Stratonovich solution for the wave equation

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Outline

1. Introduction

Wave equation

$$\begin{cases} \dfrac{\partial^2 w}{\partial t^2}(t,x) = c^2 \Delta w(t,x) & t>0, x \in \mathbb{R}^d \quad (d \leq 3) \\ w(0,x) = u_0(x), \quad \dfrac{\partial w}{\partial t}(0,x) = v_0(x) & x \in \mathbb{R}^d \end{cases}$$

Description of waves (sound, water, seismic, light,...)

$w(t,x)$ is the displacement of point x at time t; c^2 propagation speed

$d=1$: string; $d=2$: membrane; $d=3$: elastic solid

Solution ($c=1$)

$$w(t,x) = \int_{\mathbb{R}^d} G(t,x-y)v_0(y)dy + \frac{\partial}{\partial t}\int_{\mathbb{R}^d} G(t,x-y)u_0(y)dy$$

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Fundamental solution G

$$G(t,x)=\begin{cases}\frac{1}{2}1_{\{|x|<t\}} & \text{if } d=1,\\ \frac{1}{2\pi}\frac{1}{\sqrt{t^2-|x|^2}}1_{\{|x|<t\}} & \text{if } d=2,\\ \frac{1}{4\pi\sigma_t} & \text{if } d=3\end{cases}$$

$|\cdot|$ is the Euclidean norm on $\mathbb{R}^d$
σ_t is the surface measure on $\{x\in\mathbb{R}^3; |x|=t\}$

History

$d=1$: D'Alembert formula (1746)
$d=3$: Kirkhhoff formula (Euler, 1856)
$d=2$: Poisson formula; Hadamard (1923, method of descent)

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Inhomogeneous wave equation (f is smooth)

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f is a "source function" (describes the effect of the source of the waves on the medium which carries them)

Solution

$$u(t,x) = w(t,x) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) f(s,y) dyds$$

Justification: Let $L = \frac{\partial^2}{\partial t^2} - \Delta$. Then $Lw = 0$

$$LG = \delta \quad \text{and} \quad L(f * G) = f * LG = f * \delta = f$$

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What is a solution?

A (mild) **solution** of (1) satisfies

$$u(t,x) = 1 + \int_0^t \left(\int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(dy) \right) ds, \tag{2}$$

where the stochastic integral $W(dy)$ has to be interpreted in some sense.

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Covariance function γ

$\gamma : \mathbb{R}^d \to [0, \infty]$ is non-negative-definite; $\gamma = \mathcal{F}\mu$

$$\int_{\mathbb{R}^d} \varphi(x)\gamma(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi) \quad \text{for all } \varphi \in \mathcal{S}(\mathbb{R}^d)$$

Homogeneous noise with covariance γ

$W = \{W(\varphi); \varphi \in \mathcal{D}(\mathbb{R}^d)\}$ is a zero-mean Gaussian process

$$\mathbb{E}[W(\varphi)W(\psi)] = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)\psi(y)\gamma(x-y)dxdy$$
$$= \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\overline{\mathcal{F}\psi(\xi)}\mu(d\xi) := \langle \varphi, \psi \rangle_{\mathcal{H}}$$

White noise case: $\gamma = \delta_0$ and $\mu(d\xi) = (2\pi)^{-d}d\xi$

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Examples

1) (Heat kernel) For any $t > 0$,

$$\gamma(x) = (2\pi t)^{-d/2} \exp\left(-\frac{|x|^2}{2t}\right) =: p_t(x) \quad \text{and} \quad \mu(d\xi) = e^{-t|\xi|^2} d\xi$$

2) (Poisson kernel) For any $t > 0$,

$$\gamma(x) = c_d \frac{t}{(t^2 + |x|^2)^{\frac{d+1}{2}}} \quad \text{and} \quad \mu(d\xi) = e^{-t|\xi|} d\xi$$

3) (Riesz kernel) For any $\alpha \in (0, d)$,

$$\gamma(x) = |x|^{-\alpha} \quad \text{and} \quad \mu(d\xi) = C_{d,\alpha} |\xi|^{-(d-\alpha)} d\xi$$

4) (Bessel kernel) For any $\beta > 0$,

$$\gamma(x) = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-t} p_{2t}(x) dt \quad \text{and} \quad \mu(d\xi) = (1 + |\xi|^2)^{-\beta} d\xi$$

Examples (continued)

