Power-law Lévy processes, power-law vector random fields, and some extensions

Chunsheng Ma

Department of Mathematics, Statistics, and Physics
Wichita State University
Wichita, Kansas 67260, USA

E-mail: chunsheng.ma@wichita.edu

Outline

1. Various (exact or asymptotic) “power-law"s in probability and statistics
   — A brief review

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2. Power-law subordinator, power-law Lévy process, and some extensions
3. Power-law vector random fields and some extensions

1. Various (exact or asymptotic) "power-law"s in probability and statistics —- A brief review

Data with power-law decaying statistics are observed over various complex systems and stochastic dynamics with power-law behavior are investigated in diverse fields of science.

There are several types of power-law statistical models. Examples are:

1. Various (exact or asymptotic) "power-law"s in probability and statistics —- A brief review

Data with power-law decaying statistics are observed over various complex systems and stochastic dynamics with power-law behavior are investigated in diverse fields of science.

There are several types of power-law statistical models. Examples are:

(i). Power-law decaying correlation structure (of a stochastic process)

P. Carpena, P. A. Bernaola-Galván, M. Gómez-Extremera, and A. V. Coronado, Transforming Gaussian correlations. Applications to generating long-range power-law correlated time series with arbitrary distribution, *Chaos* **30** (2020), no. 8, 083140.

M. Fernández-Martínez, M. A. Sánchez-Granero, M. P. Casado Belmonte, and J. E. Trinidad Segovia, A note on power-law cross-correlated processes, *Chaos Solitons Fractals* **138** (2020), 109914.

J. Lee, Generalized Bernoulli process with long-range dependence and fractional binomial distribution, *Depend. Model.* **9** (2021), 1-12.

C. Ma, Student's t vector random fields with power-law and log-law decaying direct and cross covariances, *Stoch. Anal. Appl.* **31** (2012), 167-182.

(ii). Power-law decaying spectrum (of a stochastic process)

S. C. Lim and L. P. Teo, Generalized Whittle-Matérn random field as a model of correlated fluctuations, *J . Phys A: Math. Theor.* **42** (2009), 105202

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S. C. Lim and L. P. Teo, Generalized Whittle-Matérn random field as a model of correlated fluctuations, *J . Phys A: Math. Theor.* **42** (2009), 105202

(iii). Distribution or density functions have power-law tails both at zero and at infinity

X. Gabaix, P. Gopikrishnan, V. Plerou, and H. E. Stanley, A theory of power-law distributions in financial market fluctuations, *Nature* **423** (2003), 267-270.

F. Prieto and J. M. Sarabia, A generalization of the power law distribution with nonlinear exponent, *Commun. Nonlinear Sci. Numerical Simul.* **42** (2017), 215-228.

(iv). Power-law characteristic function

A (generalized) Linnik distribution is a univariate distribution with characteristic function

$$\varphi(\omega) = \frac{1}{(1+\alpha|\omega|^{\nu})^{\kappa}}, \omega \in \mathbb{R},$$

where $\alpha, \nu \in (0,2]$ and κ are positive constants. In case of $\kappa = 1$, $\varphi(\omega)$ was proved by Ju. V. Linnik in 1953 as the characteristic function of a symmetric distribution on $\mathbb{R}$, and by A. G. Pakes (1998) for every $\kappa > 0$.

In the extreme case $\nu = 2$, the distribution corresponds to the variance Gamma distribution, which reduces to the Laplace (double exponential) distribution if $\kappa = 1$.

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the Laplace motion (S. Kotz, T. J. Kozubowski, and K. Podgoorski, *The Laplace Distribution and Generalizations: A Revisit with Applications to Communications, Economics, Engineering, and Finance*, Springer, 2001)
the variance Gamma Lévy process (D. B. Madan and E. Seneta, *The variance gamma (V.G.) model for share market returns*, J. Business **63** (1990), 511-524)
the Linnik Lévy process (A. Kumar, A. Maheshwari, and A. Wylomanska, *Linnik Lévy process and some extensions*, Physica A **529** (2019), 121539)

A subordinator is a non-negative Lévy process. The law of a subordinator is better specified by the Laplace transforms of its one-dimensional distributions.

