# Optimal regularity of SPDEs with additive noise

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Joint work with Marta Sanz-Solé

Many thanks to the
United States' NSF & Spain's *Ministerio de Ciencia e Innovación*

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- The solution is of course

$$u(t\,,x) = \int_{(0,t)\times\mathbf{R}} p(t-s\,,y-x)\, W(\mathrm{d}s\,\mathrm{d}y),$$

where $p(t\,,x) = (4\pi t)^{-1/2} \exp\{-x^2/(4t)\}$ = heat kernel

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**Theorem (Krylov-Rozovskii, 1972; Pardoux, 1972)**

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# Other SPDEs

$\partial_t u = \Delta u + b(u) + \dot F \qquad \mathrm{Cov}[\dot F(t\,,x), \dot F(s\,,y)] = \delta_0(t-s)\Gamma(x-y)$

- $\dot F$= centered generalized Gaussian noise with $[\phi_1, \phi_2 \in \mathscr{S}(\mathbf{R}^d)]$

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$$= (s \wedge t)\,\langle \phi_1\, \phi_2 * \Gamma\rangle_{L^2(\mathbf{R}^d)} \qquad s, t > 0,$$

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- $b : \mathbf{R} \to \mathbf{R}$ is non-random and Lipschitz continuous

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*The SPDE has a random-field solution $u$ if*

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- Both theories also allow multiplicative-noise models

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$\partial_t H = \mathscr{L} H + \dot{F}, \quad H(0) = 0, \quad \mathrm{Cov}[\dot{F}(t\,, x), \dot{F}(s\,, y)] = \delta_0(t-s)\Gamma(x-y)$

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# A harmless non-degeneracy condition

- We must assume from now on that $X$ is *genuinely $d$-dimensional*; that is,

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$$\psi(\xi) = -i\boldsymbol{a} \cdot \xi + \tfrac{1}{2}\xi \cdot \boldsymbol{A}\xi + \int_{\mathbf{R}^d} \left[1 - e^{iy\cdot\xi} + i(y \cdot \xi)\mathbf{1}_{B(0,1)}(y)\right] \nu(dy),$$

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- If $\psi(\xi) = 0$ for some $\xi \neq 0$, then use

$$\mathrm{Re}\psi(\xi) = \tfrac{1}{2}\xi \cdot \boldsymbol{A}\xi + \int_{\mathbf{R}^d} [1 - \cos(y \cdot \xi)]\, \nu(\mathrm{d}y),$$

  to see that $\mathbf{A}$ does not have full rank AND $X$ can only have jumps with magnitude of form $y$ where $\xi \cdot y \in (2\pi\mathbf{Z})^d$. Moreover, $\mathbf{a} \perp \xi$

# On $H$ and $W$

Recall that

$$\begin{aligned} \partial_t H &= \mathscr{L} H + \dot{F}, \qquad H(0) = 0 \\ \partial_t^2 W &= \mathscr{L} W + \dot{F}, \qquad W(0) = \partial_t W(0+) = 0, \qquad d = 1, 2, 3 \end{aligned}$$

**Theorem (Brzézniak-van Neerven, 2003; K-Kim, 2015)**

*Assume that $X$ is genuinely $d$-dimensional. Then, $H$ and/or $W$ are well-defined random fields if and only if*

$$\int_{\mathbf{R}^d} \frac{\hat{\Gamma}(d\xi)}{1 + \mathrm{Re}\psi(\xi)} < \infty.$$

$2\mathrm{Re}\psi =$ characteristic exponent of $X - X'$ [replica symmetry]

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- *$H$ and/or $W$ are Hölder continuous in $x$ for one – hence every – $t > 0$ iff*

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- These NASCs remain valid for nonlinear SPDEs with additive noise

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*If $\{X_t\}_{t\in T}$ is a real-valued stochastic process and $T \subset \mathbf{R}^N$ is compact, and if there exist $\alpha_1, \ldots, \alpha_N \in (0\,, 1]$ and $k > \sum_{j=1}^N \alpha_j^{-1}$ such that*

$$\|X_t - X_s\|_{L^k(\Omega)} \lesssim \sum_{j=1}^N |t_j - s_j|^{\alpha_j} \qquad \textit{uniformly for all } s, t \in T,$$

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- Needs a slight adjustment since optimal $\alpha_j$'s aren't so easy to find

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**Lemma**

*If $\int_0^1 r^{-1-a} g(r)\,dr < \infty$ for some $a \in (0\,,1)$, then $g(r) \lesssim r^a$.*

# Proof

The proof is short and abelian/tauberian (fractal?):

- $\int_0^1 g(r)\,\frac{\mathrm{d}r}{r^{1+a}} < \infty \Leftrightarrow \sum_{n=0}^{\infty} \mathrm{e}^{na} g(\mathrm{e}^{n}) < \infty \Rightarrow \lim_{n\to\infty} \mathrm{e}^{na} g(\mathrm{e}^{-n}) = 0$

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**Corollary**

*If*

$$\int_0^1 \sup_{\substack{s,t\in[0,1]\\ |s-t|\le r}} \|X_t - X_s\|_{L^k(\Omega)} \frac{dr}{r^{1+a}} < \infty \quad \textit{for some } a \in (1/k\,,1),$$

*then $X$ is Hölder continuous.*

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- **Proof.**
  1. WLOG $f$ is increasing; else replace $f(r)$ by $\sup_{s\in(0,r)} f(s)$
  2. Reverse the previous proof:

$$\int_0^1 f(r)\,\frac{\mathrm{d}r}{r^{1+a}} = \infty \Leftrightarrow \sum_{n=0}^{\infty} \mathrm{e}^{na} f(\mathrm{e}^{-na}) = \infty \Rightarrow f(\mathrm{e}^{-na}) \ge \mathrm{e}^{-nb},$$

for infinitely many values of $n$; for otherwise,

$$\sum_{n=0}^{\infty} \mathrm{e}^{na} f(\mathrm{e}^{-n}) \le \sum_{n=0}^{\infty} \mathrm{e}^{n(a-b)} < \infty. \qquad \square$$

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- Finally, by the Paley-Zygmund inequality (1932),

$$\mathrm{P}\left\{|X_t - X_s| \geq \tfrac{1}{2}\|X_t - X_s\|_{L^1(\Omega)}\right\} \geq \frac{\|X_t - X_s\|^2_{L^1(\Omega)}}{4\|X_t - X_s\|^2_{L^2(\Omega)}}$$

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- If $X$ is Gaussian then it has good 0-1 laws (Kallianpur, 1970; Cambanis-Rajput, 1973).