Explicit Solutions of the Extended Skorokhod Problems in Affine Transformations of Time-Dependent Strata

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Outline

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Inspired by paper *An Explicit Formula for the Skorokhod Map on* $[0, a]$
by L. Kruk, J. Lehoczky, K. Ramanan and S. Shreve
Annals of Probability 2007, Vol. 35, No 5, pages 1740-1768.

Skorokhod problem in $\mathbb{R}$

Let $\alpha, \beta \in D\,[0,\infty)$ be such that $\alpha \leq \beta$. Given $\psi \in D[0,\infty)$, a pair of functions $(\phi,\eta) \in D[0,\infty) \times BV[0,\infty)$ is said to be a solution of the Skorokhod problem on $[\alpha,\beta]$ for ψ, if the following two properties are satisfied:

i. For every $t \in [0,\infty) \quad \phi(t) = \psi(t) + \eta(t) \in [\alpha(t),\beta(t)]$;

ii. $\eta\,(0-) = 0$ and η has the decomposition $\eta = \eta_l - \eta_u$, where $\eta_l, \eta_u \in I[0,\infty)$,

$$\int_0^\infty I_{\{\phi(s)>\alpha(t)\}} d\eta_l\,(s) = 0 \text{ and } \int_0^\infty I_{\{\phi(s)<\beta(s)\}} d\eta_u\,(s) = 0$$

Skorokhod Map

The mapping $\psi \to \phi$ from $D\,[0,\infty)$ to $D\,[0,\infty)$ is called the Skorokhod map on $[\alpha,\beta]$ for ψ.
The original Skorokhod map was introduced by Skorokhod in 1961 as a tool for solving stochastic differential equations on the half-line $\mathbb{R}_+$ with a reflecting boundary condition at 0. In other words in the original Skorohod's paper $\alpha = 0$ and $\beta = \infty$.
The explicit representation for the original Skorokhod map is

$$\phi(t) = \psi(t) + \sup_{0\le s\le t} [-\psi(s)]^+, \qquad \psi \in D[0,\infty).$$

An explicit formula for the Skorokhod Map on $[0, a]$

Kruk, Lehoczky, Ramanan and Shreve published in 2007 an explicit formula and studied the properties of the two sided Skorokhod map (reflection map) constraining the process in $D[0, \infty)$ to remain in the interval $[0, a]$, where a is a positive constant. From the applications point of view, it is desirable to allow the reflection boundary to be dependent on time.

Skorokhod Map with time dependent boundaries (1 of 2)

The concept of Skorokhod map with time dependent boundaries have been studied by K. Burdzy, K. Kang and K. Ramanan in *The Skorokhod problem in a time dependent interval* that appeared in Stochastic Processes and their Applications in 2009. They considered a general case, where both the lower and the upper boundaries are time dependent. They also developed an explicit formula for the Skorokhod map with such boundaries. In addition, their analysis includes a more relaxed version of the Skorokhod map called the extended Skorokhod map.

Skorokhod Map with time dependent boundaries (2 of 2)

Somewhat different explicit formulas were developed independently by M. Slaby for the Skorokhod map in "Explicit representation of the Skorokhod map with time dependent boundaries." *Probability and Mathematical Statistics* **2010** and for the extended Skorokhod map in " An explicit representation of the extended Skorokhod map with two time dependent boundaries." *Journal of Probability and Statistics* **2010**. These papers provide also detailed study of the properties of the SM and ESM.

Definition 1 (Extended Skorokhod problem in $\mathbb{R}$)

Let $\alpha, \beta \in D[0,\infty)$ be such that $\alpha \leq \beta$, and let $\psi \in D[0,\infty)$. A pair of real valued càdlàg functions (ϕ, η), is said to be a solution of the extended Skorokhod problem (ESP) on $[\alpha, \beta]$ for ψ, if the following three properties are satisfied:

i. For every $t \geq 0 \quad \phi(t) = \psi(t) + \eta(t) \in [\alpha(t), \beta(t)]$;

ii. For every $0 \leq s \leq t$

$$\eta(t) - \eta(s) \geq 0, \qquad \text{if } \phi(r) < \beta(r) \text{ for all } r \in (s,t],$$