2.1 *Power-law subordinator*

Theorem 1

If $\nu \in (0, 1]$, κ_1 and κ_2 are positive constants, α_1 and α_2 are nonnegative constants, $\kappa_1 \le \kappa_2$, and $\alpha_1 < \alpha_2$, then there is a power-law subordinator $\{Z(x), x \ge 0\}$ with Laplace transform

$$\mathrm{E}\exp(-\omega Z(x)) = \frac{(1+\alpha_1\omega^{\nu})^{\kappa_1 x}}{(1+\alpha_2\omega^{\nu})^{\kappa_2 x}}, \qquad \omega \ge 0,\ x \ge 0. \tag{1}$$

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Furthermore, this power-law subordinator has the following stochastic representations.

(i) If $\alpha_1 = 0$, then $\{Z(x), x \geq 0\}$ can be represented as the subordination of a positive stable subordinator with a Gamma process, i.e.,

$$Z(x) = Z_2(Z_1(x)), \qquad x \geq 0,$$

where $\{Z_1(x), x \geq 0\}$ is a Gamma process with Laplace transform

$$\mathrm{E}\exp(-\omega Z_1(x)) = (1 + \alpha_2 \omega)^{-\kappa_2 x}, \quad \omega \geq 0,\ x \geq 0,$$

$\{Z_2(x), x \geq 0\}$ is a positive stable subordinator with Laplace transform

$$\mathrm{E}\exp(-\omega Z_2(x)) = \exp\left(-\omega^{\nu} x\right), \quad \omega \geq 0,\ x \geq 0,$$

and $\{Z_1(x), x \geq 0\}$ and $\{Z_2(x), x \geq 0\}$ are independent.

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Special case: $\alpha_1 = 0$ and $\nu = 1$

$\{Z(x), x \geq 0\}$ is a Gamma process.

(ii) When $\alpha_1 > 0$ and $\kappa_1 = \kappa_2$, $\{Z(x), x \geq 0\}$ enjoys a compounded Poisson process representation

$$Z(x) = \sum_{n=1}^{\Lambda(x)} W_n Y_n^{\frac{1}{\nu}}, \qquad x \geq 0,$$

where $\{\Lambda(x), x \geq 0\}$ is a Poisson process with mean

$$\mathrm{E}\Lambda(x) = \kappa_1(\ln \alpha_2 - \ln \alpha_1)x,$$

$\{W_n, n \in \mathbb{N}\}$ is a sequence of independent and identically distributed positive stable random variables with Laplace transform

$$\mathrm{E}\exp(-\omega W_n) = \exp\left(-\omega^{\nu}\right), \quad \omega \geq 0, \; n \in \mathbb{N},$$

$\{Y_n, n \in \mathbb{N}\}$ is a sequence of independent and identically distributed random variables with density function

$$f_{Y_n}(y) = \frac{\exp\left(-\frac{y}{\alpha_2}\right) - \exp\left(-\frac{y}{\alpha_1}\right)}{(\ln \alpha_2 - \ln \alpha_1)y} I_{(0,\infty)}(y), \tag{2}$$

and $\{\Lambda(x), x \geq 0\}$, $\{W_n, n \in \mathbb{N}\}$, and $\{Y_n, n \in \mathbb{N}\}$ are independent.

(iii) In case of $\alpha_1 > 0$ and $\kappa_1 < \kappa_2$, $\{Z(x), x \geq 0\}$ admits a representation

$$Z(x) = Z_2(Z_1(x)) + Z_3(x), \qquad x \geq 0,$$

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and $\{Z_k(x), x \geq 0\}$ $(k = 1, 2, 3)$ are independent.

