Intermittency Property of Stochastic Heat and Wave Equation with Dobrić Ojeda process

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Settings for Parabolic SPDE

$$\begin{cases} \partial_t u(t,x) = (\Delta u)(t,x) + \lambda\sigma(u(t,x))\dot{V}(t,x), \\ u(0,x) = u_0, \end{cases} \quad (1)$$

.

- σ is a constant or bounded function with vanishing initial data,$u_0 \in \mathbb{R}, \lambda \in \mathbb{R}$
- $\dot{V}$ is the Dobrić-Ojeda noise, which is a generalization of white noise and captures many properties of noise that fractional in time and correlated in space.
- All the proofs in this presentation are based on the Stochastic Heat Equation,The result of Stochastic Wave Equation follows in a similar manner.

Dobrić-Ojeda noise

Definition (Dobrić-Ojeda noise)

$$\int_0^t \int_0^L F(y,s)V(dyds) = \int_0^t \int_0^L F(y,s)s^{H-1/2}W(dyds)$$

where $1/4 < H < 1$

- This process appears in a work of Dobrić-Ojeda (2009) and Conus-Dobrić-Wildman (2016) motivated by financial mathematics applications. One can show that adding a drift term yields a reasonable approximation of fBM.
- $\dot{V}(t,x) = V(dt,dx) = t^{H-1/2}\dot{W}$, $\dot{W}$ is the spatially homogeneous Gaussian noise that is white in time.
- $E(V(dt,dx)V(dt,dy)) = t^{2H-2}f(x-y)$, here $f(x-y)$ is the homogeneous space correlation function .

Lyapunov exponent

Intuitively, a random field is intermittent if:

- The field develops very high-valued peaks when t gets large.
- Those peaks are concentrated on small spatial islands.

Definition (Lyapunov exponent)

The upper pth-moment Lyapunov exponent $\bar{\gamma}(p)$ of u at x_0 as

$$\bar{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t} lnE(|u(t,x_0)|^p) \text{for all } p \in (0,\infty)$$

we say u is weakly intermittent if $\bar{\gamma}(p) > 0$ and $\bar{\gamma}(p) < \infty$ for all $p > 2$. $\frac{\bar{\gamma}(p)}{p}$is nondecreasing by Jensen's inequality.If $\bar{\gamma}(1) = 0$, the strictly increasing function $f(p) = \frac{\bar{\gamma}(p)}{p}$ will imply full intermittency.

The Existence and Uniqueness of the solution

Theorem (Dalang 1999, Q2021)

The stochastic heat equation (1) has an almost-sure unique solution u

$$u_0 + \int_0^t \int_{-\infty}^{\infty} \lambda u(s, y) \cdot p_h(t - s, x - y) s^{H-1/2} W(dyds)$$

when $H > \alpha/4$. $p_h(t, x)$ is the fundamental solution to the Stochastic heat equation .

$$p_h(t, x) = \frac{1}{(2\pi t)^{1/2}} \exp(-\frac{|x|^2}{2t})$$

The second moment formula for the expectation of u

- $X_t^{(1)}, X_t^{(2)}$ are two independent Brownian motions starting from x, y.
- N_t is a rate 1 Poisson process with jump times τ_i

Theorem (Second moment formula for the expectation of u)

The 2nd moment formula for Stochastic heat equation with DO process

$$E[u(t,x)u(t,y)]$$
$$= e^t E_{x,y}\left[u_0^2 \prod_{i=1}^{N(t)} [p_h(\tau_i - \tau_{i-1}, \mathbb{R})^2 f(X_{\tau_i}^{(1)} - X_{\tau_i}^{(2)}) \lambda^2 (t-\tau_i)^{2H-1}] \right]$$

This formula is similar to Dalang-Mueller-Tribe (2009) .
Main idea: replace integrals in the moment formula by expectations with respect to X_t, N_t

Some settings of nth moment formula

- $X_t^{(1)}, \ldots, X_t^{(n)}$ are n independent Brownian motion starting from x.
- Consider the set of pairs $\{(j_1, j_2) : 1 \leq j_1 < j_2 \leq n\}$.
- For each pair(j_1, j_2), $N_t^{(j_1, j_2)}$ is a Poisson process with rate 1.
- N_t^{ℓ} is the sum of Poisson processes involving ℓ, with jump times τ_i^{ℓ}.
- $N_t = \sum_{j_1, j_2} N_t^{(j_1, j_2)}$, with Poisson rate $\nu_n = \frac{n(n-1)}{2}$, with jump times σ_i
- (R_1^i, R_2^i) is the index of pairs that jumps at σ_i

