existence of a unique solution for equation (4.1)*.*

Continuity of the solution $u_t^\beta(x)$

The following continuity theorem of the unique solution $u_t^\beta(x)$ of equation (1.1) with respect to the parameter β is Theorem 1.3 of Foondun [2021].

Theorem 16 (Foondun [2021])

Let $u_t^{(\beta)}$ and u_t denote the solution to equation (1.1) and the solution to equation (1.1) for $\beta = 1$ respectively. The initial condition u_0 is the same for both equations. Then, for any $p \geq 2$, we have

$$\lim_{\beta \to 1} \sup_{x \in [0, L]} \mathbb{E}|u_t(x) - u_t^{(\beta)}(x)|^p = 0.$$

Theorem 17 (Mijena, N., Negash, 2022+)

Assume that $\{\varphi_n\}_{n\geq 1}$ are uniformly bounded by a constant $C(B)$. For $d < \alpha$, let $u_t^{(\gamma)}$ and $u_t^{(\beta)}$ denote the solution to (2.2) for parameters $\gamma \in (0, 1)$ and $\beta \in \left(\frac{1}{2}, 1\right)$ respectively with $\gamma \to \beta$. The initial condition u_0 is the same for both equations. Then, for any $p \geq 2$, we have

$$\lim_{\gamma \to \beta} \sup_{x \in B} \mathbb{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p = 0,$$

- Theorem 17 extends Theorem 1.3 in Foondun [2021] to fractional Laplacian case, and Theorem 4.3(b) of Dang *et al.* [2018] to stochastic case.

Theorem 18 (N. and Nguyen, 2022+)

Let $0 < \beta < 1$ and $p \geq 2$. Let u and $u^{(\beta)}$ denote two solutions of problem (2.1) with fractional parameters $\beta = 1, \beta$, and $\alpha = 2$ and $B = (0, L)$. Let d be a positive number such that $p(1-\delta) < d < \frac{p}{2}$ with $\frac{1}{2} < \delta < \frac{3}{4}$. Let us assume that $u_0 \in L^2(0, L)$. Then, for any $p \geq 2$ and b large enough, we have

$$\sup_{0\leq t\leq T} \sup_{x\in[0,L]} t^d e^{-bt}\mathbb{E}|u(t,x) - u^{(\beta)}(t,x)|^p \lesssim T^{p\beta\delta - p\beta + d}|\mathcal{E}_\beta(\gamma,\theta)|^p \|u_0\|^p_{L^2(0,L)}.$$

Where

$$\mathcal{E}_\beta(\gamma,\theta) = \left[\max\left(\frac{\Gamma(2) - \Gamma(\beta+1)}{\Gamma(\beta+1)}, 0\right) + (1-\beta)^\beta + (1-\beta)^{\frac{\gamma(\theta-1)}{\theta}}\right]. \tag{4.2}$$

Theorem 19 (N. and Nguyen, 2022+)

Let $\beta, \beta' \in [\beta_0, \beta_1]$ such that $\beta \leq \beta'$ and $0 < \beta_0 < \beta_1 < 1$. Let $u^{(\beta)}$ and $u^{(\beta')}$ denote two solutions of problem (2.1) with fractional parameters β, β', and $\alpha = 2$ and $B = (0, L)$. Let us assume that $u_0 \in V_{\delta,k}$. If θ large enough then the following estimate holds

$$\sup_{0 \leq t \leq T} \sup_{x \in [0,L]} e^{-\theta t} \mathbb{E} |u^{(\beta')}(t,x) - u^{(\beta)}(t,x)|^p$$
$$\lesssim \left[(\beta' - \beta) + (\beta' - \beta)^{\epsilon} + \left(T^{\beta' - \beta} - 1 \right) \right]^p \|u_0\|^p_{V_{\delta,k}}$$

where

$$\|u_0\|_{V_{\delta,k}} = \left(\sum_{n=1}^{\infty} \lambda_n^{\delta k^*} \langle u_0, e_n \rangle^{k^*} \right)^{\frac{1}{k^*}} < \infty$$

and $k, k^* > 0$ satisfies that $\frac{1}{k} + \frac{1}{k^*} = 1$ with $\delta k > 1/2$.

Continuity for Space fractional noise case

Theorem 20 (Mijena, N., Negash, 2022+)

Assume that $\{\varphi_n\}_{n\geq 1}$ *are uniformly bounded by a constant* $C(B)$, *and* $\int_{B\times B} f(x,y)\,\mathrm{d}x\,\mathrm{d}y < \infty$. *For* $d < \frac{1}{2}\min\{\frac{1}{\beta}, \frac{1}{\gamma}\}\alpha$, *let* $u_t^{(\beta)}$ *and* $u_t^{(\gamma)}$ *denote solutions to* (3.1) *for parameters* $\beta, \gamma \in (\frac{1}{2}, 1)$ *with* $\gamma \to \beta$. *The initial condition* u_0 *is the same for both equations. Then, for any* $p \geq 2$, *we have*

$$\lim_{\gamma\to\beta}\sup_{x\in B} \mathrm{E}|u_t^{(\gamma)}(x) - u_t^{(\beta)}(x)|^p = 0.$$

Future work

1. Continuity results the case when the noise $\dot{W}$ is colored in time and space.
2. Neumann (other) boundary condition.
3. Study explicit rates of continuity!


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BÏYAN
SHUKRIA
TINGKI
YAQHANYELAY
TASHAKKUR ATU
GRACIAS
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SHUKURIA
GOZAIMASHITA
EFCHARISTO
JUSPAXAR
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