Simply speaking,

Part (iii) = Part (i) +Part (ii) an independent summation

Theorem 2

Suppose that $\nu \in (0, 1]$, κ_1 and κ_2 are positive constants, α_1 and α_2 are nonnegative constants, $\kappa_1 \leq \kappa_2$, and $\alpha_1 < \alpha_2$.

If $\{Z_2(x), x \geq 0\}$ is Brownian motion with zero mean and covariance function $2\min(x_1, x_2)$, $x_1 \geq 0, x_2 \geq 0$,

and is independent with a power-law subordinator $\{Z_1(x), x \geq 0\}$ whose Laplace transform is identical to (1),

then

$$Z(x) = Z_2(Z_1(x)), \quad x \geq 0,$$

is a power-law Lévy process with characteristic function

$$\mathrm{E} \exp(\imath \omega Z(x)) = \frac{\left(1 + \alpha_1 |\omega|^{2\nu}\right)^{\kappa_1 x}}{\left(1 + \alpha_2 |\omega|^{2\nu}\right)^{\kappa_2 x}}, \qquad \omega \in \mathbb{R}, \ x \geq 0, \tag{3}$$

where $\imath$ denotes the imaginary unit.

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Moreover, $\{Z(x), x \ge 0\}$ has the following stochastic representations.

(i) If $\alpha_1 = 0$, then $\{Z(x), x \geq 0\}$ can be represented as the subordination of a stable process with a Gamma process, and, more precisely,

$$Z(x) = Z_2(Z_1(x)), \qquad x \geq 0, \tag{4}$$

where $\{Z_1(x), x \geq 0\}$ is a Gamma process with Laplace transform $(1 + \alpha_2 \omega)^{-\kappa_2 x}$, $\{Z_2(x), x \geq 0\}$ is a stable process with characteristic function $\exp\left(-|\omega|^{2\nu} x\right)$, and $\{Z_1(x), x \geq 0\}$ and $\{Z_2(x), x \geq 0\}$ are independent.

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In this case $\{Z(x), x \geq 0\}$ is called a Linnik Lévy process

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$$\mathrm{E} \exp(\imath\, \omega W_n) = \exp\left(-|\omega|^{2\nu}\right), \quad \omega \in \mathbb{R},\ n \in \mathbb{N},$$

$\{Y_n, n \in \mathbb{N}\}$ is a sequence of independent and identically distributed random variables with density function (2),
and $\{\Lambda(x), x \geq 0\}$, $\{W_n, n \in \mathbb{N}\}$, and $\{Y_n, n \in \mathbb{N}\}$ are independent.

(iii) In case of $\alpha_1 > 0$ and $\kappa_1 < \kappa_2$, $\{Z(x), x \geq 0\}$ admits a representation

$$Z(x) = Z_2(Z_1(x)) + Z_3(x), \qquad x \geq 0,$$

where $\{Z_1(x), x \geq 0\}$ is a Gamma process with Laplace transform

$$\mathrm{E} \exp(-\omega Z_1(x)) = (1 + \alpha_2 \omega)^{-(\kappa_2 - \kappa_1)x}, \quad \omega \geq 0, x \geq 0,$$

$\{Z_2(x), x \geq 0\}$ is a stable process with characteristic function

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$\{Z_3(x), x \geq 0\}$ is the power-law Lévy process in Part (ii),
and $\{Z_k(x), x \geq 0\}$ $(k = 1, 2, 3)$ are independent.

Simply speaking,

Part (iii) = Part (i) +Part (ii)   an independent summation

2.3 Some extensions

The pair of power-law Lévy processes in Theorems 1 and 2 have not only their own roles but also their important contributions to be used as the building blocks to construct other pairs of Lévy processes on $[0, \infty)$.

