On generic boundary-value problems for differential systems in Sobolev spaces

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We consider

the characteristics of solvability and continuity in a parameter of solutions of the most general (generic) classes of one-dimensional inhomogeneous boundary-value problems for systems of linear ordinary differential equations of an arbitrary order in Sobolev spaces on a finite interval.

The mathematician Samoilenko A. dealt with this topic. And now it is actively engaged in such mathematicians as:

Boichuk O.,
Kiguradze I., Ashordia M.
Mikhailets V., Murach O.

Let a finite interval $(a,b) \subset \mathbb{R}$ and parameters $\{m, n, r, l\} \subset \mathbb{N}$, $1 \leq p \leq \infty$, be given.

Linear boundary-value problem

$$(Ly)(t) := y^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) y^{(r-j)}(t) = f(t), \quad t \in (a,b), \tag{1}$$

$$By = c. \tag{2}$$

Here matrix-valued functions $A_{r-j}(\cdot) \in (W_p^n)^{m\times m}$, vector-valued function $f(\cdot) \in (W_p^n)^m$, vector $c \in \mathbb{C}^l$, linear continuous operator

$$B\colon (W_p^{n+r})^m \to \mathbb{C}^l \tag{3}$$

are arbitrarily chosen; vector-valued function $y(\cdot) \in (W_p^{n+r})^m$ is unknown.

The solutions of equation (1) fill the space $(W_p^{n+r})^m$ if its right-hand side $f(\cdot)$ runs through the space $(W_p^n)^m$. Hence, the condition (2) with operator (3) is generic condition for this equation.

It includes all known types of classical boundary conditions and numerous nonclassical conditions containing the derivatives (in general fractional) $y^{(k)}(\cdot)$ with $0 < k \leq n + r$.

Complex Sobolev space $W_p^{n+r} := W_p^{n+r}\big([a,b];\mathbb{C}\big)$

$$W_p^{n+r}\big([a,b];\mathbb{C}\big) := \big\{y \in C^{n+r-1}[a,b] :\, y^{(n+r-1)} \in AC[a,b],\, y^{(n+r)} \in L_p[a,b]\big\}$$

This space is Banach relative to the norm

$$\|y\|_{n+r,p} = \sum_{k=0}^{n+r-1} \big\|y^{(k)}\big\|_p + \big\|y^{(n+r)}\big\|_p,$$

where $\|\cdot\|_p$ is the norm in $L_p\big([a,b];\mathbb{C}\big)$.
By $\|\cdot\|_{n+r,p}$, we also denote the norms in Banach spaces

$$(W_p^{n+r})^m := W_p^{n+r}([a,b];\mathbb{C}^m) \quad \text{and} \quad (W_p^{n+r})^{m\times m} := W_p^{n+r}([a,b];\mathbb{C}^{m\times m}).$$

They consist of the vector-valued functions and matrix-valued functions, respectively, all components of which belong to W_p^{n+r}.

With problem (1), (2), we associate the linear operator

$$(L,B)\colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l. \tag{4}$$

A linear continuous operator $T\colon X \to Y$, where X and Y are Banach spaces, is called a Fredholm operator if its kernel $\ker T$ and cokernel $Y/T(X)$ are finite-dimensional. If this operator is Fredholm, then its range $T(X)$ is closed in Y and the index is finite:

$$\operatorname{ind} T := \dim\ker T - \dim(Y/T(X)) \in \mathbb{Z}.$$

Theorem 1.

The linear operator (4) is a bounded Fredholm operator with index $mr - l$.

Family of matrix Cauchy problems with the initial conditions

$$Y_k^{(r)}(t) + \sum_{j=1}^{r} A_{r-j}(t) Y_k^{(r-j)}(t) = O_m, \quad t \in (a,b),$$

$$Y_k^{(j-1)}(a) = \delta_{k,j} I_m, \qquad j \in \{1,\ldots,r\}.$$

By $[BY_k]$, we denote the numerical $m \times l$ matrix, in which j-th column is result of the action of B on j-th column of $Y_k(\cdot)$.

Definition 1.