$$\eta(t) - \eta(s) \leq 0, \qquad \text{if } \phi(r) > \alpha(r) \text{ for all } r \in (s,t],$$

iii. For every $t \geq 0$

$$\eta(t) - \eta(t-) \geq 0, \qquad \text{if } \phi(t) < \beta(t),$$

$$\eta(t) - \eta(t-) \leq 0, \qquad \text{if } \phi(t) > \alpha(t).$$

Solution of the extended Skorokhod Problem in $\mathbb{R}$

Let $\alpha, \beta \in D[0,\infty)$ be such that $\alpha \leq \beta$, and let $\psi \in D[0,\infty)$. The evolving ESP for ψ on $[\alpha, \beta]$ has the unique solution (ϕ, η) given by $\eta = -\Xi_{\alpha,\beta}(\psi)$ and $\phi = \psi + \eta$, where

$$\begin{aligned}\Xi_{\alpha,\beta}(\psi)(t) = {} & I_{\left\{\tau^\beta \leq \tau_\alpha\right\}} I_{[\tau^\beta,\infty)}(t) H_{\alpha,\beta}(\psi)(t) \\ & + I_{\left\{\tau_\alpha < \tau^\beta\right\}} I_{[\tau_\alpha,\infty)}(t) L_{\alpha,\beta}(\psi)(t).\end{aligned}$$

$$\tau_\alpha = \inf\left\{t > 0 | \alpha(t) - \psi(t) > 0\right\}, \tau^\beta = \inf\left\{t > 0 | \psi(t) - \beta(t) > 0\right\}$$

$$H_{\alpha,\beta}(\psi)(t) = \sup_{0 \leq s \leq t} [(\psi(s) - \beta(s)) \wedge \inf_{s \leq r \leq t} (\psi(r) - \alpha(r))],$$

$$L_{\alpha,\beta}(\psi)(t) = \inf_{0 \leq s \leq t} [(\psi(s) - \alpha(s)) \vee \sup_{s \leq r \leq t} (\psi(r) - \beta(r))].$$

A stratum and a block in $\mathbb{R}^n$

A closed set G in $\mathbb{R}^n$ will be called a stratum if it admits the following representation

$$G = \left\{(\mathbf{x} : x^i \in [a^i, b^i], i < n, x^n \in [A(x^1, \ldots, x^{n-1}), B(x^1, \ldots, x^{n-1})]\right\},$$

where $a^i \leq b^i$ for $i = 1, 2, \ldots, n-1$ and A, B are two real valued continuous functions on $[a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}]$ such that $A(x) \leq B(x)$ for every x. In such case we will shortly write

$$G = S\left([a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}], [A, B]\right).$$

In the special case when A and B are constant functions G will be called a block. In other words a block is a cross product of n intervals.

Hausdorff distance

We are interested in the constraining domains in $\mathbb{R}^n$ that change with time and so we shall need to introduce the convergence for sets. This will be defined in the sense of the Housdorff metric. For any two sets $G_1, G_2 \subset \mathbb{R}^n$ their Hausdorff distance is defined by

$$d_H(G_1, G_2) = \left(\sup_{x \in G_1} d(x, G_2)\right) \vee \left(\sup_{x \in G_2} d(x, G_1)\right),$$

where $d(x, G) = \inf_{y \in G} \|x - y\|$ and where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.

It is well known that the set of all non-empty compact subsets of $\mathbb{R}^n$ forms a complete metric space with d_H.

Definition 2 (A càdlàg family of strata)

A family $\{G_t : t \geq 0\}$ of closed subsets of $\mathbb{R}^n$ will be called càdlàg if the function $t \mapsto G_t$ is càdlàg with respect to the Hausdorff metric d_H.

To represent càdlàg family of strata we shall use the following notation

$$G_t = S\left([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^{n-1}, \beta_t^{n-1}], [A_t, B_t]\right),$$

where $\alpha_t^i \leq \beta_t^i$ for $i = 1, 2, ..., d$ and $A_t \leq B_t$.