For instance, suppose that $\{a_k, k \in \mathbb{N}\}$ is a summable sequence of nonnegative numbers, and that $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is a sequence of independent copies of power-law subordinators with Laplace transform (1). Then

$$Z(x) = \sum_{k=1}^{\infty} a_k Y_k(x), \qquad x \geq 0,$$

is a subordinator with Laplace transform

$$\mathrm{E} \exp\left(-\omega Z(x)\right) = \prod_{k=1}^{\infty} \frac{\left(1 + \alpha_1 a_k^{\nu} \omega^{\nu}\right)^{\kappa_1 x}}{\left(1 + \alpha_2 a_k^{\nu} \omega^{\nu}\right)^{\kappa_2 x}}, \quad \omega \geq 0, \; x \geq 0.$$

Similarly, a Lévy process is obtained with characteristic function

$$\mathrm{E} \exp\left(\imath \omega Z(x)\right) = \prod^{\infty} \frac{\left(1 + \alpha_1 |a_k|^{2\nu} |\omega|^{2\nu}\right)^{\kappa_1 x}}{\left(1 + \alpha_2 |a_k|^{2\nu} |\omega|^{2\nu}\right)^{\kappa_2 x}}, \quad \omega \in \mathbb{R}, \; x \geq 0.$$

Example 1

Let β be a nonnegative constant, and $Z(x) = \sum_{k=1}^{\infty} \frac{Y_k(x)}{\left(\pi^2\left(k-\frac{1}{2}\right)^2+\beta\right)^{\frac{1}{\nu}}}, \qquad x \geq 0.$

If $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is an independent sequence of power-law subordinators with Laplace transform (1), then $\{Z(x), x \geq 0\}$ is a hyperbolic cosine ratio subordinator with Laplace transform

$$\mathrm{E}\exp\left(-\omega Z(x)\right) = \left(\cosh(\sqrt{\beta})\right)^{(\kappa_2-\kappa_1)x} \frac{\left(\cosh\left((\alpha_1\omega^{\nu}+\beta)^{\frac{1}{2}}\right)\right)^{\kappa_1 x}}{\left(\cosh\left((\alpha_2\omega^{\nu}+\beta)^{\frac{1}{2}}\right)\right)^{\kappa_2 x}}, \; \omega \geq 0, x \geq 0.$$

Alternatively, a hyperbolic cosine ratio Lévy process with characteristic function

$$\mathrm{E}\exp\left(\imath\omega Z(x)\right) = \left(\cosh(\sqrt{\beta})\right)^{(\kappa_2-\kappa_1)x} \frac{\left(\cosh\left((\alpha_1|\omega|^{2\nu}+\beta)^{\frac{1}{2}}\right)\right)^{\kappa_1 x}}{\left(\cosh\left((\alpha_2|\omega|^{2\nu}+\beta)^{\frac{1}{2}}\right)\right)^{\kappa_2 x}}, \; \omega \in \mathbb{R}, x \geq 0,$$

is obtained if $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is an independent sequence of power-law Lévy processes with characteristic function (3).

Example 2

Let β be a nonnegative constant. If $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is an independent sequence of power-law subordinators with Laplace transform (1), where $\kappa_1 = \kappa_2 = \kappa$, then

$$Z(x) = \sum_{k=1}^{\infty} \frac{Y_k(x)}{(\pi^2 k^2 + \beta)^{\frac{1}{\nu}}}, \qquad x \geq 0,$$

is a hyperbolic sine ratio subordinator with Laplace transform

$$\mathrm{E} \exp(-\omega Z(x)) = \left(\frac{(\alpha_2 \omega^\nu + \beta)^{\frac{1}{2}} \sinh\left((\alpha_1 \omega^\nu + \beta)^{\frac{1}{2}} \right)}{(\alpha_1 \omega^\nu + \beta)^{\frac{1}{2}} \sinh\left((\alpha_2 \omega^\nu + \beta)^{\frac{1}{2}} \right)} \right)^{\kappa x}, \ \omega \geq 0, \ x \geq 0.$$