A block numerical matrix

$$M(L,B) := ([BY_0], \dots, [BY_{r-1}]) \in \mathbb{C}^{mr \times l} \tag{5}$$

is characteristic matrix to problem (1), (2). It consists of r rectangular block columns $[BY_k(\cdot)] \in \mathbb{C}^{m \times l}$.

Theorem 2.

The dimensions of kernel and cokernel of the operator (4) *are equal to the dimensions of kernel and cokernel of matrix* (5), *respectively:*

$$\dim\ker(L,B) = \dim\ker\left(M(L,B)\right),$$

$$\dim\operatorname{coker}(L,B) = \dim\operatorname{coker}\left(M(L,B)\right).$$

Corollary 1.

The operator (4) *is invertible if and only if* $l = mr$ *and the matrix* $M(L,B)$ *is nondegenerate.*

Consider problem (1), (2) putting $A(t) \equiv 0$ with the next boundary conditions:

$$By = \sum_{k=0}^{n-1} \alpha_k y^{(k)}(a) + \int_a^b \Phi(t) y^{(n)}(t) \mathrm{d}t, \quad y(\cdot) \in (W_p^n)^m.$$

Then we have

$$BY = \sum_{s=0}^{n-1} \alpha_s Y^{(s)}(a) + \int_a^b \Phi(t) Y^{(n)}(t) \mathrm{d}t, \quad Y(\cdot) = I_m,$$

$$M(L,B) = \alpha_0.$$

The numerical matrix α_0 does not depend on p, $\alpha_1, \ldots, \alpha_{n-1}$, and $\Phi(\cdot)$. Thus, the statement of Theorem 2 holds:

$$\dim\ker(M(L,B)) = \dim\ker(\alpha_0),$$
$$\dim\operatorname{coker}(M(L,B)) = \dim\operatorname{coker}(\alpha_0).$$

Boundary-value problems depending on the parameter $k \in \mathbb{N}$

$$L(k)y(t,k) := y^{(r)}(t,k) + \sum_{j=1}^{r} A_{r-j}(t,k)y^{(r-j)}(t,k) = f(t,k), \quad t \in (a,b), \quad (6)$$

$$B(k)y(\cdot,k) = c(k), \quad k \in \mathbb{N}, \quad (7)$$

where $A_{r-j}(\cdot,k)$, $f(\cdot,k)$, $c(k)$, and linear continuous operator $B(k)$ satisfy the above conditions to problem (1), (2).

The sequence of linear continuous operators

$$(L(k),B(k)) \colon (W_p^{n+r})^m \to (W_p^n)^m \times \mathbb{C}^l,$$

and characteristic matrices

$$M\big(L(k),B(k)\big) := \big([B(k)Y_0(\cdot,k)],\ldots,[B(k)Y_{r-1}(\cdot,k)]\big) \subset \mathbb{C}^{mr\times l}.$$

Theorem 3.

If the sequence of operators $\big(L(k),B(k)\big)$ converges strongly to the operator $\big(L,B\big)$ then the sequence of characteristic matrices $M\big(L(k),B(k)\big)$ converges to the matrix $M\big(L,B\big)$ for $k \to \infty$.

Corollary 2.

Under assumptions in Theorem 3, the following inequalities hold starting with sufficiently large k:

$$\dim\ker(L(k),B(k)) \leq \dim\ker(L,B),$$
$$\dim\operatorname{coker}(L(k),B(k)) \leq \dim\operatorname{coker}(L,B).$$

In particular, for sufficiently large k, we have:

1) if $l = mr$ and operator (L,B) is invertible, then the operators $(L(k),B(k))$ are also invertible;
2) if problem (1), (2) has a solution, then problems (6), (7) also have a solution;
3) if problem (1), (2) has a unique solution, then problems (6), (7) also have a unique solution [1, 3, 4].

Boundary-value problem depending on a parameter $\varepsilon \in [0, \varepsilon_0)$

$$L(\varepsilon)y(t,\varepsilon) := y^{(r)}(t,\varepsilon) + \sum_{j=1}^{r} A_{r-j}(t,\varepsilon)y^{(r-j)}(t,\varepsilon) = f(t,\varepsilon), \; t \in (a,b), \tag{8}$$

$$B(\varepsilon)y(\cdot;\varepsilon) = c(\varepsilon), \tag{9}$$

where a linear continuous operator

$$B(\varepsilon) \colon (W_p^{n+r})^m \to \mathbb{C}^{rm}.$$

According to Theorem 1, problem (8), (9) is a Fredholm one with zero index for every $\varepsilon \in [0, \varepsilon_0)$.