5) (Fractional kernel) $H_1, \ldots, H_d \in (\frac{1}{2}, 1)$

$$\gamma(x) = \prod_{i=1}^{n} \alpha_{H_i} |x^{(i)}|^{2H_i - 2}, \qquad x = (x^{(1)}, \ldots, x^{(d)})$$

$$\mu(d\xi) = \prod_{i=1}^{n} c_{H_i} |\xi^{(i)}|^{1-2H_i} d\xi, \qquad \xi = (\xi^{(1)}, \ldots, \xi^{(d)})$$

$$\alpha_H = H(2H-1) \quad \text{and} \quad c_H = \frac{\Gamma(2H+1)\sin(\pi H)}{2\pi}$$

$\{W(x) = W(1_{[0,x]})\}_{x \in \mathbb{R}^d}$ is a **fractional Brownian sheet** with Hurst indices $H_1, \ldots, H_d$

If $d = 1$, $\{W(x)\}_{x \in \mathbb{R}}$ is a **fractional Brownian motion** (fBm)

Malliavin calculus

$\mathcal{H}$ is the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to $\langle\cdot,\cdot\rangle_{\mathcal{H}}$

$W = \{W(\varphi); \varphi \in \mathcal{H}\}$ is an isonormal Gaussian process

Any $F \in L^2(\Omega)$ (measurable w.r.t. W) has the Wiener chaos expansion:

$$F = E(F) + \sum_{n\geq 1} I_n(f_n)$$

$I_0 : \mathbb{R} \to \mathbb{R}$ is the identity map ; $I_1(\varphi) = W(\varphi)$ for any $\varphi \in \mathcal{H}$

$I_n : \mathcal{H}^{\otimes n} \to \mathcal{H}_n$ is the **multiple Wiener integral** of order n

$\mathcal{H}_n$ is the n-th Wiener chaos space corresponding to W

$$\mathbb{E}|F|^2 = \big(\mathbb{E}(F)\big)^2 + \sum_{n\geq 1} n!\|\widetilde{f}_n\|^2_{\mathcal{H}^{\otimes n}}$$

$\widetilde{f}_n$ is the symmetrization of f_n

Skorohod integral

$$\delta(u) := \int_{\mathbb{R}^d} u(x) W(\delta x) \qquad u \in \mathrm{Dom}\,\delta$$

$\delta : \mathrm{Dom}\,\delta \subset L^2(\Omega; \mathcal{H}) \to L^2(\Omega)$ is the adjoint of the Malliavin derivative $D : \mathbb{D}^{1,2} \subset L^2(\Omega) \to L^2(\Omega; \mathcal{H})$

Stratonovich integral

$$\int_{\mathbb{R}^d} u(x) W^{\circ}(dx) \stackrel{P}{=} \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^d} u(x) \dot{W}^{\varepsilon}(x) dx \quad \text{(if it exists)}$$

where $\dot{W}^{\varepsilon}(x) = W(p_{\varepsilon}(x - \cdot))$ and $p_{\varepsilon}(x)$ is the heat kernel

If $W = \{W(t)\}_{t \in [0,T]}$ is a fBm of index $H > 1/2$

$$\int_0^T u(t) W^{\circ}(dt) = \int_0^T u(t) W(\delta t) + \alpha_H \int_0^T \int_0^T D_s u(t) |t-s|^{2H-2} dt ds$$

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Adapted integrands

$u(t) = u(0) + A(t) + M(t)$ is a semi-martingale: A is bounded variation process, M is a square-integrable martingale (both continuous)

$$\int_0^T u(t) W^{\circ}(dt) = \int_0^T u(t) W(dt) + \frac{1}{2}\langle W, M\rangle_T$$

Non-adapted integrands

$$\int_0^T u(t) W^{\circ}(dt) = \int_0^T u(t) W(\delta t) + L_T$$

$$\sum_{i=0}^{n-1} \frac{1}{t_{i+1}-t_i}\int_{t_i}^{t_{i+1}}\int_{t_i}^{t_{i+1}} D_s u(t)\,dt\,ds \xrightarrow[|\pi|\to 0]{P} L_T$$

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Definition 1

u is a (mild) **Skorohod solution** of stochastic wave equation (1) if

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta y) ds$$

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v is a (mild) **Stratonovich solution** of stochastic wave equation (1) if