Alternatively, a hyperbolic sine ratio Lévy process with characteristic function

$$\mathrm{E} \exp(\imath \omega Z(x)) = \left(\frac{\left(\alpha_2 |\omega|^{2\nu} + \beta\right)^{\frac{1}{2}} \sinh\left(\left(\alpha_1 |\omega|^{2\nu} + \beta\right)^{\frac{1}{2}} \right)}{\left(\alpha_1 |\omega|^{2\nu} + \beta\right)^{\frac{1}{2}} \sinh\left(\left(\alpha_2 |\omega|^{2\nu} + \beta\right)^{\frac{1}{2}} \right)} \right)^{\kappa x}, \ \omega \in \mathbb{R}, \ x \geq 0,$$

results from assuming that $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is an independent sequence of power-law Lévy processes with characteristic function (3), where $\kappa_1 = \kappa_2 = \kappa$.

Example 3

Let

$$Z(x) = \left(4\pi^4\right)^{-\frac{1}{\nu}} \sum_{k=1}^{\infty} \frac{Y_k(x)}{\left(k - \frac{1}{2}\right)^{\frac{4}{\nu}}}, \qquad x \geq 0.$$

If $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is an independent sequence of power-law subordinators with Laplace transform (1), then $\{Z(x), x \geq 0\}$ is a subordinator with Laplace transform

$$\mathrm{E}\exp\left(-\omega Z(x)\right) = 2^{(\kappa_2-\kappa_1)x} \frac{\left(\cosh\left(\alpha_1^{\frac{1}{4}}\omega^{\frac{\nu}{4}}\right) + \cos\left(\alpha_1^{\frac{1}{4}}\omega^{\frac{\nu}{4}}\right)\right)^{\kappa_1 x}}{\left(\cosh\left(\alpha_2^{\frac{1}{4}}\omega^{\frac{\nu}{4}}\right) + \cos\left(\alpha_2^{\frac{1}{4}}\omega^{\frac{\nu}{4}}\right)\right)^{\kappa_2 x}},\ \omega \geq 0,\ x \geq 0.$$

Alternatively, a Lévy process with characteristic function

$$\mathrm{E}\exp\left(\imath\omega Z(x)\right) = 2^{(\kappa_2-\kappa_1)x} \frac{\left(\cosh\left(\alpha_1^{\frac{1}{4}}|\omega|^{\frac{\nu}{2}}\right) + \cos\left(\alpha_1^{\frac{1}{4}}|\omega|^{\frac{\nu}{2}}\right)\right)^{\kappa_1 x}}{\left(\cosh\left(\alpha_2^{\frac{1}{4}}|\omega|^{\frac{\nu}{2}}\right) + \cos\left(\alpha_2^{\frac{1}{4}}|\omega|^{\frac{\nu}{2}}\right)\right)^{\kappa_2 x}},\ \omega \in \mathbb{R},\ x \geq 0,$$

is obtained if $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is an independent sequence of power-law Lévy processes with characteristic function (3).

Example 4

Let $\kappa_1 = \kappa_2 = \kappa$ and

$$Z(x) = \left(4\pi^4\right)^{-\frac{1}{\nu}} \sum_{k=1}^{\infty} \frac{Y_k(x)}{k^{\frac{4}{\nu}}}, \qquad x \geq 0.$$

If $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is an independent sequence of power-law subordinators with Laplace transform (1), then $\{Z(x), x \geq 0\}$ is a subordinator with Laplace transform

$$\mathrm{E}\exp\left(-\omega Z(x)\right) = \left(\frac{\cosh\left(\alpha_1^{\frac{1}{4}}\omega^{\frac{\nu}{4}}\right) - \cos\left(\alpha_1^{\frac{1}{4}}\omega^{\frac{\nu}{4}}\right)}{\cosh\left(\alpha_2^{\frac{1}{4}}\omega^{\frac{\nu}{4}}\right) - \cos\left(\alpha_2^{\frac{1}{4}}\omega^{\frac{\nu}{4}}\right)}\right)^{\kappa x}, \quad \omega \geq 0,\ x \geq 0.$$

Alternatively, a Lévy process with characteristic function

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results from the assumption that $\{Y_k(x), x \geq 0, k \in \mathbb{N}\}$ is an independent sequence of power-law Lévy processes with characteristic function (3).

