Some extreme value problems arising with ship motions

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NSF/CBMS Conference, August 3, 2021

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1Based on joint work and papers with V. Belenky, K. Weems and others (NSWCCD), T. Sapsis (MIT), (under)graduate students.
Ship motions

3 rotational \((roll, pitch, yaw)\) and 3 linear \((surge, sway, heave)\) motions. The focus here will typically be on roll, pitch, heave motions.

What do motions depend on? *Ship geometry and loading, operational parameters* (heading, speed) and *surrounding sea* (spectrum involving significant wave height, modal period, and so on). *Computer code* used as well. Randomness is due to random waves.²

²Waves are typically generated from a spatio-temporal Gaussian (or thought so) field.
Ship motions

The time series of **roll** (left) and **pitch** (right) obtained by the SimpleCode for 10mins at 0.5s sampling rate. The *ship geometry* is the ONR tumblehome top (THT). The *heading* is at 45°, the *speed* is 6 knots, the waves are modeled using Bretschneider spectrum in open ocean (with the *significant height* of 9m, the *mean zero-crossing period* of 10.65s, etc).

![Roll](image1.png)

![Pitch](image2.png)
Extreme events: These include:

A. *Extreme ship motion* (e.g. roll exceeding a certain large angle);
B. *Capsizing of a ship* (through pure loss of stability);
C. *Surfriding and broaching-to*;
D. *Extreme loads* (due to waves and/or slamming)

Basic problems: These include:

- Estimate the (small) probabilities of these rare events from the data obtained through a ship motion simulator (SimpleCode, LAMP, etc);
- Understand distribution tails of the “metrics” associated with these rare events, through reduced-order models.

This is important in e.g. ship design and concerning ship stability. The talk is also about the interplay of *Statistics, Stochastic Dynamics, Multifidelity Methods, Naval Architecture* in addressing/understanding these problems.
Direct estimation of probabilities of rare events from long records of ship motion should be avoided for several reasons: events are rarely observed, many conditions to consider, a desire to have methodology that can be applied to other data (e.g. from a basin) where long records are not available. (Analytic calculation is not a possibility either. Other approaches though exist.)

Instead of direct estimation, estimation of the probability of a rare event is first recast as the problem of estimation of the exceedance probability

$$P(Y > y^*) ,$$

where $Y$ is some variable ("metric") related to the rare event of interest and $y^*$ is some large critical value, given independent observations $Y_1, \ldots, Y_n$ of $Y$, with no values of $Y_i$ typically larger than $y^*$. Then, “standard” methods of Extreme Value Theory in Statistics are used to extrapolate data distribution into the tail to estimate the exceedance probability. (This is also about a distribution tail.)
**Exceedance probability for extreme motion**

**Extreme motion:** For the probability of extreme motion as e.g. roll exceeding a certain large angle $y^*$, $Y$ is taken as a *suitable envelope peak* of a ship motion.
Peaks-over-threshold (PoT) approach from Extreme Value Theory:
The idea is to fit a generalized Pareto distribution (GPD) to the data above an intermediate threshold and use it for extrapolation:

\[ P(Y > y^*) = P(Y > u)P(Y > y^* | Y > u) \approx P(Y > u)F_{u,\xi,\sigma}(y^*), \]

where \( F_{u,\xi,\sigma}(y) \) is the complementary CDF of the GPD given by

\[
F_{u,\xi,\sigma}(y) = \begin{cases} 
\left(1 + \frac{\xi(y-u)}{\sigma}\right)^{-1/\xi}, & \text{if } \xi > 0, \\
\exp\left(-\frac{y-u}{\sigma}\right), & \text{if } \xi = 0, \\
\left(1 + \frac{\xi(y-u)}{\sigma}\right)^{-1/\xi}, & \text{if } \xi < 0,
\end{cases}
\]

where \( \xi, \sigma \) and \( u \) are the shape, scale and threshold parameters. Some issues: (i) estimating GPD parameters, (ii) setting a threshold \( u \), (iii) constructing a confidence interval for exceedance probability, etc.
Application to peaks/extreme motion

THT hull, Heading $45^\circ$, etc. **Roll motion**, target $60^\circ$. Left: The probability (rate) estimates with confidence intervals for 100 runs of 100hrs each. Right: Same but using different confidence intervals.

