Convergence of Densities of Spatial Averages for Stochastic Heat Equation

Gaussian Random Fields, Fractals, SPDEs, and Extremes
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Work in progress, joint with David Nualart
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Stochastic Heat Equation

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + \sigma(u) \dot{W}, \quad x \in \mathbb{R}, \ t > 0,
\]  

(1.1)

- \( u(0, x) = u_0(x) = 1 \)
- \( \dot{W} \) is space-time white noise
- \( \sigma \) nonrandom, Lipschitz
### Stochastic Heat Equation

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#### Proposition ([Walsh, 1986])

*There exists a unique mild solution \( u = \{u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\} \) such that*

\[
\sup_{(t,x) \in [0,T] \times \mathbb{R}} \mathbb{E} [\left| u(t, x) \right|^p] = C_{T,p} \tag{1.2}
\]
Spatial Averages

Fix $t > 0$. The process $x \rightarrow u(t, x)$ is stationary.
Spatial Averages

Fix $t > 0$. The process $x \rightarrow u(t, x)$ is stationary. Consider

$$F_{R, t} := \frac{1}{\sigma_{R, t}} \left( \int_{-R}^{R} u(t, x) \, dx - 2R \right)$$
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**Theorem ([Huang et al., 2020])**

$$d_{TV}(F_{R,t}, N) \leq \frac{C_t}{\sqrt{R}}. \quad (1.3)$$
Question

What about convergence in densities?
Theorem ([Caballero et al., 1998, Hu et al., 2014])

Assume that

- $v \in \mathbb{D}^{1,6}(\Omega; \mathcal{H})$
- $F = \delta(v) \in \mathbb{D}^{2,6}$ with $\mathbb{E}[F] = 0$, $\mathbb{E}[F^2] = 1$.
- $(D_vF)^{-1} \in L^4(\Omega)$

Then,

$$\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq \left( \|F\|_4 \|(D_vF)^{-1}\|_4 + 2 \right) \|1 - D_vF\|_2$$

$$+ \|(D_vF)^{-1}\|_4^2 \|D_v(D_vF)\|_2.$$  \hspace{1cm} (1.4)
Theorem (K. & Nualart (2021+))

Assume

- **H1**: $\sigma : \mathbb{R} \to \mathbb{R} \in C^2$ with $\sigma'$ bounded and $|\sigma''(x)| \leq C(1 + |x|^m)$, for some $m > 0$.
- **H2**: For some $q > 10$, $\mathbb{E}\left[|\sigma(u(t,0))|^{-q}\right] < \infty$.

Then,

$$\sup_{x \in \mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \leq \frac{C_t}{\sqrt{R}}.$$
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Then,

$$\sup_{x \in \mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \leq \frac{C_t}{\sqrt{R}}.$$

Remark

**H2** holds if $\sigma$ is bounded away from zero or if $|\sigma(x)| \leq \Lambda |x|$ [Chen et al., 2016].
Comments on the Proof

For \( t \in [0, T] \) and \( r < s < t \)

\[
\| D_{s,y}u(t, x) \|_p \leq C_{T,p}p_{t-s}(x - y)
\]
Comments on the Proof

For $t \in [0, T]$ and $r < s < t$

- \[ \|D_{s,y}u(t, x)\|_p \leq C_{T,p}p_{t-s}(x - y) \]

- [Chen et al., 2020a] If $\sigma(x) = x$ then
  \[ \|D_{r,z}D_{s,y}u(t, x)\|_p \leq C_{T,p}p_{t-s}(x - y)p_{s-r}(y - z) \]
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- Under \( H1 \):
  \[ \| D_{r,z} D_{s,y} u(t, x) \|_p \leq C_{T,p} \Phi_{r,z,s,y}(t, x) \]
  where
  \[ \Phi_{r,z,s,y}(t, x) := p_{t-s}(x - y) \times \left( p_{s-r}(y - z) + \frac{p_{t-r}(z - y) + p_{t-r}(z - x) + 1_{|y-x|>|z-y|}}{(s-r)^{1/4}} \right) \]
Comments on the Proof Continued

- Under **H2**: there exists $R_0 > 0$ such that

$$
\sup_{R \geq R_0} E \left[ \left| D_{v_R,t} F_{R,t} \right|^{-p} \right] < \infty.
$$

(1.5)
Parabolic Anderson Model with Delta Initial Condition
Parabolic Anderson Model

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u \dot{W}, \quad x \in \mathbb{R}, \ t > 0, \tag{2.1}
\]

- \( u(0, x) = u_0(x) = \delta_0 \)
- \( \dot{W} \) is space-time white noise
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\[ \frac{\partial u}{\partial t} = \frac{1}{2} \frac{\partial^2 u}{\partial x^2} + u \dot{W}, \quad x \in \mathbb{R}, \ t > 0, \] (2.1)

- \( u(0, x) = u_0(x) = \delta_0 \)
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**Proposition ([Chen and Dalang, 2015])**

*There exists a unique mild solution \( u = \{u(t, x) : (t, x) \in \mathbb{R}_+ \times \mathbb{R}\} \) such that*

\[ \sup_{t \in [0, T]} \mathbb{E} [|u(t, x)|^p] \leq C_{T,p} p_t(x). \] (2.2)
Spatial Averages

Fix $t > 0$. The process $x \rightarrow U(t, x) := u(t, x)/p_t(x)$ is stationary [Amir et al., 2011].
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Fix $t > 0$. The process $x \rightarrow U(t, x) := u(t, x)/p_t(x)$ is stationary [Amir et al., 2011]. Consider

$$G_{R,t} := \frac{1}{\Sigma_{R,t}} \left( \int_{-R}^{R} U(t, x)dx - 2R \right)$$
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\Sigma_{R, t}^2 := \text{Var} \left( \int_{-R}^{R} U(t, x) dx \right) \sim R \log R, \quad [\text{Chen et al., 2020b}]
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**Theorem** ([Chen et al., 2020b])

$$d_{TV}(G_{R,t}, N) \leq \frac{C_t \sqrt{\log R}}{\sqrt{R}}. \quad (2.3)$$
Theorem (K. & Nualart (2021+))

Fix $\gamma > \frac{19}{2}$. Then, there exists an $R_0 \geq 1$ such that for all $R \geq R_0$

$$\sup_{x \in \mathbb{R}} |f_{G_{R,t}}(x) - \phi(x)| \leq \frac{C_t (\log R)^\gamma}{\sqrt{R}}.$$
References


😊 Thank you for your attention!