Phase Analysis of a Stochastic Reaction-Diffusion Equation

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The basic problem in a simplified setting

$\partial_t \psi = \partial_x^2 \psi + \psi - \psi^3 + \lambda \psi \dot{W}$

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- To be concrete we will find $\psi = \{\psi(t\,, x);\ t \geq 0, x \in \mathbb{T}\}$ such that

$$\partial_t \psi = \partial_x^2 \psi + \psi - \psi^3 + \lambda \psi \dot{W} \qquad \text{on } (0\,, \infty) \times \mathbb{T},$$

and $\psi(0) = \psi_0 \in C_+(\mathbb{T})$ independent of $\dot{W}$. More general SPDEs can be studied as well.

Spatio-temporal intermittency: Zimmerman et al (2000)

$\partial_t \psi = \partial_x^2 \psi + \psi - \psi^3 + \lambda \psi \dot{W}$ here, $u \leftrightarrow \psi$

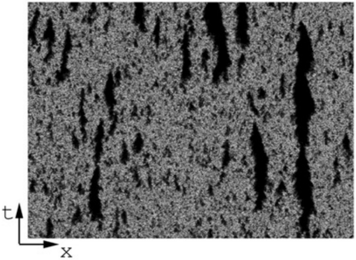

FIG. 1. Space-time evolution of $u(x,t)$ in a persistent STI regime. Black: $u = 0$, grey: $u > 0$. x and t ranges are $(0, 400)$ and $(0, 90)$ with periodic spatial boundary positions. The initial condition is random in the interval $u_0(x) \in (0, 2.4)$. The other parameter values are $\epsilon = 0.95$, $a = 0.5$, $D = 2.0$, $h = 0.22$, $\Delta x = 1$, $\Delta t = 0.001$.

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- Recall that $\mu \in M_1(C(\mathbb{T}))$ is an invariant measure if

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- Not hard to see that if $\psi_0 \equiv 0$ then $\psi(t) \equiv 0$; i.e., $\delta_{\mathbf{0}}$ is invariant, where $\mathbf{0}(x) = 0$ for all $x \in \mathbb{T}$. Is $\delta_{\mathbf{0}}$ unique?

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- This proves predictions of Zimmerman et al (2000).
- One can replace $\psi - \psi^3$ by a more general reaction term $V(\psi)$. For example, when $V(\psi) = \psi - \psi^2$, everything is the same except $\forall \alpha \in (0\,,1/2)\ \exists q > 0:\ \int \exp\left(q\|\omega\|^{1/4}_{C^\alpha(\mathbb{T})}\right) \mu_+(\mathrm{d}\omega) < \infty$

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- Is there a sharp phase transition?

Sketch of proof when noise is high

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- By a comparison argument, $0 \le \psi \le v$ where

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$$\limsup_{t\to\infty} t^{-1}\log\|u(t)\|_{C(\mathbb{T})} \le -c\lambda^4 \quad \text{a.s.}$$

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- Therefore, $\limsup_{t\to\infty} t^{-1}\log\|v(t)\|_{C(\mathbb{T})} \le 1 - c\lambda^4 \quad \text{a.s.}$

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$$\limsup_{t\to\infty} t^{-1}\log\|u(t)\|_{C(\mathbb{T})} \le -c\lambda^4 \quad \text{a.s.}$$

- Therefore, $\limsup_{t\to\infty} t^{-1}\log\|v(t)\|_{C(\mathbb{T})} \le 1 - c\lambda^4$ a.s.
- $\Rightarrow$ if $\lambda > c^{-1/4}$ then $\limsup_{t\to\infty} t^{-1}\log\|\psi(t)\|_{C(\mathbb{T})} < 0$ a.s.

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- This would yield a contradiction, though we can't rigorize this method

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1. $\mathrm{P}(B_t)\ge 1-c\exp(-ct)$ *for all* $t\ge t_0$*; and for all* $t\ge t_0$,
2. $e^{-c_1\lambda^4 t}\lesssim \mathrm{E}\left(\inf_{\mathbb{T}}|u(t)|^k; B(t)\right)\le \mathrm{E}\left(\sup_{\mathbb{T}}|u(t)|^k; B(t)\right)\lesssim e^{-c_2\lambda^4 t}$.

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- So we can prove tightness and deduce existence of invariant measures

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- B_δ is compact [Arzéla–Ascoli], A_ε is closed, $\therefore$ it suffices to prove that

$$\lim_{\varepsilon,\delta \downarrow 0} \limsup_{T \to \infty} \frac{1}{T} \mathrm{E}_{\mathbf{1}} \left[\int_0^T \mathbf{1}_{\{\psi(t) \notin A_\varepsilon \cap B_\delta\}}\, \mathrm{d}t \right] = 0.$$

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Proposition (KKMS 2020+)

$$\sup_{t \geq \eta} \mathrm{E}_{\mathbf{1}} \left(\|\psi(t)\|_{C^\alpha(\mathbb{T})}^k \right) < \infty \qquad \forall \eta > 0, \alpha \in (0\,, 1/2), k \geq 2.$$

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- Uniqueness will use a "coupling argument" (à la Mueller, 1993 + a lemma from stoch analysis)

A large deviations lemma

The following is used in order to make the "random walk argument" work:

Lemma (KKMS 2020+)

Let $\{Z_n\}_{n=1}^{\infty}$ *be i.i.d. with* $p = \mathrm{P}\{Z_1 = 1\} > 1/2$ *and* $q = \mathrm{P}\{Z_1 = -1\} = 1 - p$. *Then,*

$$\sum_{n=1}^{\infty} \mathrm{P}\left\{Z_1 + \cdots + Z_n \leq -k\right\} \leq \frac{\sqrt{4pq}}{1 - \sqrt{4pq}} \left(\frac{q}{p}\right)^{k/2} \qquad \forall k \geq 0.$$

Proof is a nice exercise.

A stochastic analysis lemma

The following fact about "stochastic differential inequalities" is used in order to make the "coupling argument" work. Suppose:

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Lemma (KKMS 2020+)

Uniformly for all $b, t > 0$,

$$\mathrm{P}\left\{ \inf_{s\le t} X_s \neq 0\ ,\ \int_0^t e^{-s}\,\frac{\mathrm{d}\langle X\rangle_s}{X_s} \ge b^2 \right\} \lesssim \frac{a}{b}.$$

Proof is a nice exercise.