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$$\mathcal{F}G(t,\cdot)(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x} G(t,x)dx = \frac{\sin(t|\xi|)}{|\xi|}$$

$$|\mathcal{F}G(t,\cdot)(\xi)| \leq C_t \left(\frac{1}{1+|\xi|^2}\right)^{1/2}$$

Condition (C):

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{1/2} \mu(d\xi) < \infty$$

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Dalang's condition (D):

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If (D) holds, the **Skorohod solution exists** and has the chaos expansion

$$u(t,x) = 1 + \sum_{n\geq 1} I_n\big(f_n(\cdot,x;t)\big), \quad \text{with}$$

$$f_n(x_1,\ldots,x_n,x;t) = \int_{T_n(t)} \prod_{i=1}^{n} G(t_{i+1}-t_i, x_{i+1}-x_i)dt_1\ldots dt_n$$

where $T_n(t) = \{0 < t_1 < \ldots < t_n < t\}$. Here $t_{n+1} = t$ and $x_{n+1} = x$.

Goal of this talk

Show that the **Stratonovich solution exists**, if (C) holds.

Note: (C) does not hold if W is white noise and $d = 1$

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Necessity of conditions (C) and (D)

Conjecture 1

The Skorohod solution of the **stochastic wave equation** (1) exists if and only if (D) holds.

Remark: This fact is known to be true for the stochastic heat equation driven by space-time homogeneous Gaussian noise with spatial covariance kernel given by the Riesz kernel.

[For the stochastic heat equation, both Skorohod and Stratonovich solutions exist, if (D) holds.]

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2. Equation with mollified noise

Equation with mollified noise $\dot{W}^{\varepsilon}(x) = W(p_{\varepsilon}(x - \cdot))$

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Definition

v^{ε} is a **solution** of (3) if it satisfies

$$v^{\varepsilon}(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) v^{\varepsilon}(s,y) \dot{W}^{\varepsilon}(y) dy ds.$$

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Series expansion

Intuitively, v^ε should be given by $v^\varepsilon(t,x) = 1 + \sum_{n\geq 1} H_n^\varepsilon(t,x)$ where

$$H_n^\varepsilon(t,x) = \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^n G(t_{i+1}-t_i, x_{i+1}-x_i) \prod_{i=1}^n \dot{W}^\varepsilon(x_i) d\mathbf{x} d\mathbf{t}.$$

Lemma 1.

If (D) holds, for any $\varepsilon > 0, p \geq 1, T > 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|v_n^\varepsilon(t,x) - v^\varepsilon(t,x)|^p \to 0 \quad \text{as } n \to \infty,$$

where

$$v_n^\varepsilon(t,x) = 1 + \sum_{k=1}^n H_k^\varepsilon(t,x).$$

Moreover, v^ε is a solution of equation (3).

Series expansion

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$$H_n^{\varepsilon}(t,x) = \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^{n} G(t_{i+1} - t_i, x_{i+1} - x_i) \prod_{i=1}^{n} \dot{W}^{\varepsilon}(x_i) d\mathbf{x} d\mathbf{t}.$$

Lemma 1.

If (D) holds, for any $\varepsilon > 0, p \geq 1, T > 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|v_n^{\varepsilon}(t,x) - v^{\varepsilon}(t,x)|^p \to 0 \quad \text{as } n \to \infty,$$

where

$$v_n^{\varepsilon}(t,x) = 1 + \sum_{k=1}^{n} H_k^{\varepsilon}(t,x).$$

Moreover, v^{ε} is a solution of equation (3).

Lemma 2. (Feynman-Kac-type representation)

$$v^{\varepsilon}(t,x) = e^{t}\mathbb{E}^{N,X}\Big[\prod_{i=1}^{N_t}(\tau_i - \tau_{i-1})\prod_{i=1}^{N_t}\dot{W}^{\varepsilon}(X_{\tau_i}^{x})\Big]$$

• $N = (N_t)_{t\geq 0}$ is a Poisson process of rate 1, with jump times

$$0 = \tau_0 < \tau_1 < \tau_2 < \dots$$

• $(U_i)_{i\geq 1}$ are i.i.d. random variables with density $G(1,\cdot)$.
(If $d = 1$, $U_i \sim \mathrm{Uniform}(-1,1)$.)