Definition 2.

The solution to the problem (8), (9) depends continuously on a parameter ε at $\varepsilon = 0$ if the conditions are satisfied:

- $(*)$ there exists a positive number $\varepsilon_1 < \varepsilon_0$ such that, for any $\varepsilon \in [0, \varepsilon_1)$ and arbitrary chosen $f(\cdot;\varepsilon) \in (W_p^n)^m$, $c(\varepsilon) \in \mathbb{C}^{rm}$, this problem has a unique solution $y(\cdot;\varepsilon) \in (W_p^{n+r})^m$;
- $(**)$ the convergence of right-hand sides $f(\cdot;\varepsilon) \to f(\cdot;0)$ and $c(\varepsilon) \to c(0)$ implies the convergence of solutions

$$y(\cdot;\varepsilon) \to y(\cdot;0) \quad \text{in} \quad (W_p^{n+r})^m \quad \text{as} \quad \varepsilon \to 0+.$$

Consider the following conditions:

(0) the homogeneous boundary-value problem

$$L(0)y(t,0)=0, \quad t\in(a,b), \quad B(0)y(\cdot,0)=0$$

has only the trivial solution;

(I) $A_{r-j}(\cdot;\varepsilon)\to A_{r-j}(\cdot;0)$ in $(W_p^n)^{m\times m}$ for every $j\in\{1,\dots,r\}$;

(II) $B(\varepsilon)y\to B(0)y$ in $\mathbb{C}^{rm}$ for every $y\in(W_p^{n+r})^m$.

Theorem 4.

The solution to the problem (8), (9) depends continuously on the parameter ε at $\varepsilon=0$ if and only if this problem satisfies Conditions (0), (I), and (II).

We supplement our result with a two-sided estimate of the error $\|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p}$ of solution $y(\cdot;\varepsilon)$ via its discrepancy

$$\widetilde{d}_{n,p}(\varepsilon) := \|L(\varepsilon)y(\cdot;0) - f(\cdot;\varepsilon)\|_{n,p} + \|B(\varepsilon)y(\cdot;0) - c(\varepsilon)\|_{\mathbb{C}^{rm}}.$$

Here, we interpret $y(\cdot;0)$ as an approximate solution to problem (8), (9).

Theorem 5.

Let the problem (8), (9) satisfies Conditions (0), (I), and (II). Then there exist positive numbers $\varepsilon_2 < \varepsilon_1$, γ_1, and γ_2, such that

$$\gamma_1\, \widetilde{d}_{n,p}(\varepsilon) \leq \|y(\cdot;0) - y(\cdot;\varepsilon)\|_{n+r,p} \leq \gamma_2\, \widetilde{d}_{n,p}(\varepsilon)$$

for any $\varepsilon \in (0, \varepsilon_2)$. Here, the numbers ε_2, γ_1, and γ_2 do not depend on $y(\cdot;0)$, and $y(\cdot;\varepsilon)$.

Thus, the error and discrepancy of the solution to problem (8), (9) are of the same degree of smallness [2, 6, 7].

For any $\varepsilon \in [0, \varepsilon_0)$, $\varepsilon_0 > 0$, we associate with the system (8)

multi-point Fredholm boundary condition

$$B(\varepsilon)y(\cdot,\varepsilon) = \sum_{j=0}^{N} \sum_{k=1}^{\omega_j(\varepsilon)} \sum_{l=0}^{n+r-1} \beta_{j,k}^{(l)}(\varepsilon) y^{(l)}(t_{j,k}(\varepsilon),\varepsilon) = q(\varepsilon), \qquad (10)$$

where the numbers $\{N, \omega_j(\varepsilon)\} \subset \mathbb{N}$, vectors $q(\varepsilon) \in \mathbb{C}^{rm}$, matrices $\beta_{j,k}^{(l)}(\varepsilon) \in \mathbb{C}^{m \times m}$, and points $\{t_j, t_{j,k}(\varepsilon)\} \subset [a,b]$ are arbitrarily given.