Orthogonal evolving stratum constraining system

A family of pairs $\{(G_t, d_t(\cdot)) : t \geq 0\}$ will be called an orthogonal evolving stratum constraining system if G_t is a stratum for every $t \geq 0$, $\{G_t : t \geq 0\}$ is càdlàg, and

$$d_t(x) = \left\{ \sum_{i \in I_t^+} r^i \mathbf{e}_i - \sum_{i \in I_t^-} r^i \mathbf{e}_i : r_i \geq 0, \text{ for } i \in I_t^+(x) \cup I_t^-(x) \right\},$$

where $I_t^+(x)$
$= \left\{ i : 1 \leq i < n \text{ and } x^i = \alpha_t^i \text{ or } i = n \text{ and } x^n = A\left(x^1, ..., x^{n-1}\right) \right\},$

and $I_t^-(x) =$
$\left\{ i : 1 \leq i < n \text{ and } x^i = \beta_t^i \text{ or } i = n \text{ and } x^n = B\left(x^1, ..., x^{n-1}\right) \right\}.$

Orthogonal evolving block constraining system

In the special case when G_t is a block for every t the orthogonal evolving stratum constraining system will be called an orthogonal evolving block constraining system.

Definition 3 (Solution of ESP on an orthogonal evolving stratum constraining system)

Given an orthogonal evolving stratum constraining system $\{(G_t, d_t(\cdot)) : t \geq 0\}$ and a càdlàg function $\psi \in D_{G_0}([0,\infty), \mathbb{R}^n)$, the pair $(\phi, \eta) \in D_{G_0}([0,\infty), \mathbb{R}^n) \times D_{\{0\}}([0,\infty), \mathbb{R}^n)$ is the solution of the evolving ESP for ψ with respect to $(G_t, d_t(\cdot))$ if the following conditions hold for every $t \geq 0$:

(i) $\phi(t) = \psi(t) + \eta(t)$;

(ii) $\phi(t) \in G_t$;

(iii) $\eta(t) - \eta(s) \in \overline{\text{co}}\left[\bigcup_{u \in (s,t]} d_u(\phi(u))\right]$ for every $s \in [0, t]$;

(iv) $\eta(t) - \eta(t-) \in d_t(\phi(t))$.

Explicit formula for the solutions of ESP on an orthogonal evolving stratum constraining system

Let $\{(G_t, d_t(\cdot)) : t \geq 0\}$ be an orthogonal evolving stratum constraining system with
$G_t = S\left([\alpha_t^1, \beta_t^1] \times \dots [\alpha_t^{n-1}, \beta_t^{n-1}], [A_t, B_t]\right)$. Then, the evolving ESP for any $\psi \in D_{G_0}([0,\infty), \mathbb{R}^n)$ on $(G_t, d(\cdot))$ has a unique solution (ϕ, η) given by
$\eta = \left(-\Xi_{\alpha^1,\beta^1}(\psi^1), -\Xi_{\alpha^2,\beta^2}(\psi^2), ..., -\Xi_{\alpha^n,\beta^n}(\psi^n)\right)$ and $\phi = \psi + \eta$, where

$$\alpha_t^n = A_t\left(\psi^1(t) - \Xi_{\alpha_t^1,\beta_t^1}, \psi^2(t) - \Xi_{\alpha_t^2,\beta_t^2}, ..., \psi^{n-1}(t) - \Xi_{\alpha_t^{n-1},\beta_t^{n-1}}\right),$$

$$\beta_t^n = B_t\left(\psi^1(t) - \Xi_{\alpha_t^1,\beta_t^1}, \psi^2(t) - \Xi_{\alpha_t^2,\beta_t^2}, ..., \psi^{n-1}(t) - \Xi_{\alpha_t^{n-1},\beta_t^{n-1}}\right),$$

and for every $i = 1, 2, \dots, n$

$$\Xi_{\alpha^i,\beta^i}(\psi^i)(t) = I_{\{\tau^{\beta^i} \leq \tau_{\alpha^i}\}} I_{[\tau^{\beta^i},\infty)} H_{\alpha^i,\beta^i}\left(\psi^i\right)(t) + I_{\{\tau^{\alpha^i} < \tau_{\beta^i}\}} I_{[\tau^{\alpha^i},\infty)} L_{\alpha^i,\beta^i}\left(\psi^i\right)(t).$$