• N, $(U_i)_{i\geq 1}$ and W are independent

• The **linearly-interpolated process** $(X_t)_{t\geq 0}$ is given by:

$$X_t = X_{\tau_i} + (t-\tau_i)U_{i+1} \quad \text{for } \tau_i < t \leq \tau_{i+1},\ i \geq 0,$$

with $X_0 = 0$. We let $X_t^x = x + X_t$.

3. Existence of Stratonovich solution

Theorem 1

If (C) holds, there exists a process v such that for any $T > 0, p \geq 2$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|v^{\varepsilon}(t,x) - v(t,x)|^p \to 0 \quad \text{as } \varepsilon \downarrow 0. \tag{4}$$

Moreover, v is a Stratonovich solution of equation (1), and

$$v(t,x) = 1 + \sum_{n\geq 1} H_n(t,x),$$

where $H_n(t,x) = I_n^{\circ}\big(f_n(\cdot, x; t)\big)$ is the **multiple Stratonovich integral** of

$$f_n(x_1,\ldots,x_n,x;t) = \int_{T_n(t)} \prod_{i=1}^{n} G(t_{i+1} - t_i, x_{i+1} - x_i) d\mathbf{t}.$$

Idea for the proof of (4):

Step 1. Show that, for any n, uniformly in $t \in [0, T]$ and $x \in \mathbb{R}^d$,

$$v_n^{\varepsilon}(t,x) \stackrel{L^p(\Omega)}{\longrightarrow} \text{some } v_n(t,x) \quad \text{as} \quad \varepsilon \downarrow 0$$

Step 2. Show that, uniformly in $t \in [0, T]$ and $x \in \mathbb{R}^d$,

$$v_n(t,x) \stackrel{L^p(\Omega)}{\longrightarrow} \text{some } v(t,x) \quad \text{as} \quad n \to \infty$$

Step 3. Show that, uniformly in $t \in [0, T], x \in \mathbb{R}^d$ and $\varepsilon > 0$,

$$v_n^{\varepsilon}(t,x) \stackrel{L^p(\Omega)}{\longrightarrow} v^{\varepsilon}(t,x) \quad n \to \infty$$

Diagram summarizing the 3 steps:

$$\begin{array}{cccccl} & v_n^{\varepsilon}(t,x) & \longrightarrow_{\varepsilon} & v_n(t,x) & \text{for any } n \\ \text{(uniform in } \varepsilon) & \downarrow_n & & \downarrow_n & \\ & v^{\varepsilon}(t,x) & - - - \rightarrow_{\varepsilon} & v(t,x) & \end{array}$$

A brief review: multiple Stratonovich integrals

Definition

The multiple Stratonovich integral is given by

$$I_n^{\circ}(f) := \int_{(\mathbb{R}^d)^n} f(x_1,\ldots,x_n)\prod_{i=1}^{n} W^{\circ}(dx_i)$$

$$\overset{P}{=} \lim_{\varepsilon\downarrow 0}\int_{(\mathbb{R}^d)^n} f(x_1,\ldots,x_n)\prod_{i=1}^{n}\dot{W}^{\varepsilon}(x_i)dx_1\ldots dx_n \quad \text{(if it exists)}$$

Basic idea for proving existence of $I_n^{\circ}(f)$

Compute

$$\prod_{i=1}^{n}\dot{W}^{\varepsilon}(x_i) = \prod_{i=1}^{n} I_1(p_{\varepsilon}(x_i-\cdot))$$

using the **product formula** from Malliavin calculus.

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using the **product formula** from Malliavin calculus.

Product Formula

$$I_n(f)I_1(g) = I_{n+1}(f \otimes g) + nI_{n-1}(f \otimes_1 g)$$

where $(f \otimes g)(x_1, \ldots, x_{n+1}) = f(x_1, \ldots, x_n)g(x_{n+1})$ and

$$(f \otimes_1 g)(x_1, \ldots, x_{n-1}) = \langle f(\cdot, x_1, \ldots, x_{n-1}), g\rangle_{\mathcal{H}}$$
$$= \int_{(\mathbb{R}^d)^2} f(y, x_1, \ldots, x_{n-1})g(z)\gamma(y-z)dydz.$$

Application of product formula

$$I_1(f)I_1(g) = I_2(f \otimes g) + \langle f, g\rangle_{\mathcal{H}}$$

$$I_1(f)I_1(g)I_1(h) = I_3(f \otimes g \otimes h) + I_1(f)\langle g, h\rangle_{\mathcal{H}} + I_1(g)\langle f, h\rangle_{\mathcal{H}} + I_1(h)\langle f, g\rangle_{\mathcal{H}}$$