![LogNormal method, Coverage = 96%](image1)

![Quantile logNormal method, Coverage = 96%](image2)

Such plots are used for validation of the proposed methods. The average value of shape parameter estimates is 0.19 (with standard error of 0.13).
"Annoying" features: The PoT approach generally works but e.g. (i) automatic threshold \( u \) selection methods are available but they are not failproof, (ii) negative shape parameter estimates can lead to large upper confidence intervals: e.g., the Weibull distribution \( P(Y > y) = e^{-y^2} \) with sample size \( N = 2,000 \):

Implicit assumption in PoT: Data need to be on the “correct” distribution tail. What about tails with changing character? Any real-world examples where this not the case and the PoT fails/struggles? (In my limited experience, this is the rule rather than the exception.)
**PWL oscillator:** As a model for ship roll, the system satisfies the equation

\[ \ddot{X} + 2\delta \dot{X} + r(X) = s\dot{W}(t) \]

with white noise excitation \( \dot{W}(t) \) and restoring force

\[ r(x) = \omega_0^2 x 1\{|x| \leq x_m\} - (k\omega_0^2 x \pm \omega^2 x_m (k + 1)) 1\{|x| > x_m\}. \]

Notes: \(|x| \leq x_m\): linear regime; \(|x| > x_m\): non-linear regime; \(\omega_0\): natural frequency; \(x_m\): “knuckle” point; \(x_v\): point of vanishing stability.

\(^3\)For linear restoring, the system is Gaussian.
PDF of response/derivative: In the statistically steady state, it is given by

\[ f(x, \dot{x}) = Ce^{-\frac{4\delta}{s^2}\left(\frac{1}{2}\dot{x}^2 + V(x)\right)} \]

with normalizing constant \( C \) and potential \( V(x) = \int_0^x r(y)dy \).

“Capsizing:” Condition on the system to be inside the separatrix, that is, for \( |\dot{x}| < x_s(x) \) with \( x_s(x) = \sqrt{2(V(x_v) - V(x))} \).

PDF of (conditioned) response: \( f_s(x) \propto \int_{-x_s(x)}^{x_s(x)} f(x, \dot{x})d\dot{x} \) for \( |x| < x_v \).
PDF of (conditioned) response:

3 tail regimes:

(I) Gaussian core: \( x \in [0, x_m] \)

(II) Heavy tail (possibly power-law): \( x \in [x_m, x_f] \)

(III) Light (bounded) tail: \( x \in [x_f, x_v] \)

Cases:

- **Good**: Data in (I)-(II), target in (II)
- **Bad 1**: Data in (I), no/little data in (II), target in (II)
- **Bad 2**: Data in (I)-(II), target in (III)
Good case: Data in (I)-(II), target in (II)

Setting: Target $x^* = x_f$, $\mathbb{P}(X > x_f) = 7.22 \times 10^{-5}$.
Sample $N = 3,000$, $\mathbb{P}(X > x_m) = 0.012$, $N \cdot \mathbb{P}(X > x_m) = 36$.

Note: Several cases with small estimated probability: Automatic threshold selection methods had difficulties in finding the “right” threshold.
Bad case 1: Data in (I), no/little data in (II), target in (II)

**Setting:** Target $x^* = x_f$, $\mathbb{P}(X > x_f) = 7.22 \times 10^{-5}$.
Sample $N = 500$, $\mathbb{P}(X > x_m) = 0.012$, $N \cdot \mathbb{P}(X > x_m) = 6$.

**Note:** A number of cases with small estimated probability: (a) when no confidence interval is computed, the estimated shape parameter is typically smaller than $-0.5$, (b) if confidence interval is computed, it is very large on its upper side.
Data example of extreme ship motions

PoT analysis on envelope peaks from roll motion: THT geometry, 9m significant wave height, 15sec modal period, 90 degree heading, etc. Left: 100 hour datasets. Right: 10 hour datasets.

Notes: This is along the same lines as bad case 1. Conclusion: Need to understand the underlying physics and e.g. adjust for sample sizes or use other approaches (such as fitting the exponential tail or the heavy tail).
PoT analysis for heavy-tailed distributions

If heavy (power-law) tail is expected as for peaks/extreme motion, a natural possibility is to use PoT with Pareto or heavy tail:

$$\Pr(Y > y) \approx Cy^{-1/\xi}, \quad \text{for large } y,$$

where $\xi > 0$ is a parameter and $C$ is a constant (possibly replaced by a function which varies “slowly”). There are estimators which ensure $\xi > 0$.

Comparing POTs with heavy tail and GPD for peaks data:
Large Amplitude Motion Program (LAMP): The higher-fidelity LAMP is based on a 3-dimensional potential flow panel method integrating pressures over the submerged hull and incorporating a direct calculation of Froude-Krylov and hydrostatic (FKHS), radiation, diffraction and other forces.

<table>
<thead>
<tr>
<th>LAMP-2</th>
<th>3-D body-nonlinear method</th>
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<tbody>
<tr>
<td>LAMP-1</td>
<td>body-linearized 3-D method</td>
</tr>
<tr>
<td>LAMP-0</td>
<td>“Hydrostatics only” formulation</td>
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</tbody>
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Simple Code (SC) and reduced-order ODE models: The lower-fidelity but computationally more efficient SC and similar ODE models may assume a particular, e.g. linear and separable, form of FKHS forces, and furthermore capture radiation forces through (constant) added mass and damping coefficients. Currently, SC does not incorporate diffraction forces.
LAMP, SC and reduced-order models

Focus on the case of head seas, with the resulting heave $\zeta_g$ and pitch $\theta$.  

