Convergence of Densities of Spatial Averages for Stochastic Heat Equation

Gaussian Random Fields, Fractals, SPDEs, and Extremes
University of Alabama in Huntsville

Şefika Kuzgun

August 2, 2021

Work in progress, joint with David Nualart

Table of contents

Stochastic Heat Equation with Flat Initial Condition

Stochastic Heat Equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W}, \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

- $u(0,x) = u_0(x) = 1$
- $\dot{W}$ is space-time white noise
- σ nonrandom, Lipschitz

Stochastic Heat Equation

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + \sigma(u)\dot{W}, \qquad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

- $u(0,x) = u_0(x) = 1$
- $\dot{W}$ is space-time white noise
- σ nonrandom, Lipschitz

Proposition ([Walsh, 1986])

There exists a unique mild solution $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ *such that*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}} \mathrm{E}\left[|u(t,x)|^p\right] = C_{T,p} \tag{1.2}$$

Spatial Averages

Fix $t > 0$. The process $x \to u(t,x)$ is stationary.

Spatial Averages

Fix $t > 0$. The process $x \to u(t,x)$ is stationary. Consider

$$F_{R,t} := \frac{1}{\sigma_{R,t}} \left(\int_{-R}^{R} u(t,x)dx - 2R \right)$$

Spatial Averages

Fix $t > 0$. The process $x \to u(t,x)$ is stationary. Consider

$$F_{R,t} := \frac{1}{\sigma_{R,t}} \left(\int_{-R}^{R} u(t,x)dx - 2R \right)$$

where

$$\sigma_{R,t}^2 := \mathrm{Var} \left(\int_{-R}^{R} u(t,x)dx \right)$$

Spatial Averages

Fix $t > 0$. The process $x \to u(t,x)$ is stationary. Consider

$$F_{R,t} := \frac{1}{\sigma_{R,t}} \left(\int_{-R}^{R} u(t,x)dx - 2R \right)$$

where

$$\sigma_{R,t}^2 := \mathrm{Var}\left(\int_{-R}^{R} u(t,x)dx \right) \sim R, \quad \text{[Huang et al., 2020]}$$

Spatial Averages

Fix $t > 0$. The process $x \to u(t,x)$ is stationary. Consider

$$F_{R,t} := \frac{1}{\sigma_{R,t}} \left(\int_{-R}^{R} u(t,x)dx - 2R \right)$$

where

$$\sigma_{R,t}^2 := \text{Var} \left(\int_{-R}^{R} u(t,x)dx \right) \sim R, \quad \text{[Huang et al., 2020]}$$

Theorem ([Huang et al., 2020])

$$d_{TV}(F_{R,t}, N) \leq \frac{C_t}{\sqrt{R}}. \tag{1.3}$$

Question

What about convergence in densities?

Theorem ([Caballero et al., 1998, Hu et al., 2014])

Assume that

- $v \in \mathbb{D}^{1,6}(\Omega; \mathfrak{H})$
- $F = \delta(v) \in \mathbb{D}^{2,6}$ *with* $\mathrm{E}[F] = 0$, $\mathrm{E}\left[F^2\right] = 1$.
- $(D_v F)^{-1} \in L^4(\Omega)$

Then,

$$\sup_{x \in \mathbb{R}} |f_F(x) - \phi(x)| \leq \left(\|F\|_4 \left\|(D_v F)^{-1}\right\|_4 + 2\right) \|1 - D_v F\|_2 + \left\|(D_v F)^{-1}\right\|_4^2 \|D_v (D_v F)\|_2 . \tag{1.4}$$

Theorem (K. & Nualart (2021+))

Assume

- **H1**: $\sigma : \mathbb{R} \to \mathbb{R} \in C^2$ *with* σ' *bounded and* $|\sigma''(x)| \le C(1 + |x|^m)$, *for some* $m > 0$.
- **H2**: *For some* $q > 10$, $\mathrm{E}\left[|\sigma(u(t,0))|^{-q}\right] < \infty$.