3.1 Power-law vector random field

An m-variate random field $\{\boldsymbol{Z}(x), x \in \mathbb{D}\}$ is said to be a power-law random field, if it takes the form

$$\boldsymbol{Z}(x) = \sqrt{2U}\mathbf{Y}(x) + \boldsymbol{\mu}(x), \ \ x \in \mathbb{D}, \tag{5}$$

where U is a nonnegative random variable with with Laplace transform

$$\mathrm{E} \exp(-\omega U) = \frac{(1+\alpha_1 \omega^\nu)^{\kappa_1}}{(1+\alpha_2 \omega^\nu)^{\kappa_2}}, \qquad \omega \geq 0, \tag{6}$$

$\{\mathbf{Y}(x), x \in \mathbb{D}\}$ is an m-variate Gaussian random field
with mean zero and covariance matrix function $\mathbf{C}(x_1, x_2)$,
U and $\{Y(x), x \in \mathbb{D}\}$ are independent,
$\mu(x)$ is a (non-random) function on $\mathbb{D}$,
$\nu \in (0, 1]$,
$0 < \kappa_1 \leq \kappa_2$,
$0 \leq \alpha_1 < \alpha_2$.

Finite-dimensional characteristic functions

The power-law vector random field enjoys exact power-law finite-dimensional characteristic functions, and it is infinitely divisible.

Theorem 3

If $\{\boldsymbol{Z}(x), x \in \mathbb{D}\}$ is an m-variate power-law random field of the form (5), then, for every $n \in \mathbb{N}$ and any distinct $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$, an mn-variate random vector $(\boldsymbol{Z}'(x_1), \ldots, \boldsymbol{Z}'(x_n))'$ possesses the characteristic function

$$\mathrm{E} \exp\left(\imath \sum_{k=1}^{n} \boldsymbol{\omega}_k' \boldsymbol{Z}(x_k) \right) = \exp\left(\imath \sum_{k=1}^{n} \boldsymbol{\omega}_k' \boldsymbol{\mu}(x_k) \right) \frac{\left(1+\alpha_1 \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j \right)^{\nu} \right)^{\kappa_1}}{\left(1+\alpha_2 \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j \right)^{\nu} \right)^{\kappa_2}}, \quad \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n \in \mathbb{R}^m. \tag{7}$$

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The power-law vector random field enjoys exact power-law finite-dimensional characteristic functions, and it is infinitely divisible.

Theorem 3

If $\{\boldsymbol{Z}(x), x \in \mathbb{D}\}$ is an m-variate power-law random field of the form (5), then, for every $n \in \mathbb{N}$ and any distinct $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$, an mn-variate random vector $(\boldsymbol{Z}'(x_1), \ldots, \boldsymbol{Z}'(x_n))'$ possesses the characteristic function

$$\mathrm{E} \exp\left(\imath \sum_{k=1}^{n} \boldsymbol{\omega}_k' \boldsymbol{Z}(x_k)\right) = \exp\left(\imath \sum_{k=1}^{n} \boldsymbol{\omega}_k' \boldsymbol{\mu}(x_k)\right) \frac{\left(1+\alpha_1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j\right)^{\nu}\right)^{\kappa_1}}{\left(1+\alpha_2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j\right)^{\nu}\right)^{\kappa_2}}, \quad \boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_n \in \mathbb{R}^m. \tag{7}$$

Special cases

$\alpha_1 = 0$: a Linnik vector random field,
$\alpha_1 = 1$ and $\nu = 1$: a variance Gamma vector random field,
$\alpha_1 = 0$, $\kappa_2 = 1$ and $\nu = 1$: a Laplace (or double exponential) vector random field.