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Notation

- $[n] = \{1, \ldots, n\}$
- $|J|$ is the cardinality of J, for any set $J \subset [n]$
- J^c is the complement of J in $[n]$
- $\lfloor x \rfloor = k$ if $k \in \mathbb{Z}$ and $k \leq x < k+1$, for any $x \in \mathbb{R}$

Product of n Wiener integrals

$$\prod_{j=1}^{n} I_1(f_j) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{J \subset [n] \\ |J| = n-2k}} \sum_{\substack{\{I_1, \ldots, I_k\} \text{ partition of } J^c \\ I_i = \{\ell_i, m_i\} \forall i = 1, \ldots, k}} I_{n-2k}\Big(\bigotimes_{j \in J} f_j\Big) \prod_{i=1}^{k} \langle f_{\ell_i}, f_{m_i} \rangle_{\mathcal{H}},$$

Convention: $\otimes_{j \in \emptyset} f_j = 1$.

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Basic observation

$$\langle p_\varepsilon(x_1 - \cdot), p_\varepsilon(x_2 - \cdot)\rangle_{\mathcal{H}} = (p_{2\varepsilon} * \gamma)(x_1 - x_2)$$

$p_{2\varepsilon} * \gamma = \mathcal{F}\mu_\varepsilon$ in the sense of distributions, where

$$\mu_\varepsilon(d\xi) = e^{-\varepsilon|\xi|^2}\mu(d\xi)$$

General formula

$$\prod_{i=1}^{n} \dot{W}^\varepsilon(x_i) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{J \subset [n] \\ |J| = n-2k}} \sum_{\substack{\{I_1,\ldots,I_k\} \text{ partition of } J^c \\ I_i = \{\ell_i, m_i\} \forall i=1,\ldots,k}} I_{n-2k}\Big(\bigotimes_{j \in J} p_\varepsilon(x_j - \cdot)\Big) \prod_{i=1}^{k} (p_{2\varepsilon} * \gamma)(x_{\ell_i} - x_{m_i})$$

We multiply by $f(x_1, \ldots, x_n)$ and we integrate $dx_1 \ldots dx_n$.

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We multiply by $f(x_1, \ldots, x_n)$ and we integrate $dx_1 \ldots dx_n$.

Existence of multiple Stratonovich integral

If $I_{n-2k}(f^{\varepsilon}_{k,J,I_1,\ldots,I_k}) \stackrel{P}{\to} I_{n-2k}(f_{k,J,I_1,\ldots,I_k})$ as $\varepsilon \downarrow 0$, for any $k, J, (I_i)$, where

$$f^{\varepsilon}_{k,J,I_1,\ldots,I_k}((y_j)_{j\in J}) = \int_{(\mathbb{R}^d)^n} f(x_1,\ldots,x_n) \prod_{j\in J} p_\varepsilon(x_j - y_j) \prod_{i=1}^{k} (p_{2\varepsilon} * \gamma)(x_{\ell_i} - x_{m_i}) dx_1 \ldots dx_n$$

$$f_{k,J,I_1,\ldots,I_k}((x_j)_{j\in J}) = \int_{(\mathbb{R}^d)^{2k}} f(x_1,\ldots,x_n) \prod_{i=1}^{k} \gamma(x_{\ell_i} - x_{m_i}) d((x_j)_{j\in J^c}),$$

then $I_n^{\circ}(f)$ exists and is given by:

$$I_n^{\circ}(f) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{J\subset[n] \\ |J|=n-2k}} \sum_{\substack{\{I_1,\ldots,I_k\} \text{ partition of } J^c \\ I_i=\{\ell_i,m_i\} \forall i=1,\ldots,k}} I_{n-2k}(f_{k,J,I_1,\ldots,I_k}),$$

Case $n = 2$

$$I_2^{\circ}(f) = \int_{(\mathbb{R}^d)^2} f(x_1, x_2) W(dx_1) W(dx_2) + \int_{(\mathbb{R}^d)^2} f(x_1, x_2) \gamma(x_1 - x_2) dx_1 dx_2$$

Formally,

$$W^{\circ}(dx_1) W^{\circ}(dx_2) = W(dx_1) W(dx_2) + \gamma(x_1 - x_2) dx_1 dx_2.$$