Projections in $\mathbb{R}$

The projections $\pi_{a,b} : \mathbb{R} \to [a, b]$ are used to construct the SM or ESM in $\mathbb{R}$. They are defined by

$$\pi_{a,b} = \begin{cases} a, & \text{if } x \leq a; \\ x, & \text{if } a \leq x \leq b; \\ b, & \text{if } x \geq b. \end{cases}$$

Projections in $\mathbb{R}^n$

In the vector valued case we will need similar projections onto blocks and strata. Given a block $D = [a^1, b^1] \times \ldots \times [a^n, b^n]$ we define $\pi_D : \mathbb{R}^n \to D$ by

$$\pi_D(x) = \left(\pi_{a^1,b^1}(x^1), \pi_{a^2,b^2}(x^2), ..., \pi_{a^n,b^n}(x^n)\right).$$

Finally, the projection on a stratum $G = S\left([a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}], [A, B]\right)$ will be defined by

$$\pi_G(x) = \left(\pi_{a^1,b^1}(x^1), ..., \pi_{a^{n-1},b^{n-1}}(x^{n-1}), \pi_{\bar{A}(x^1,...,x^{n-1}),\bar{B}(x^1,...,x^{n-1})}(x^n)\right).$$

Example 1 (ESM for a simple function)

Consider the ESP for a function $\psi \in S\left([0, \infty)], \mathbb{R}^n\right)$ with an orthogonal evolving stratum constraining system $(G_t, d_t(\cdot))$ such that

$$\psi(t) = \psi(t_k) \quad \text{and} \quad G_t = G_{t_k} \quad \text{for every } t \in [t_k, t_{k+1}), k = 0, 1, ..., m,$$

where $0 = t_0 < t_1 < t_2 < ... < t_m < \infty$ and $t_{m+1} = \infty$. Then, the corresponding ESM is the function ϕ such that for $t \in [t_k, t_{k+1}), k = 0, 1, ..., m$

$$\phi(t) = \phi(t_k) = \pi_{G_{t_k}}\left(\phi(t_{k-1}) + \psi(t_k) - \psi(t_{k-1})\right).$$

Expansion from strata to quasistrata

We are ready now to expand our explicit formula from strata to a much bigger class of constraining domains that will be called quasistrata.
Let $E = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be the standard orthonormal basis and $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be any basis of $\mathbb{R}^n$. We will use $(x^1, x^2, \ldots, x^n)_V$ to represent the vector $\mathbf{x} = x^1\mathbf{v}_1 + x^2\mathbf{v}_2 + \ldots + x^n\mathbf{v}_n$ in terms of its coordinates with respect to V. The subscript will be omitted when V is the standard orthonormal basis.

Quasistratum

A closed set G in $\mathbb{R}^n$ will be called a quasi-stratum if there is a basis V such that

$$G = \Big\{(x^1, x^2, \dots, x^n)_V : x^i \in [a^i, b^i] \text{ for } i = 1, 2, \dots, n-1, \\ x^n \in [A(x^1, \dots, x^{n-1}), B(x^1, \dots, x^{n-1})]\Big\},$$

where $a^i \leq b^i$ for $i = 1, 2, \dots, n-1$ and A, B are two real valued continuous functions defined on $[a^1, b^1] \times \dots \times [a^{n-1}, b^{n-1}]$ such that $A(\mathbf{x}) \leq B(\mathbf{x})$ for every $\mathbf{x}$. For short, we will write

$$G = S^V \left([a^1, b^1] \times \dots \times [a^{n-1}, b^{n-1}], [A, B]\right).$$

The superscript will be omitted when V is a standard orthonormal basis. In the special case when A and B are constant functions G will be called a quasi-block.

Quasiblock

In two dimensions a quasi-block is simply a parallelogram, in three dimensions it is a parallelepiped. In general, a quasi-block in $\mathbb{R}^n$ is a parallelotope. This perhaps not quite popular name was introduced by H.S.M. Coxeter in his 1973 book *Regular politopes*. Alternatively, a quasi-block in $\mathbb{R}^n$ can be described as an n-dimensional parallelepiped.