The nth moment formula for the expectation of u

Theorem (Nth moment formula for the expectation of u)

The nth moment formula for Stochastic heat equation with DO process

$$
\begin{aligned}
&E(u^n(t,x)) \\
&= e^{tn(n-1)/2} E_x\left[u_0^n \prod_{\ell=1}^{n} \prod_{i=1}^{N_t^\ell} p_h(\tau_i^\ell - \tau_{i-1}^\ell, \mathbb{R}) \right. \\
&\quad \left. \times \prod_i f(X_{\sigma_i}^{R_1^i} - X_{\sigma_i}^{R_2^i}) \lambda^2 (t-\sigma_i)^{2H-1} \right]
\end{aligned}
$$

The Upper Bound for the $\mathrm{E}(u^n(t,x))$

Let a be defined as

$$a = \begin{cases} 0 & \text{Smooth noise} & f \text{ is bounded} \\ \alpha & \text{fractional type noise} & f(x-y) = |x-y|^{-\alpha} \\ 1 & \text{White noise} & f(x-y) = \delta_0(x-y) \end{cases}$$

Theorem (nth moment Upper Bound)

Stochastic Heat Equation: $$E[u^n(t,x)] \leq u_0^n \cdot \exp(A_{h_0} \lambda^2 n^{\frac{4-a}{2-a}} t^{\frac{4H-a}{2-a}})$$

Stochastic Wave Equation: $$E[u^n(t,x)] \leq u_0^n \exp(A_{w_0} \lambda^2 n^{\frac{4-a}{3-a}} t^{\frac{2H+2-a}{3-a}})$$

where A_{h_0}, A_{w_0} are universal constants

The Lower Bound for the $\mathbb{E}(u^n(t,x))$

Theorem (nth moment Lower Bound)

Stochastic Heat Equation: $E[u^n(t,x)] \geq u_0^n \cdot \exp(A_{h_1}\lambda^2 n^{\frac{4-a}{2-a}} t^{\frac{4H-a}{2-a}})$

Stochastic Wave Equation: $E[u^n(t,x)] \geq u_0^n \exp(A_{w_1}\lambda^2 n^{\frac{4-a}{3-a}} t^{\frac{2H+2-a}{3-a}})$

where A_{h_1},A_{w_1} are universal constants

Some remarks about the paper

- If H=1/2, then $\dot{V}$ becomes $\dot{W}$,i.e., the spatially homogeneous Gaussian noise that is white in time.
- The order of t and n are the same with spatially homogeneous Gaussian noise which in time like a fractional Brownian motion with Hurst index H, i.e., $E(\dot{F}(t,x)\dot{F}(s,y)) = |t-s|^{2H-2}f(x-y)$
- The whole proof for Dobric-Ojeda noise does not require Malliavin Calculus technique, which simplified the calculation.

Proof of existence and uniqueness

There are 3 requirements need to be satisfied for the kernel $\Gamma(t,s,x-y) = p_h(t-s,x-y)s^{H-1/2}$ by Dalang (1999).

Requirement 1.$\Gamma(t,s,\mathbb{R})$ is finite,i.e.,

$$\int_{-\infty}^{\infty} \Gamma(t,s,x-y)dy < \infty \text{ for all x}$$

Requirement 2. The Fourier transform of $\Gamma(t,s,x-y)$ is L^2 measurable.

$$\int_0^t ds \int_{\mathbb{R}} \mu(d\xi)|\mathcal{F}\Gamma(t,s)(\xi)|^2 < +\infty$$

This gives us the bound for H, i.e., $\alpha/4 < H < 1$

Requirement 3. Continuity of the Fourier transform of the fundamental solution.