Csse $n = 3$

$$W^{\circ}(dx_1) W^{\circ}(dx_2) W^{\circ}(dx_3) = W(dx_1) W(dx_2) W(dx_3) + W(dx_1)\gamma(x_2 - x_3) dx_2 dx_3 + W(dx_2)\gamma(x_1 - x_3) dx_1 dx_3 + W(dx_3)\gamma(x_1 - x_2) dx_1 dx_2.$$

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Csse $n = 3$

$$\begin{aligned} W^{\circ}(dx_1) W^{\circ}(dx_2) W^{\circ}(dx_3) = {} & W(dx_1) W(dx_2) W(dx_3) + \\ & W(dx_1)\gamma(x_2 - x_3) dx_2 dx_3 + W(dx_2)\gamma(x_1 - x_3) dx_1 dx_3 + \\ & W(dx_3)\gamma(x_1 - x_2) dx_1 dx_2. \end{aligned}$$

Back to our problem

Step 1: convergence as $\varepsilon \downarrow 0$

We have to show that for any n fixed, there exists some v_n such that

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|v_n^{\varepsilon}(t,x) - v_n(t,x)|^p \to 0 \quad \text{as} \quad \varepsilon \downarrow 0$$

where

$$v_n^{\varepsilon}(t,x) = 1 + \sum_{k=1}^{n} H_k^{\varepsilon}(t,x)$$

$$H_n^{\varepsilon}(t,x) = \int_{T_n(t)} \int_{(\mathbb{R}^d)^n} \prod_{i=1}^{n} G(t_{i+1}-t_i, x_{i+1}-x_i) \prod_{i=1}^{n} \dot{W}^{\varepsilon}(x_i) d\mathbf{x} d\mathbf{t}.$$

Theorem 2.

If (C) holds, then for any $n \geq 1, p \geq 2, T > 0$,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \mathbb{E}|H_n^{\varepsilon}(t,x) - H_n(t,x)|^p \to 0 \quad \text{as } \varepsilon \downarrow 0,$$

where

$$H_n(t,x) := I_n^{\circ}\big(f_n(\cdot,x;t)\big) = \sum_{k=0}^{\lfloor n/2 \rfloor} \sum_{\substack{J\subset[n] \\ |J|=n-2k}} \sum_{\substack{\{I_1,\ldots,I_k\} \text{ partition of } J^c \\ I_i=\{\ell_i,m_i\}\forall i=1,\ldots,k}}$$

$$\int_{T_n(t)} \int_{(\mathbb{R}^d)^{2k}} \left(\int_{(\mathbb{R}^d)^{n-2k}} \prod_{j=1}^{n} G(t_{j+1}-t_j, x_{j+1}-x_j) \prod_{j\in J} W(dx_j) \right)$$

$$\prod_{i=1}^{k} \gamma(x_{\ell_i} - x_{m_i}) d\big((x_j)_{j\in J^c}\big) dt_1 \ldots dt_n.$$

Sketch of proof ($n = 2$)

$$
\begin{aligned}
H_2^{\varepsilon}(t,x) &= \int_{T_2(t)} \int_{(\mathbb{R}^d)^2} G(t-t_2, x-x_2) G(t_2-t_1, x_2-x_1) \\
&\qquad \dot{W}^{\varepsilon}(x_1) \dot{W}^{\varepsilon}(x_2) d\mathbf{x} d\mathbf{t} \\
&= \int_{T_2(t)} \int_{(\mathbb{R}^d)^2} G(t-t_2, x-x_2) G(t_2-t_1, x_2-x_1) \\
&\qquad I_2\big(p_{\varepsilon}(x_1-\cdot) \otimes p_{\varepsilon}(x_2-\cdot)\big) d\mathbf{x} d\mathbf{t} + \\
&\quad \int_{T_2(t)} \int_{(\mathbb{R}^d)^2} G(t-t_2, x-x_2) G(t_2-t_1, x_2-x_1) \\
&\qquad (p_{2\varepsilon} * \gamma)(x_1-x_2) d\mathbf{x} d\mathbf{t} \\
&=: H_2'^{\varepsilon}(t,x) + H_2''^{\varepsilon}(t,x).
\end{aligned}
$$

Under (D): $\int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi)|^2 \mu(d\xi) < \infty$

$$H_2'^{\varepsilon}(t,x) = I_2\big(f_{2,\varepsilon}(\cdot,x;t) \to I_2\big(f_2(\cdot,x;t) \quad \text{in } L^2(\Omega)$$

where $f_{2,\varepsilon}(\cdot,x;t) = f_2(\cdot,x;t) * (p_\varepsilon \otimes p_\varepsilon)$