Quasistratum as a linear transformation of a stratum

Note that in the special case, when V is an orthonormal basis the quasi-stratum becomes a stratum and a quasi-block becomes a block.
By T_V we will denote the unique linear transformation $T_V : \mathbb{R}^n \to \mathbb{R}^n$ mapping the standard orthonormal basis onto V, i.e. such that $T_V(\mathbf{e}_i) = \mathbf{v}_i$ for $i = 1, 2, \ldots, n$. Then

$$S^V \left([a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}], [A, B]\right) = T_V(G),$$

where $G = S\left([a^1, b^1] \times \ldots \times [a^{n-1}, b^{n-1}], [A, B]\right)$. Note that T_V can be represented by a matrix whose columns are $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$.
Any invertible affine transformation of $\mathbb{R}^n$ can be represented as a composition of a translation with T_V for some basis V.

Evolving quasi-stratum constraining system

A family of pairs $\{(G_t, d_t(\cdot)) : t \geq 0\}$ will be called an evolving quasi-stratum constraining system if there is a basis V such that $G_t = S^V\left([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^{n-1}, \beta_t^{n-1}], [A_t, B_t]\right)$, $\{G_t : t \geq 0\}$ is càdlàg with respect to the Hausdorff distance between constraining sets, and d_t satisfies the following conditions.

For any $\mathbf{x} = (x^1, x^2, \ldots, x^n)_V$ on the boundary of G_t

$$d_t(\mathbf{x}) = d_t^V(\mathbf{x}) = \left\{ \sum_{i \in I_t^V} r^i \mathbf{v}_i - \sum_{i \in J_t^V} r^i \mathbf{v}_i : r^i \geq 0, \text{ for } i \in I_t^V(\mathbf{x}) \cup J_t^V(\mathbf{x}) \right\},$$

where $I_t^V(\mathbf{x}) =$

$$\left\{ i : 1 \leq i < n \text{ and } x^i = \alpha_t^i \text{ or } i = n \text{ and } x^n = A\left(x^1, x^2, ..., x^{n-1}\right) \right\},$$

and $J_t^V(\mathbf{x}) =$

$$\left\{ i : 1 \leq i < n \text{ and } x^i = \beta_t^i \text{ or } i = n \text{ and } x^n = B\left(x^1, x^2, ..., x^{n-1}\right) \right\}.$$

Finally, $d_t(\mathbf{x}) = \mathbf{0}$ for any $\mathbf{x}$ in the interior of G_t.

Evolving quasi-block constraining system

In the special case when G_t is a quasi-block for every $t \geq 0$, the evolving quasi-stratum constraining system will be called an evolving quasi-block constraining system.

Remark 1

Let $\{(G_t, d_t(\cdot)) : t \geq 0\}$ be an orthogonal evolving stratum constraining system with $G_t = S\left([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^{n-1}, \beta_t^{n-1}], [A_t, B_t]\right)$, let V be any basis and let $d_t^V(\mathbf{x}) = T_V\left(d_t\left(T_V^{-1}(\mathbf{x})\right)\right)$. Then $\left\{\left(T_V G_t, d_t^V\right) : t > 0\right\}$ is a quasi-stratum constraining system. In particular,

$$T_V G_t = S^V\left([\alpha_t^1, \beta_t^1] \times \ldots \times [\alpha_t^{n-1}, \beta_t^{n-1}], [A_t, B_t]\right)$$

and

$$d_t^V(\mathbf{x}) = \left\{\sum_{i \in I_t^V} r^i \mathbf{v}_i - \sum_{i \in J_t^V} r^i \mathbf{v}_i : r^i \geq 0, \text{ for } i \in I_t^V(\mathbf{x}) \cup J_t^V(\mathbf{x})\right\}.$$

Proposition 1 (How solutions of ESP are affected by affine mappings of $\mathbb{R}^n$)

Let $\{(G_t, d_t(\cdot)) : t \geq 0\}$ be an orthogonal evolving stratum constraining system, let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be an invertible affine transformation and let $T_0 = T - T(\mathbf{0})$ be its linear transformation component. For any $\psi \in D_{G_0}([0, \infty), \mathbb{R}^n)$, if (ϕ, η) is a solution of ESP for ψ with respect to $\{(G_t, d_t(\cdot)) : t \geq 0\}$, then $(T\phi, T_0\eta)$ is the unique solution of ESP for $T\psi$ with respect to $\left\{(TG_t, d_t^V(\cdot)) : t \geq 0\right\}$, where $V = T_0(E)$ and E is the standard orthonormal basis.