Proof of Upper Bound with smooth noise

The initial condition should satisfy

$$|E[u^n(t,x)]| \leq u_0^n e^{tn(n-1)/2} E_{(0,x),\dots,(0,x)}[(A_0\lambda^2)^{N_t} Z_{n,t}]$$

where $Z_{n,t} = \prod_{\ell \in \mathcal{L}_n} \prod_{i=1}^{N_t(\ell)} ((t-\tau_i^\ell)^{H-1/2})$

By arithmetic–geometric inequality, $Z_{n,t} \leq (\frac{1}{N_t} \sum_{\ell \in \mathcal{L}_n} \sum_{i=1}^{N_t(\ell)} (t-\tau_i)^{H-1/2})^{N_t}$

Since we know τ_i is the ordered statistics from a uniform distribution on $[0,t]$, we have

$$\sum_{i=1}^{N_t} (t-\tau_i)^{H-1/2} = t^{H-1/2} \sum_{i=1}^{N_t} (1-U_{(i)})^{H-1/2}$$

$$\stackrel{\text{law}}{=} t^{H-1/2} \sum_{i=1}^{N_t} U_{(i)}^{H-1/2}$$

With $N_t = k$, $\quad t^{H-1/2} \int_0^1 \int_0^1 \dots \int_0^1 \sum_{i=1}^{k} s_i^{H-1/2} ds_1 \dots ds_k = t^{H-1/2} \frac{1}{H+\frac{1}{2}} \cdot k$

Proof of Upper Bound with smooth noise Cont.

Then by the following lemma, we complete the proof.

Lemma

$$e^{tn(n-1)/2}E_{(0,x),\dots,(0,x)}[(A_0\lambda^2)^{N_t}Z_{n,t}] \le \exp(A_{h_0}\lambda^2 t^{2H}n^2)$$

Proof:

$$\begin{aligned}
&E_{(0,x)\dots(0,x)}[(A_0\lambda^2)^{N_t}Z_{n,t}]\\
&=\sum_{k=0}^{\infty}E_{(0,x),\dots,(0,x)}[(A_0\lambda^2)^{N_t}Z_{n,t}|N_t=k]\cdot P(N_t=k)\\
&\le e^{-tn(n-1)/2}+\sum_{k=1}^{\infty}\left(\frac{1}{2k}\cdot 2\cdot(A_0\lambda^2)^{1/2}\frac{k}{H+\frac{1}{2}}t^{H-1/2}\right)^{2k}\cdot\frac{(\nu_n t)^k e^{-\nu_n t}}{k!}\\
&=e^{-\nu_n t}+\exp(-\nu_n t+\frac{\nu_n\cdot(A_0\lambda^2)t^{2H}}{(H+\frac{1}{2})^2})
\end{aligned}$$

$$e^{\nu_n t}E_{(0,x),\dots,(0,x)}[(A_0\lambda^2)^{N_t}Z_{n,t}] \le \exp((A_{h_0}\lambda^2)n^2t^{2H}) \qquad \square$$

Proof of Upper Bound with fractional type noise

By Burkholder's inequality, we have

$$\mathbb{E}[(u(t,x))^k]^{1/k}$$
$$\leq 1 + c_k \cdot \mathbb{E}[(\int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} S_{t,s}(x,y,z)\lambda^2 u(s,y)u(s,z)dydzds)^{k/2})]^{1/k}$$

where

$$S_{t,s}(x,y,z) = p_h(t-s,x-y)p_h(t-s,x-z)f(y-z)s^{2H-1}$$

We need some notations for simplicity.

$$\mathcal{N}_{\beta,\gamma,k}(u) := \sup_{s,y}(e^{-\beta s^{\gamma}} \|u(s,y)\|_k)$$

$$\Upsilon_{\beta}(t) = \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} S_{t,s}(x,y,z)e^{2\beta(s^{\gamma})}dydzds$$

Proof of Upper Bound with fractional type noise Cont.

$$\begin{aligned}
&\mathcal{N}_{\beta,\gamma,k}(u) \\
&\leq 1+\mathcal{N}_{\beta,\gamma,k}(u)\cdot(c_k\lambda)\Big(\int_0^t\int_{\mathbb{R}}\int_{\mathbb{R}} S_{t,s}(x,y,z)e^{-2\beta(t^\gamma-s^\gamma)}dydzds\Big)^{1/2} \\
&= 1+\mathcal{N}_{\beta,\gamma,k}(u)(c_k\lambda)[e^{-2\beta t^\gamma}\Upsilon_\beta(t)]^{1/2}.
\end{aligned}$$

Then by the technique of incomplete gamma function and Stirling's formula ,we managed to get

$$\Upsilon_\beta(t)\approx\gamma^*(-\frac{1}{2}\alpha+1,\beta t^\gamma)t^{2H-1-\frac{1}{2}\alpha+1}$$