It is not assumed that the coefficients $A_{r-j}(\cdot,\varepsilon)$, $\beta_{j,k}^{(l)}(\varepsilon)$ or points $t_{j,k}(\varepsilon)$ have a certain regularity on the parameter ε as $\varepsilon > 0$. It will be required that for each fixed $j \in \{1, \ldots, N\}$ all the points $t_{j,k}(\varepsilon)$ have a common limit as $\varepsilon \to 0+$, but for the zero-point series $t_{0,k}(\varepsilon)$ this requirement will not be necessary.

The solution $y = y(\cdot,\varepsilon)$ of the multi-point boundary-value problem (8), (10) is continuous on the parameter ε if it exists, is unique, and satisfies the limit relation

$$\left\| y(\cdot,\varepsilon) - y(\cdot,0) \right\|_{n+r,p} \to 0 \quad \text{as} \quad \varepsilon \to 0+. \qquad (11)$$

Assumptions as $\varepsilon \to 0+$:

(α) $t_{j,k}(\varepsilon) \to t_j$ for all $j \in \{1,\dots,N\}$, and $k \in \{1,\dots,\omega_j(\varepsilon)\}$;

(β) $\sum_{k=1}^{\omega_j(\varepsilon)} \beta_{j,k}^{(l)}(\varepsilon) \to \beta_j^{(l)}$ for all $j \in \{1,\dots,N\}$, and $l \in \{0,\dots,n+r-1\}$;

(γ) $\sum_{k=1}^{\omega_j(\varepsilon)} \left\|\beta_{j,k}^{(l)}(\varepsilon)\right\| \left|t_{j,k}(\varepsilon) - t_j\right| \to 0$ for all $j \in \{1,\dots,N\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n+r-1\}$;

(δ) $\sum_{k=1}^{\omega_0(\varepsilon)} \left\|\beta_{0,k}^{(l)}(\varepsilon)\right\| \to 0$ for all $k \in \{1,\dots,\omega_0(\varepsilon)\}$, and $l \in \{0,\dots,n+r-1\}$.

Assumptions (β) and (γ) imply that the norms of the coefficients $\beta_{j,k}^{(l)}(\varepsilon)$ can increase as $\varepsilon \to 0+$, but not too fast.

Theorem 6.

Let the boundary-value problem (8), (10) for $p = \infty$ satisfy the assumptions (α), (β), (γ), (δ). Then it satisfies the limit condition (II). If, moreover, the conditions (0) and (I) are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation (11).

Assumptions as $\varepsilon \to 0+$:

(γ_p) $\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(n+r-1)}(\varepsilon)\| \left|t_{j,k}(\varepsilon) - t_j\right|^{1/p'} = O(1)$ for all $j \in \{1,\dots,N\}$, and $k \in \{1,\dots,\omega_j(\varepsilon)\}$;

(γ') $\sum_{k=1}^{\omega_j(\varepsilon)} \|\beta_{j,k}^{(l)}(\varepsilon)\| \left|t_{j,k}(\varepsilon) - t_j\right| \to 0$ for all $j \in \{1,\dots,N\}$, $k \in \{1,\dots,\omega_j(\varepsilon)\}$, and $l \in \{0,\dots,n+r-2\}$.

Theorem 7.

Let the boundary-value problem (8), (10) for $1 \le p < \infty$ satisfy the assumptions (α), (β), (γ_p), (γ'), (δ). Then it satisfies the limit condition (II). If, moreover, the conditions (0) and (I) are fulfilled, then for a sufficiently small ε its solution exists, is unique and satisfies the limit relation (11) [5, 8].

The systems of conditions (α), (β), (γ), (δ) and (α), (β), (γ_p), (γ'), (δ) do not guarantee uniform convergence of continuous operators $B(\varepsilon)$ to $B(0)$ as $\varepsilon \to 0+$. Therefore, Theorems 6, 7 do not follow from the general facts of the theory of linear operators.

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Thank you for your attention!