Idea: expanding the results for the orthogonal constraining systems via affine transformations

The above result suggests that through the use of affine transformations the orthogonal evolving constraining systems can generate much larger class of constraining systems. Moreover, the affine transformation provides the link between the solutions of ESP with respect to the image constraining system and the solutions of ESP with respect to the original orthogonal constraining system.

Constraining system generated by an orthogonal constraining system

A time-dependent constraining system $\left\{(\tilde{G}_t, \tilde{d}_t(\cdot)) : t \geq 0\right\}$ in $\mathbb{R}^n$ is generated by an orthogonal constraining system, if there is an orthogonal evolving stratum constraining system $\{(G_t, d_t(\cdot)) : t \geq 0\}$ and an affine mapping $T : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that, for every $\psi \in D_{G_0}([0,\infty), \mathbb{R}^n)$, if (ϕ, η) is the solution of ESP for ψ with respect to $\{(G_t, d_t(\cdot)) : t \geq 0\}$ then $(T \circ \phi, T_0 \circ \eta)$ is the solution of ESP for $T \circ \psi$ with respect to $\left\{\left(T(G_t), T_0\left(d_t\left(T_0^{-1}(\cdot)\right)\right)\right) : t \geq 0\right\}$. Such a mapping will be referred to as preserving the solutions of ESP.

Proposition 2 (Every quasi-stratum constraining system is generated by an orthogonal constraining system)

Let $\left\{\left(\tilde{G}_t, \tilde{d}_t\right) : \ t \geq 0\right\}$ be an evolving quasi-stratum constraining system in $\mathbb{R}^n$, let V be the associated basis as described in the definition, and let $T_V : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be the linear mapping such that $T_V\left(\mathbf{e}_i\right) = \mathbf{v}_i$ for $i = 1, 2, \ldots, n$. Then $\left\{\left(\tilde{G}_t, \tilde{d}_t\right) : \ t \geq 0\right\}$ is generated by an orthogonal evolving stratum constraining system and T_V is preserving the solutions of the ESP.

Theorem 1 (Explicit solutions of ESP on quasi-stratum constraining systems) Page 1 of 2

Let $\left(\tilde{G},\tilde{d}\right)$ be an evolving quasi-stratum constraining system in $\mathbb{R}^n$, i.e. there is a basis $V=\{\mathbf{v}_1,\mathbf{v}_1,\ldots,\mathbf{v}_n\}$, such that $\tilde{G}_t = S^V\left([\alpha_t^1,\beta_t^1]\times\ldots\times[\alpha_t^{n-1},\beta_t^{n-1}],[A_t,B_t]\right)$ and $\tilde{d}_t=d_t^V$, for every $t\geq 0$. Then for any $\tilde{\psi}\in D_{\tilde{G}_0}\left([0,\infty),\mathbb{R}^n\right)$ the evolving ESP on $\left(\tilde{G},\tilde{d}\right)$ has the unique solution $\left(\tilde{\phi},\tilde{\eta}\right)$ given by

$$\tilde{\eta}=T_V\left(-\Xi_{\alpha^1,\beta^1}(\psi^1),-\Xi_{\alpha^2,\beta^2}(\psi^2),...,-\Xi_{\alpha^n,\beta^n}(\psi^n)\right)\text{ and }\tilde{\phi}=\tilde{\psi}+\tilde{\eta},$$

where $T_V:\mathbb{R}^n\longrightarrow\mathbb{R}^n$ is the linear transformation defined by $T_V\left(\mathbf{e}_i\right)=\mathbf{v}_i$ for every $i=1,2,\ldots,n$,

$$\psi(t)=\left(\psi^1(t),\psi^2(t),\ldots,\psi^n(t)\right)=\left(T_V\right)^{-1}\left(\left(\tilde{\psi}^1(t),\tilde{\psi}^2(t),\ldots,\tilde{\psi}^n(t)\right)_V\right.$$