Here $\gamma^*(\nu,z)=e^{-z}\sum_{m=0}^{\infty}\frac{z^m}{\Gamma(\nu+m+1)}$

Proof of Upper bound with fractional type noise Cont.2

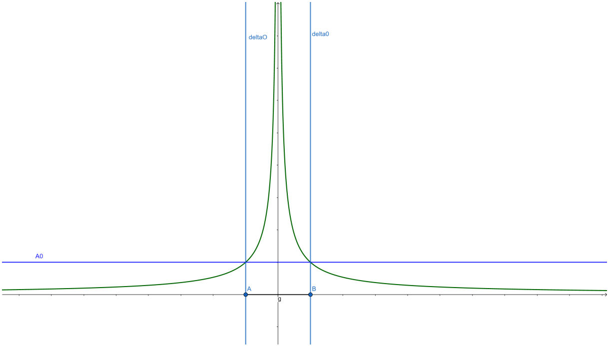
For incomplete gamma function $\gamma^*(\nu, z)$, If $z \to \infty$, then $\gamma^*(\nu, z) \sim z^{-\nu}$, this gives $\gamma = \frac{4H-\alpha}{2-\alpha}$
The next definition was inspired by Foondun-Khoshnevisan (2009)

$$\Upsilon(\beta) = \lim_{t\to\infty} e^{-2\beta t^{\gamma}} \Upsilon_{\beta}(t)$$

Finally ,since we need to choose β such that $(c_k\lambda)[\Upsilon(\beta)]^{1/2} < 1$, we have the upper bound .

Basic Settings in the proof of Lower Bound

We can describe the event $|X_{\sigma_i}^{R_i^1} - X_{\sigma_i}^{R_i^2}| < 2\delta_0$ as the intersection of two event $D(t)$ and $C(k, n, t)$, where $D(t)$ restricts the amplitude of the process and $C(k, n, t)$ restricts the length of time interval between each jump.



The Lower Bound with smooth noise

The covariance function f has the following property: there exist $\delta > 0$ and $A > 0$ such that for $|x| < 2\delta, f(x) \geq A$

$$E[u^n(t,x)] \\ \geq e^{\nu_n t} E_{(0,x),\ldots,(0,x)}[(A\lambda^2)^{N_t} Z_{n,t} \mathbb{1}_{D(t)} \mathbb{1}_{C_{k,n,t}} | N_t = k] \\ \cdot P(N_t = k)$$

$$E[u^n(t,x)] \\ \geq \left(\frac{A\lambda^2(\frac{t}{2})^{2H-1} q^2(\sqrt{2}c)^{2/n} e^{-1} n\nu_n t}{8kn}\right)^k$$

Now we want the fraction in the parenthesis equal to e, we can choose the value of k,i.e., the order of the lower bound.

The Lower Bound with fractional noise

Recall that $f(x-y) \geq A = |x-y|^{-\alpha} = (2\delta)^{-\alpha}$

$$E[u^n(t,x)] \geq \left(\frac{((2\delta)^{-\alpha}\lambda^2)(\frac{t}{2})^{2H-1}q^2(\sqrt{2}c)^{2/n}e^{-1}n\nu_n t}{8kn}\right)^k$$
$$= \left(\frac{((2(\frac{M\cdot tn}{2k})^{1/2})^{-\alpha}\lambda^2)(\frac{t}{2})^{2H-1}q^2(\sqrt{2}c)^{2/n}e^{-1}n\nu_n t}{8kn}\right)^k$$

Now we need to set $k = (C(\lambda^2)t^{2H-\alpha/2}n^{2-\alpha/2})^{\frac{1}{1-\alpha/2}}$
Finally, $E[u^n(t,x)] \geq \exp(C\lambda^{\frac{4}{2-\alpha}}t^{\frac{4H-\alpha}{2-\alpha}}n^{\frac{4-\alpha}{2-\alpha}})$
Here then by Conus-Balan's paper, let $\alpha \to 1$, we have the white noise $E[u^n(t,x)] \geq \exp(Ct^{4H-1}\lambda^4 n^3)$

Further Questions

- Deduce information as regards the behavior of the size of the islands, space-time scaling, etc. and compare both noises.
- what property of a noise impacts its intermittency level?

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