Then,

$$\sup_{x\in\mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \le \frac{C_t}{\sqrt{R}}.$$

Theorem (K. & Nualart (2021+))

Assume

- **H1**: $\sigma : \mathbb{R} \to \mathbb{R} \in C^2$ *with* σ' *bounded and* $|\sigma''(x)| \leq C(1 + |x|^m)$, *for some* $m > 0$.
- **H2**: *For some* $q > 10$, $\mathrm{E}\left[|\sigma(u(t,0))|^{-q}\right] < \infty$.

Then,

$$\sup_{x \in \mathbb{R}} |f_{F_{R,t}}(x) - \phi(x)| \leq \frac{C_t}{\sqrt{R}}.$$

Remark

H2 *holds if* σ *is bounded away from zero or if* $|\sigma(x)| \leq \Lambda |x|$ *[Chen et al., 2016].*

Comments on the Proof

For $t \in [0,T]$ and $r < s < t$

- $$\|D_{s,y}u(t,x)\|_p \leq C_{T,p}p_{t-s}(x-y)$$

Comments on the Proof

For $t \in [0, T]$ and $r < s < t$

- $$\|D_{s,y}u(t,x)\|_p \leq C_{T,p}p_{t-s}(x-y)$$

- [Chen et al., 2020a] If $\sigma(x) = x$ then

$$\|D_{r,z}D_{s,y}u(t,x)\|_p \leq C_{T,p}p_{t-s}(x-y)p_{s-r}(y-z)$$

Comments on the Proof

For $t \in [0,T]$ and $r < s < t$

- $$\|D_{s,y}u(t,x)\|_p \leq C_{T,p}p_{t-s}(x-y)$$

- [Chen et al., 2020a] If $\sigma(x) = x$ then

  $$\|D_{r,z}D_{s,y}u(t,x)\|_p \leq C_{T,p}p_{t-s}(x-y)p_{s-r}(y-z)$$

- Under **H1**:

  $$\left\|D_{r,z}D_{s,y}u(t,x)\right\|_p \leq C_{T,p}\Phi_{r,z,s,y}(t,x)$$

  where

  $$\Phi_{r,z,s,y}(t,x) := p_{t-s}(x-y) \times \left(p_{s-r}(y-z) + \frac{p_{t-r}(z-y) + p_{t-r}(z-x) + \mathbf{1}_{\{|y-x|>|z-y|\}}}{(s-r)^{1/4}}\right).$$

Comments on the Proof Continued

- Under **H2**: there exists $R_0 > 0$ such that

$$\sup_{R \geq R_0} \mathrm{E}\left[\left| D_{v_{R,t}} F_{R,t} \right|^{-p} \right] < \infty. \tag{1.5}$$

Parabolic Anderson Model with Delta Initial Condition

Parabolic Anderson Model

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u\dot{W}, \qquad x \in \mathbb{R},\ t > 0, \tag{2.1}$$

- $u(0, x) = u_0(x) = \delta_0$
- $\dot{W}$ is space-time white noise

Parabolic Anderson Model

$$\frac{\partial u}{\partial t} = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} + u\dot{W}, \qquad x \in \mathbb{R},\ t > 0, \tag{2.1}$$

- $u(0,x) = u_0(x) = \delta_0$
- $\dot{W}$ is space-time white noise

Proposition ([Chen and Dalang, 2015])

There exists a unique mild solution $u = \{u(t,x) : (t,x) \in \mathbb{R}_+ \times \mathbb{R}\}$ *such that*

$$\sup_{t\in[0,T]} \mathrm{E}\left[|u(t,x)|^p\right] \leq C_{T,p}\, p_t(x). \tag{2.2}$$

Spatial Averages

Fix $t > 0$. The process $x \to U(t,x) := u(t,x)/p_t(x)$ is stationary [Amir et al., 2011].