Theorem 1 (Explicit solutions of ESP on quasi-stratum constraining systems) Page 2 of 2

$$\alpha_t^n = A_t\left(\psi^1(t) - \Xi_{\alpha_t^1,\beta_t^1}, \psi^2(t) - \Xi_{\alpha_t^2,\beta_t^2}, ..., \psi^{n-1}(t) - \Xi_{\alpha_t^{n-1},\beta_t^{n-1}}\right),$$

$$\beta_t^n = B_t\left(\psi^1(t) - \Xi_{\alpha_t^1,\beta_t^1}, \psi^2(t) - \Xi_{\alpha_t^2,\beta_t^2}, ..., \psi^{n-1}(t) - \Xi_{\alpha_t^{n-1},\beta_t^{n-1}}\right).$$

In the above, for every $i = 1, 2, \ldots, n$,

$$\Xi_{\alpha^i,\beta^i}(\psi^i)(t) = I_{\{\tau^{\beta^i} \le \tau_{\alpha^i}\}} I_{[\tau^{\beta^i},\infty)} H_{\alpha^i,\beta^i}\left(\psi^i\right)(t) + I_{\{\tau_{\alpha^i} < \tau^{\beta^i}\}} I_{[\tau_{\alpha^i},\infty)} L_{\alpha^i,\beta^i}\big($$

where $\tau_{\alpha^i} = \inf\left\{t > 0 \mid \alpha^i(t) - \psi^i(t) > 0\right\}$,
$\tau^{\beta^i} = \inf\left\{t > 0 \mid \psi^i(t) - \beta^i(t) > 0\right\}$, while

$$H_{\alpha,\beta}(\psi)(t) = \sup_{0\le s\le t}[(\psi(s) - \beta(s)) \wedge \inf_{s\le r\le t}(\psi(r) - \alpha(r))],$$

$$L_{\alpha,\beta}(\psi)(t) = \inf_{0\le s\le t}[(\psi(s) - \alpha(s)) \vee \sup_{s\le r\le t}(\psi(r) - \beta(r))].$$

Continuity and Lipschitz Conditions - constant C_V

Let Λ be the set of all permutations $\lambda : \{1, 2, \ldots, n\} \longrightarrow \{1, 2, \ldots, n\}$. Given any sequence of n independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and any $\lambda \in \Lambda$ we define

$$C_k^{\lambda}(V) = \tan\left(\frac{1}{2}\angle\left(\sum_{i=1}^{k}\mathbf{v}_{\lambda(i)}, \sum_{i=k+1}^{n}\mathbf{v}_{\lambda(i)}\right)\right) \text{ for } k = 1, 2, \ldots, n-1,$$

$$C_n^{\lambda}(V) = 1.$$

Finally, we set

$$C_V = \max_{1 \leq k \leq n} \max_{\lambda \in \Lambda} C_k^{\lambda}(V).$$

Continuity and Lipschitz Conditions - constant K_V

Given a sequence of n independent vectors $V = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, let V_j denote the linear subspace spanned by vectors $V \setminus \{\mathbf{v}_j\}$, let $K_j = \sin \angle (\mathbf{v}_j, V_j)$ and let $K_V = \min_{1 \le j \le n} K_j$. It will be also convenient to use the following notations: $\bar{\mathbf{v}} = \frac{\mathbf{v}}{\|\mathbf{v}\|}$ and $\bar{V} = \{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \ldots, \bar{\mathbf{v}}_n\}$. It is important to notice two things. First, for every $j = 1, 2, \ldots, n$

$$K_V \le K_j = \sin\left(\angle (\mathbf{v}_j, V_j)\right) \le \sin\left(\angle\left(\mathbf{v}_j, \sum_{i=1}^{n} \mathbf{v}_i - \mathbf{v}_j\right)\right).$$

Second, K_V depends only on the directions of vectors in V and not on their magnitudes, i.e.

$$K_V = K_{\bar{V}}.$$

Theorem 2 (Continuity and Lipschitz Conditions)