3.2 Some extensions

It may be employed as the building block to construct other random fields.

Suppose that $\{a_k, k \in \mathbb{N}\}$ is a summable sequence of real numbers, and that $\{\boldsymbol{Z}_k(x), x \in \mathbb{D}, k \in \mathbb{N}\}$ is a sequence of independent copies of m-variate power-law random fields with finite-dimensional characteristic functions (7), where $\boldsymbol{\mu}(x) \equiv \mathbf{0}$. Define an m-variate random field

$$\boldsymbol{Z}(x) = \sum_{k=1}^{\infty} a_k \boldsymbol{Z}_k(x), \qquad x \in \mathbb{D}. \tag{8}$$

It is an m-variate elliptically contoured random field with finite-dimensional characteristic functions

$$\mathrm{E} \exp\left(\imath \sum_{j=1}^{n} \boldsymbol{\omega}_j' \boldsymbol{Z}(x_j)\right) = \prod_{k=1}^{\infty} \frac{\left(1 + \alpha_1 |a_k|^{2\nu} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j\right)^{\nu}\right)^{\kappa_1}}{\left(1 + \alpha_2 |a_k|^{2\nu} \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j\right)^{\nu}\right)^{\kappa_2}},$$

$$\boldsymbol{\omega}_k \in \mathbb{R}^m,\ x_k \in \mathbb{D},\ k \in \{1, \ldots, n\}, n \in \mathbb{N}.$$

Example 1

In (8) taking $a_k = \left(\pi^2 \left(k - \frac{1}{2}\right)^2 + \beta\right)^{-\frac{1}{2\nu}}$ $(k \in \mathbb{N})$ yields an m-variate random field with finite-dimensional characteristic functions

$$\mathrm{E} \exp\left(\imath \sum_{j=1}^{n} \boldsymbol{\omega}_j' \boldsymbol{Z}(x_j)\right)$$
$$= \frac{\left(\cosh\left(\sqrt{\beta}\right)\right)^{\kappa_2 - \kappa_1} \left\{\cosh\left(\left(\alpha_1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\nu} + \beta\right)^{\frac{1}{2}}\right)\right\}^{\kappa_1}}{\left\{\cosh\left(\left(\alpha_2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\nu} + \beta\right)^{\frac{1}{2}}\right)\right\}^{\kappa_2}},$$
$$\boldsymbol{\omega}_k \in \mathbb{R}^m,\ x_k \in \mathbb{D},\ k \in \{1, \ldots, n\}, n \in \mathbb{N}.$$

In particular, it reduces to an m-variate hyperbolic cosine ratio random field if $\kappa_1 = \kappa_2$, and an m-variate hyperbolic secant random field if $\alpha_1 = 0$.

Example 2

For $\alpha_1 = 0$ and $\alpha_2 = \alpha^2$, if $a_k = (\pi k)^{-\frac{1}{\nu}}$, $k \in \mathbb{N}$, then $\{\boldsymbol{Z}(x), x \in \mathbb{D}\}$ is an m-variate (generalized) logistic random field with finite-dimensional characteristic functions

$$\mathrm{E} \exp\left(\imath \sum_{j=1}^{n} \boldsymbol{\omega}_j' \boldsymbol{Z}(x_j) \right) = \left\{ \frac{\alpha \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j \right)^{\frac{\nu}{2}}}{\sinh \left(\alpha \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j) \boldsymbol{\omega}_j \right)^{\frac{\nu}{2}} \right)} \right\}^{\kappa},$$

$$\boldsymbol{\omega}_k \in \mathbb{R}^m,\ x_k \in \mathbb{D},\ k \in \{1, \ldots, n\}, n \in \mathbb{N}.$$