Spatial Averages

Fix $t > 0$. The process $x \to U(t,x) := u(t,x)/p_t(x)$ is stationary [Amir et al., 2011]. Consider

$$G_{R,t} := \frac{1}{\Sigma_{R,t}} \left(\int_{-R}^{R} U(t,x)dx - 2R \right)$$

Spatial Averages

Fix $t > 0$. The process $x \to U(t,x) := u(t,x)/p_t(x)$ is stationary [Amir et al., 2011]. Consider

$$G_{R,t} := \frac{1}{\Sigma_{R,t}} \left(\int_{-R}^{R} U(t,x)dx - 2R \right)$$

where

$$\Sigma^2_{R,t} := \mathrm{Var} \left(\int_{-R}^{R} U(t,x)dx \right)$$

Spatial Averages

Fix $t > 0$. The process $x \to U(t,x) := u(t,x)/p_t(x)$ is stationary [Amir et al., 2011]. Consider

$$G_{R,t} := \frac{1}{\Sigma_{R,t}} \left(\int_{-R}^{R} U(t,x)dx - 2R \right)$$

where

$$\Sigma_{R,t}^2 := \operatorname{Var}\left(\int_{-R}^{R} U(t,x)dx \right) \sim R \log R, \quad \text{[Chen et al., 2020b]}$$

Spatial Averages

Fix $t > 0$. The process $x \to U(t,x) := u(t,x)/p_t(x)$ is stationary [Amir et al., 2011]. Consider

$$G_{R,t} := \frac{1}{\Sigma_{R,t}} \left(\int_{-R}^{R} U(t,x)dx - 2R \right)$$

where

$$\Sigma_{R,t}^2 := \mathrm{Var}\left(\int_{-R}^{R} U(t,x)dx \right) \sim R \log R, \quad \text{[Chen et al., 2020b]}$$

Theorem ([Chen et al., 2020b])

$$d_{TV}(G_{R,t}, N) \leq \frac{C_t \sqrt{\log R}}{\sqrt{R}}. \tag{2.3}$$

Theorem (K. & Nualart (2021+))

Fix $\gamma > \frac{19}{2}$. *Then, there exists an* $R_0 \geq 1$ *such that for all* $R \geq R_0$

$$\sup_{x \in \mathbb{R}} |f_{G_{R,t}}(x) - \phi(x)| \leq \frac{C_t (\log R)^{\gamma}}{\sqrt{R}}.$$

References

Amir, G., Corwin, I., and Quastel, J. (2011).
Probability distribution of the free energy of the continuum directed random polymer in 1+ 1 dimensions.
Communications on pure and applied mathematics, 64(4):466–537.

Caballero, M. E., Fernández, B., and Nualart, D. (1998).
Estimation of densities and applications.
Journal of Theoretical Probability, 11(3):831–851.

Chen, L. and Dalang, R. C. (2015).
Moments and growth indices for the nonlinear stochastic heat equation with rough initial conditions.
The Annals of Probability, 43(6):3006–3051.

Chen, L., Hu, Y., and Nualart, D. (2016).
Regularity and strict positivity of densities for the nonlinear stochastic heat equation.
arXiv preprint arXiv:1611.03909.

Chen, L., Khoshnevisan, D., Nualart, D., and Pu, F. (2020a).
Central limit theorems for spatial averages of the stochastic heat equation via malliavin-stein's method.
arXiv preprint arXiv:2008.02408.

Chen, L., Khoshnevisan, D., Nualart, D., and Pu, F. (2020b).
Spatial ergodicity and central limit theorems for parabolic anderson model with delta initial condition.
arXiv preprint arXiv:2005.10417.

Hu, Y., Lu, F., and Nualart, D. (2014).
Convergence of densities of some functionals of gaussian processes.
Journal of Functional Analysis, 266(2):814–875.

Huang, J., Nualart, D., and Viitasaari, L. (2020).
A central limit theorem for the stochastic heat equation.
Stochastic Processes and their Applications, 130(12):7170–7184.

Walsh, J. B. (1986).
An introduction to stochastic partial differential equations.
In *École d'Été de Probabilités de Saint Flour XIV-1984*, pages 265–439. Springer.

☺ Thank you for your attention!