Under (C): $\int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi)| \mu(d\xi) < \infty$

$$H_2''^{\varepsilon}(t,x) = \int_{T_2(t)} (t-t_2) K^{\varepsilon}(t_2-t_1) dt_1 dt_2 \to \int_{T_2(t)} (t-t_2) K(t_2-t_1) dt_1 dt_2$$

where

$$K^{\varepsilon}(t) := \int_{\mathbb{R}^d} G(t,x)(p_{2\varepsilon} * \gamma)(x) dx = \int_{\mathbb{R}^d} \mathcal{F}G(t,\cdot)(\xi) e^{-\varepsilon|\xi|^2} \mu(d\xi)$$

$$\to \int_{\mathbb{R}^d} \mathcal{F}G(t,\cdot)(\xi)\mu(d\xi) = \int_{\mathbb{R}^d} G(t,x)\gamma(x) dx =: K(t),$$

Under (D): $\int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi)|^2 \mu(d\xi) < \infty$

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Under condition (C):

$$H_2^{\varepsilon}(t,x) \to H_2(t,x) \quad \text{in} \quad L^2(\Omega), \quad \text{as} \quad \varepsilon \downarrow 0,$$

where

$$\begin{aligned} H_2(t,x) &= \int_{T_2(t)} \int_{(\mathbb{R}^d)^2} G(t-t_2, x-x_2) G(t_2-t_1, x_2-x_1) W(dx_1) W(dx_2) d\mathbf{t} \\ &\quad + \int_{T_2(t)} \int_{(\mathbb{R}^d)^2} G(t-t_2, x-x_2) G(t_2-t_1, x_2-x_1) \gamma(x_1-x_2) d\mathbf{t} \\ &= I_2^{\circ}(f_2(\cdot, x; t)). \end{aligned}$$

Steps 2 and 3: convergence as $n \to \infty$

Theorem 3

If (C) holds, then for any $T > 0$ and $p \geq 2$, the limit

$$v(t,x) := \lim_{n\to\infty} v_n(t,x) \quad \text{exists in } L^p(\Omega),$$

uniformly in $[0,T] \times \mathbb{R}^d$. Moreover,

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \sup_{\varepsilon>0} \mathbb{E}|v_n^{\varepsilon}(t,x) - v^{\varepsilon}(t,x)|^p \to 0 \text{ as } \quad n \to \infty.$$

Sketch of proof:

$$\sum_{n\geq 0} \sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \|H_n(t,x)\|_p < \infty$$

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4. The case of smooth noise: $\mu(\mathbb{R}^d) < \infty$

Remark

$\{p_\varepsilon(x-\cdot)\}_{\varepsilon>0}$ is Cauchy in $\mathcal{H}$, uniformly in $x \in \mathbb{R}^d$: when $\varepsilon, \varepsilon' \to 0$,

$$\|p_\varepsilon(x-\cdot) - p_{\varepsilon'}(x-\cdot)\|_{\mathcal{H}}^2 = \int_{(\mathbb{R}^d)2} \left(e^{-\varepsilon|\xi|^2/2} - e^{-\varepsilon'|\xi|^2/2}\right)^2 \mu(d\xi) \to 0.$$

Say $p_\varepsilon(x-\cdot) \to A(x)$ in $\mathcal{H}$, as $\varepsilon \downarrow 0$.

Definition

$$\dot{W}(x) := W(A(x)) \quad \text{is well-defined in } L^2(\Omega)$$

$\dot{W}^\varepsilon(x) \to \dot{W}(x)$ in $L^2(\Omega)$, as $\varepsilon \downarrow 0$, uniformly in x.

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Feynman-Kac-type representation

$$v(t,x)=e^t\mathbb{E}^{N,X}\Big[\prod_{i=1}^{N_t}(\tau_i-\tau_{i-1})\prod_{i=1}^{N_t}\dot{W}(X_{\tau_i}^x)\Big].$$

This is proved similarly to the Feynman-Kac-type representation of v^ε (**Lemma 2**), replacing $\dot{W}^\varepsilon$ by $\dot{W}$.

General Case

A similar Feynman-Kac-type representation can be written down for v in the case of a general spectral measure μ (not necessarily finite).

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Thank you!