Example 3

Let $\alpha_1 = 0$ and $\kappa_1 = \kappa_2 = \kappa$. Given $a_k = \left(\pi^2 k^2 + \beta\right)^{-\frac{1}{2\nu}}$, $k \in \mathbb{N}$, (8) is an m-variate random field with finite-dimensional characteristic functions

$$\mathrm{E} \exp\left(\imath \sum_{j=1}^{n} \boldsymbol{\omega}_j' \boldsymbol{Z}(x_j)\right)$$

$$= \left\{ \frac{\left(\alpha_2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\nu} + \beta\right)^{\frac{1}{2}} \sinh\left(\left(\alpha_1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\nu} + \beta\right)^{\frac{1}{2}}\right)}{\left(\alpha_1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\nu} + \beta\right)^{\frac{1}{2}} \sinh\left(\left(\alpha_2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\nu} + \beta\right)^{\frac{1}{2}}\right)} \right\}^{\kappa},$$

$$\boldsymbol{\omega}_k \in \mathbb{R}^m,\ x_k \in \mathbb{D},\ k \in \{1, \ldots, n\}, n \in \mathbb{N}.$$

It tends to an m-variate hyperbolic sine ratio random field, as $\beta \to 0_+$ and $\nu \to 1$.

Example 4

For $\kappa_1 = \kappa_2 = \kappa$ and $\beta_k = \alpha_k^{\frac{1}{4}} > 0$ $(k = 1, 2)$, in (8) taking $a_k = \left(2\pi^2 k^2\right)^{-\frac{1}{\nu}}$ $(k \in \mathbb{N})$ yields an m-variate random field with finite-dimensional characteristic functions

$$
\begin{aligned}
& \mathrm{E} \exp\left(\imath \sum_{j=1}^{n} \boldsymbol{\omega}_j' \boldsymbol{Z}(x_j)\right) \\
= & \left\{ \frac{\cosh\left(\beta_1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\frac{\nu}{4}}\right) - \cos\left(\beta_1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\frac{\nu}{4}}\right)}{\cosh\left(\beta_2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\frac{\nu}{4}}\right) - \cos\left(\beta_2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \boldsymbol{\omega}_i' \mathbf{C}(x_i, x_j)\boldsymbol{\omega}_j\right)^{\frac{\nu}{4}}\right)} \right\}^{\kappa}, \\
& \boldsymbol{\omega}_k \in \mathbb{R}^m,\ x_k \in \mathbb{D},\ k \in \{1, \ldots, n\}, n \in \mathbb{N}.
\end{aligned}
$$

Example 5

For $\kappa_1 = \kappa_2 = \kappa$ and $\beta_k = \alpha_k^{\frac{1}{4}}$ $(k = 1, 2)$, letting $a_k = \left(2\pi^2\left(k - \frac{1}{2}\right)\right)^{-\frac{1}{\nu}}$ $(k \in \mathbb{N})$ in (8) results in an m-variate random field with finite-dimensional characteristic functions

$$\begin{aligned} & \mathrm{E}\exp\left(\imath\sum_{j=1}^{n}\boldsymbol{\omega}_j'\boldsymbol{Z}(x_j)\right) \\ = & \left\{\frac{\cosh\left(\beta_1\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j\right)^{\frac{\nu}{4}}\right)+\cos\left(\beta_1\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j\right)^{\frac{\nu}{4}}\right)}{\cosh\left(\beta_2\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j\right)^{\frac{\nu}{4}}\right)+\cos\left(\beta_2\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\boldsymbol{\omega}_i'\mathbf{C}(x_i,x_j)\boldsymbol{\omega}_j\right)^{\frac{\nu}{4}}\right)}\right\}^{\kappa}, \\ & \boldsymbol{\omega}_k \in \mathbb{R}^m,\ x_k \in \mathbb{D},\ k \in \{1,\ldots,n\}, n \in \mathbb{N}. \end{aligned}$$