**LAMP:** The equations of motion can be expressed as

\[
\begin{align*}
\ddot{m} \zeta_g &= F_{3,fkhs} + F_{3,hd} \\
\dot{I}_Y \dot{\theta} &= F_{5,fkhs} + F_{5,hd}
\end{align*}
\]

with $F_{i,hd} = F_{i,rad} + F_{i,dif}$.

**SC, etc:** Assuming $-F_{i,rad} \simeq A_{i3} \ddot{\zeta}_g + A_{i5} \dot{\theta} + B_{i3} \dot{\zeta}_g + B_{i5} \dot{\theta}$, the SC program solves

\[
\begin{align*}
(m + A_{33}) \ddot{\zeta}_g + A_{35} \dot{\theta} + B_{33} \dot{\zeta}_g + B_{35} \dot{\theta} + F_3(\zeta_g, \theta, t) &= 0 \\
(I_Y + A_{55}) \ddot{\theta} + A_{53} \dot{\zeta}_g + B_{53} \dot{\zeta}_g + B_{55} \dot{\theta} + F_5(\zeta_g, \theta, t) &= 0.
\end{align*}
\]

An assumption of small motions leads to the separation of restoring and excitation in FKHS forces, for example, as

\[
\begin{align*}
(m + A_{33}) \ddot{\zeta}_g + A_{35} \dot{\theta} + B_{33} \dot{\zeta}_g + B_{35} \dot{\theta} + C_{33} \zeta_g + C_{35} \theta + F_{3k}(t) &= 0 \\
(I_Y + A_{55}) \ddot{\theta} + A_{53} \dot{\zeta}_g + B_{53} \dot{\zeta}_g + B_{55} \dot{\theta} + C_{53} \zeta_g + C_{55} \theta + F_{5k}(t) &= 0.
\end{align*}
\]

Calibration is another problem of interest.

\footnote{E.g. the elastic beam equation $M \partial^2 w/\partial t^2 + EI \partial^4 w/\partial x^4 = q(t, x)$ is of interest for slamming loads.}
**Setting:** ONR topsides flared hull geometry; 180° heading; Sea state 8; etc.

**Basic question:** Can SC data help answer (e.g. extreme value) questions about LAMP?
**Extrapolation framework:** If parametric distributions (e.g. Weibull, Gumbel, Gaussian, etc.) are used for extrapolation, we have a framework when under suitable conditions, more lower-fidelity data (e.g. SC) can help in estimation of extremal quantities of high-fidelity model (e.g. LAMP).
MFMC estimation: non-rare problem

Setting: Have (independent) samples \((X^{(1)}_1, X^{(2)}_1), \ldots, (X^{(1)}_{n_1}, X^{(2)}_{n_1})\) from \((X^{(1)}, X^{(2)})\), and additional samples \(X^{(2)}_{n_1+1}, \ldots, X^{(2)}_{n_2}\) for some \(n_2 > n_1\).

MF estimator: To estimate the unknown mean \(\mathbb{E}X^{(1)}\), set

\[
\hat{\mu}_{mf} = \overline{X}^{(1)}_{n_1} + \alpha (\overline{X}^{(2)}_{n_2} - \overline{X}^{(2)}_{n_1}) = \alpha \overline{X}^{(2)}_{n_2} + (\overline{X}^{(1)}_{n_1} - \alpha \overline{X}^{(2)}_{n_1}),
\]

where \(\alpha \in \mathbb{R}\) and \(\overline{X}_n = \frac{1}{n} \sum_{i=1}^{n} X_i\).

Baseline estimator: \(\hat{\mu}_{bl} = \overline{X}^{(1)}_{n_1}\).

Punchline: If \(X^{(1)}\) and \(X^{(2)}\) are correlated enough, then

\[
\text{Var}(\hat{\mu}_{mf}) < \text{Var}(\hat{\mu}_{bl}),
\]

so that the resulting confidence intervals are smaller for MF estimator. Naturally, \(\alpha = \arg\min_a \text{Var}(\hat{\mu}_{mf})\).
Sampling framework: LAMP could be run only for selective SC records, for example, with largest record maxima/minima among many records.

20 top and 20 bottom SC record maxima are from the total of 2,000 SC records, but many more could easily be considered.
**Importance sampling:** Illustration with 300 points of bivariate standard Gaussian for SC, LAMP and correlation $\rho = 0.8$. SC records selected uniformly over the value range and kernel density estimation with suitable weights for LAMP data.

**Notes:** No such benefit when e.g. $\rho = 0$. Other methods based on GPR, multi-armed bandits are also being investigated.
Some references

Extreme motions:

Capsizing:

Extreme loads:

Multifidelity:

Calibration: