Stochastic Lotka-Volterra Competitive Reaction-Diffusion Systems Perturbed by Space-Time Noise: Modeling and Analysis

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Outline

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- The dynamics of many systems in Biology and Ecology such as: epidemic models, tumor-immune models, chemostat models, prey-predator models, competitive models, and among others can be mathematically described.
- The earliest and simplest mathematical models are given by ordinary differential equations (ODEs)

$$dX_i(t) = f_i(X_1(t), \ldots, X_n(t))dt, \quad i = 1, \ldots, n.$$

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Model: Stochastic differential equations. Therefore, one has corresponding stochastic equations

$$dX_i(t) = f_i(X_1(t), \dots, X_n(t))dt + \sum_{k=1}^{d} g_{ik}(X_1(t), \dots, X_n(t))dB_k(t),$$

$$\text{for } i = 1, \dots, n.$$

Here $B_k(t), k = 1, \dots, d$ are standard Brownian motions, which may or may not independent.

Stochastic perturbation and stochastic differential equations frameworks

Overview of literature

- Such stochastic models are modeled in 1970s and have studied widely for 30-40 years.
- The models under SDEs framework are now relatively well-understood. For example, see [Benaïm, 2018] for general abstract theory, Hening and D. Nguyen (AAP 18), and A. Hening, D. Nguyen and Schreiber (AAP 21) for Kolmogorov type equations, N. N. and N. Du (JDE 20) for epidemic SIR model, D. Nguyen, N.N. and G. Yin (SPA 20) for chemostat model, N.N., T. Tuong, G. Yin (SCL 20) for tumor-immune model, N.N., T. Tuong (CNSNS 20) for NP model, D. Nguyen, G. Yin and C. Zhu (SIAP 20) for SIRS model, Benaïm and Schreiber (JMB 19) for discrete setting, Benaïm and Strickler (AAP 19) for PMDPs setting, among others.
- The general methodology is that: looking at the solution on the boundary and then defining a threshold from the Lyapunov exponent (on the boundary problem).

Past dependence and stochastic functional (delay) differential equations frameworks

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Motivation.

- The delays or past dependence are unavoidable in natural phenomena and dynamical systems. So, the framework of SFDEs is more realistic, more effective, and more general for the population dynamics in real life than SDEs counterpart.

Model:

- Consider a stochastic delay Kolmogorov system

$$dX_i(t) = X_i(t)f_i(\mathbf{X}_t)dt + X_i(t)g_i(\mathbf{X}_t)dB_i(t), \quad i = 1, \dots, n, \tag{1.1}$$

where for function $\varphi(t), t \in \mathbb{R}$,

$$\varphi_t := \{\varphi(t+s) : s \in [-r, 0]\} \in C([-r, 0])$$

is a segment function; and

$$\mathbf{X}_t = (X_{1,t}, \dots, X_{n,t}) \in \mathscr{C} := C([-r, 0], \mathbb{R}^n),$$

with $X_{i,t}$ being segment function of $X_i(t)$.

Past dependence and stochastic functional (delay) differential equations frameworks

Overview of literature

- In contrast to Kolmogorov stochastic differential equations, the work on Kolmogorov stochastic differential equations with delay is relatively scarce excepting a few studies. Nevertheless, other than the specific models and applications treated, there has not been a unified framework and a systematic treatment for Kolmogorov SFDEs yet. Moreover, most of the existing results involving delays are not as sharp as desired.
- Recently, D. Nguyen, N. Nguyen, G. Yin develop a unified frameworks and a systematic treatment for general stochastic delay systems for wide range of application; provide almost complete characterization of the longtime behaviors; and present how to apply our proposed results to common systems.

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Model: Stochastic partial differential equations

$$\begin{aligned}\frac{\partial X_i(t,x)}{\partial t} =& A_i X_i(t,x) + f_i(t,x,X_1(t,x),\ldots,X_n(t,x)) \\ &+ g_i(t,x,X_1(t,x),\ldots,X_n(t,x))\frac{\partial^2 W_i(t,x)}{\partial t \partial x}, \quad t \in \mathbb{R}_+, x \in \mathscr{O},\end{aligned}$$

for $i = 1,\ldots,n$. Here $\mathscr{O}$ is a bounded domain, A_i are (second-order) differential operators (w.r.t x) endowed with Neumann boundary condition, which indicate the diffusion in space; and $\frac{\partial^2 W_i(t,x)}{\partial t \partial x}, i = 1,\ldots,d$ are two-parameters (space-time) noises.

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Overview of literature

- Without the noise, i.e., $g_i = 0, \forall i$, the system is deterministic partial differential equations (PDEs). Such problems are so-called reaction-diffusion system, and studied widely in PDE community, for example
  - The studies of coexistence states in the Volterra-Lotka competitive model by Cosner and Lazer (SIAP 84), Gui and Lou (CPAM 94), Hutson, Lou and Mischaikow (JDE 02), Lam and Ni (SIAP 12).
  - The studies of small diffusion by Hutson, Lou, and Mischaikow (JDE 05), and some other studies by He and Ni (JDE 13, CPAM 16, Calc. Var. PDE 16, 17).
- There seems to have no works before considering the ecological and biological systems involving both random effects and spatial inhomogeneity.

Stochastic reaction-diffusion Lotka-Volltera competitive models

This talk focuses on stochastic reaction-diffusion Lotka-Volltera competitive models perturbed by space-time noises, given by

$$\begin{cases} \dfrac{\partial U(t,x)}{\partial t} = \Delta U(t,x) + U(t,x)(m_1(x) - a_1(x)U(t,x) - b_1(x)V(t,x)) \\ \qquad\qquad + \sigma_1(x)U(t,x)\dfrac{\partial^2 W_1(t,x)}{\partial t \partial x}, \quad 0 \le x \le 1, t \ge 0, \\ \dfrac{\partial V(t,x)}{\partial t} = \Delta V(t,x) + V(t,x)(m_2(x) - a_2(x)V(t,x) - b_2(x)U(t,x)) \\ \qquad\qquad + \sigma_2(x)V(t,x)\dfrac{\partial^2 W_2(t,x)}{\partial t \partial x}, \quad 0 \le x \le 1, t \ge 0, \\ \dfrac{\partial U}{\partial x}(t,0) = \dfrac{\partial U}{\partial x}(t,1) = \dfrac{\partial V}{\partial x}(t,0) = \dfrac{\partial V}{\partial x}(t,1) = 0, \quad t \ge 0, \\ U(0,x) = U_0(x), V(0,x) = V_0(x), \quad 0 \le x \le 1., \end{cases} \tag{1.2}$$

where $U(t,x), V(t,x)$ represent the densities of species at time t and location x, $m_i(x)$, $a_i(x)$, $b_i(x)$, for $i = 1,2$ are functions defined on $[0,1]$, and Δ is the Laplace operator. The use of Neumann boundary condition is motivated by applications in biology and ecology, namely, the population will not leave a finite domain.

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- Well-posedness of the problem: existence, uniqueness, continuity on initial data of the solution; and regularity of solution.
- Absolute continuity of the law and the existence of a density
- The existence of an invariant measure
- Sufficient conditions for the permanence and extinction.

Space-time white driving noise.

Assume that $\{\beta_{1,k}(t)\}_{k=1}^{\infty}$, and $\{\beta_{2,k}(t)\}_{k=1}^{\infty}$ are two sequences of independent $\{\mathscr{F}_t\}_{t\geq 0}$-adapted one-dimensional standard Wiener processes. Now, let $\{e_k\}_{k=1}^{\infty}$ be a complete orthonormal system in $L^2((0,1),\mathbb{R})$ including eigenfunctions of Neumann Laplace operator in $[0,1]$. It is seen that they are uniformly bounded. That is,

$$\sup_{k\in\mathbb{N}}\sup_{x\in[0,1]}|e_k(x)|<\infty.$$

We define the standard cylindrical Q-Winner processes $W_i(t), i=1,2$ as follows

$$W_i(t)=\sum_{k=1}^{\infty}\beta_{k,i}(t)e_k,\quad i=1,2.$$

In higher dimension, we will need to use colored noise in space to obtain more regularity but do not need to require it be a finite-trace Q-Wiener process.

Definition of solution

Now, we define a mild solution of (1.2) as a processes satisfying

$$
\begin{cases}
U(t,x) = \int_0^1 G_t(x,y)U_0(y)dy \\
\qquad + \int_0^t \int_0^1 G_{t-s}(x,y)U(s,y)\left(m_1(y) - a_1(y)U(s,y) - b_1(y)V(s,y)\right)dyds \\
\qquad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma_1(y)U(s,y)W_1(ds,dy), \\
V(t,x) = \int_0^1 G_t(x,y)V_0(y)dy \\
\qquad + \int_0^t \int_0^1 G_{t-s}(x,y)V(s,y)\left(m_2(y) - a_2(y)V(s,y) - b_2(y)U(s,y)\right)dyds \\
\qquad + \int_0^t \int_0^1 G_{t-s}(x,y)\sigma_2(y)V(s,y)W_2(ds,dy),
\end{cases}
\tag{2.1}
$$

where the stochastic integrals are in Walsh's sense with respect to the corresponding Brownian sheets of $W_1(t)$, $W_2(t)$ (denoted by $W_1(t,y), W_2(t,y)$ for simplicity of notation);

Definition of solution (cont)

or satisfying the following stochastic integral equation

$$\begin{cases} U(t) = e^{t\Delta_N} U_0 + \int_0^t e^{(t-s)\Delta_N} U(s)\,(m_1 - a_1 U(s) - b_1 V(s))\,ds \\ \qquad + \int_0^t e^{(t-s)\Delta_N} \sigma_1 U(s) dW_1(s), \\ V(t) = e^{t\Delta_N} V_0 + \int_0^t e^{(t-s)\Delta_N} V(s)\,(m_2 - a_2 V(s) - b_2 U(s))\,ds \\ \qquad + \int_0^t e^{(t-s)\Delta_N} \sigma_2 V(s) dW_2(s), \end{cases} \tag{2.2}$$

where the stochastic integrals, in which $\sigma_1 U(s)$ and $\sigma_2 V(s)$ as multiplication operators, are defined as in infinite-dimensional integration theory and $U(t) = U(t,x)$, $V(t) = V(t,x)$, $m_i = m_i(x)$, $a_i = a_i(x)$, $b_i = b_i(x)$, $\sigma_i = \sigma_i(x)$ $(i = 1, 2)$ are understood as elements in a Hilbert space $L^2((0,1), \mathbb{R})$.

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- We need to work in infinite dimensional spaces and then it is not easy to obtain the tightness.
- The classical Itô formula is not valid for the mild solutions in SPDEs.
- The evolution involves both time-flow and space-flow in random environment, the analysis is much different from either SDEs or PDEs setting.

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- Due to the lack of linear-growth property, we could not obtain the (uniformly) boundedness of truncated solutions. However, we can overcome by establishing the non-negativity and then, ignoring the negative terms.
- Due to we are studying the system in infinite dimensional space, to establish the tightness of a family of measures, we need to establish estimates as well as regularities of the solution in an appropriate Hölder space, which is embedded compactly to some continuous spaces.
- We use an approximation of mild solution by strong solutions and a novel mild Itô formula developed recently by Da Prato, Jentzen and Röckner (Trans. AMS 2019) to study the longtime behavior.

Our formal results

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- The law of the solution (for fixed time and state) is continuous (w.r.t Lebesgue measure) and admit a density.
- There is an invariant measure.

The mild Itô formula

Theorem 4.1

(Da Prato, Jentzen and Röckner (Trans. AMS 2019)) *Let* $X : [0,\infty) \times \Omega \to H$ *be a mild Itô formula with evolution family* $S : \{(t_1, t_2) : 0 \le t_1 < t_2\} \to L(\widehat{H}, \check{H})$, *mild drift* $F : [0,\infty] \times \Omega \to \widehat{H}$ *and mild diffusion* $G : [0,\infty) \times \Omega \to HS(U_0, \widehat{H})$. *Let* V *be a real separable Hilbert space and* $\mathbb{U} \subset U_0$ *be an arbitrary orthonormal basis of* U_0. *Then, for all* $\varphi \in C^{1,2}([0,\infty) \times \check{H}, V)$, $t_0 < t \in [0,\infty)$, *it holds a.s. that* $\int_0^t \| \frac{\partial \varphi}{\partial X}(s, S_{s,t}X_s) S_{s,t} F_s \|_V + \| \frac{\partial \varphi}{\partial X}(s, S_{s,t}X_s) S_{s,t} G_s \|^2_{HS(U_0,V)} ds < \infty$, *and* $\int_0^t \| \frac{\partial \varphi}{\partial t}(s, S_{s,t}X_s) \|_V + \| \frac{\partial^2 \varphi}{\partial X^2}(s, S_{s,t}X_s) \| \| S_{s,t} G_s \|^2_{HS(U_0,\check{H})} ds < \infty$, *and*

$$\begin{aligned}\varphi(t, X_t) =& \varphi(t_0, S_{t_0,t}X_{t_0}) + \int_{t_0}^t \frac{\partial \varphi}{\partial t}(s, S_{s,t}X_s) ds + \int_{t_0}^t \frac{\partial \varphi}{\partial X}(s, S_{s,t}X_s) S_{s,t} F_s ds \\ &+ \frac{1}{2} \sum_{u \in \mathbb{U}} \int_{t_0}^t \frac{\partial^2 \varphi}{\partial X^2}(s, S_{s,t}X_s)(S_{s,t}G_s u, S_{s,t}G_s u) ds \\ &+ \int_{t_0}^t \frac{\partial \varphi}{\partial X}(s, S_{s,t}X_s) S_{s,t} Z_s dW_s.\end{aligned}$$

Longtime behavior

Theorem 4.2

Assume that $\sup_{x\in[0,1]} m_1(x) < \frac{1}{2}\inf_{x\in[0,1]} \sigma_1^2(x)$*. For any initial* $(U_0, V_0) \in E$*,* $U_0, V_0 \geq 0$*, one has that*

$$\limsup_{t\to\infty} \mathbb{E} \ln \int_0^1 U(t,x)dx = -\infty.$$

Similarly, if $\sup_{x\in[0,1]} m_2(x) < \frac{1}{2}\inf_{x\in[0,1]} \sigma_2^2(x)$ *then*

$$\limsup_{t\to\infty} \mathbb{E} \ln \int_0^1 V(t,x)dx = -\infty.$$

Further Remarks

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- Strictly positivity, in sense of existence negative moments.
- The regularity of the density
- Uniqueness of the invariant measure
- Provide sufficient but also almost necessary condition for the persistence and extinction. Formally, we expect to introduce a Hypothesis (E) such that under (E),

$$\limsup_{t\to\infty}\sup_{x\in[0,1]} U(t,x)=0, \quad \limsup_{t\to\infty}\sup_{x\in[0,1]} V(t,x)=0,$$

in some sense (almost surely or in expectation or in probability); and a Hypothesis (C) such that under C,

$$\liminf_{t\to\infty}\inf_{x\in[0,1]} U(t,x)>\delta, \quad \liminf_{t\to\infty}\inf_{x\in[0,1]} V(t,x)>\delta,$$

for some positive constant δ (independent of the initial value) in some sense (almost surely or in expectation or in probability). Moreover, the Hypotheses (E) and (C) cover almost all possible cases and only critical cases are left.

Thanks for your attention!