Let $\left(\tilde{G}, \tilde{d}\right)$ be an evolving quasi-block constraining system in $\mathbb{R}^n$ that can be represented as an image of an orthogonal evolving block constraining system via an invertible linear transformation T and let V be the image of the orthonormal basis through T. If $\left(\tilde{\phi}_1, \tilde{\eta}_1\right)$ and $\left(\tilde{\phi}_2, \tilde{\eta}_2\right)$ are the solutions of the ESP for $\tilde{\psi}_1$ and $\tilde{\psi}_2$ with respect to $\left(\tilde{G}, \tilde{d}\right)$ then the following Lipschitz conditions hold

$$\|\tilde{\eta}_1 - \tilde{\eta}_2\| \leq L_V \cdot \|\tilde{\psi}_1 - \tilde{\psi}_2\|,$$

$$\|\tilde{\phi}_1 - \tilde{\phi}_2\| \leq (1 + L_V)\, \|\tilde{\psi}_1 - \tilde{\psi}_2\|,$$

where

$$L_V = \frac{C_{\bar{V}}}{K_{\bar{V}}} \left\| \sum_{i=1}^{n} \bar{\mathbf{v}}_i \right\|.$$

Remark 2

The Lipschitz constants of Theorem 2 depend only on the angles defining the shape of the quasi-block and not on its size.

Remark 3

In the special case of an orthogonal block constraining system, all the relevant angles are right angles and therefore $C_{\bar{V}} = 1$ and $K_{\bar{V}} = 1$. Thus, the Lipschitz constant in Theorem 2 becomes $\| \sum_{i=1}^{n} \bar{\mathbf{v}}_i \| = \sqrt{n}$ matching the result of Proposition 3.1 in 2013 paper "Explicit Solutions of the extended Skorokhod problems in time-dependednt bounded regions with orthogonal reflection fields",*Probability and Mathematical Statistics* by M. Slaby.

The following example will show that the Lipschitz constant L_V in Theorem 2 is tight in $\mathbb{R}^2$. Essentially, it will demonstrate that for any quasi-block constraining system in $\mathbb{R}^2$ there are functions ψ_1 and ψ_2 such that $\|\eta_1 - \eta_2\| = L_V \|\psi_1 - \psi_2\|$.

Example 2 (1 of 2)

Let $\left(\tilde{G}, \tilde{d}\right)$ be an arbitrary non-evolving quasi-block constraining system in $\mathbb{R}^2$. Then $\tilde{G}$ is a parallelogram. Let α be the obtuse angle in $\tilde{G}$ and let $\bar{\alpha} = \pi - \alpha$. We can assume without the loss of generality that $\left(\tilde{G}, \tilde{d}\right)$ is generated by vectors $\mathbf{v}_1 = a\mathbf{e}_1$ and $\mathbf{v}_2 = b\left(\cos\alpha\mathbf{e}_1 + \sin\alpha\mathbf{e}_2\right)$. More specifically, we assume that $\tilde{G}$ has vertices at $\mathbf{0}$, $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_1 + \mathbf{v}_2$. Then
$C_{\bar{V}} = \max\left\{\tan\frac{\alpha}{2}, 1\right\} = \tan\frac{\alpha}{2} = \cot\frac{\bar{\alpha}}{2}$,
$K_{\bar{V}} = \sin\alpha = \sin\bar{\alpha} = 2\sin\frac{\bar{\alpha}}{2}\cos\frac{\bar{\alpha}}{2}$ and
$\|\bar{\mathbf{v}}_1 + \bar{\mathbf{v}}_2\| = 2\cos\frac{\alpha}{2} = 2\sin\frac{\bar{\alpha}}{2}$ and so $L_V = \csc\frac{\bar{\alpha}}{2}$.
Let $\tilde{\psi}_1 = -r\left(\cot\alpha\bar{\mathbf{v}}_1 I_{[0,1)} + \csc\alpha\bar{\mathbf{v}}_1 I_{[1,\infty)}\right)$ and
$\tilde{\psi}_2 = -r\left(\csc\alpha\bar{\mathbf{v}}_2 I_{[0,1)} + \cot\alpha\bar{\mathbf{v}}_2 I_{[1,\infty)}\right)$, where
$r < \min\left\{-a\tan\alpha, b\cot\frac{\alpha}{2}\right\}$. Then, using projections as in Example 1, we can evaluate ϕ_1 and ϕ_2.

Example 2 (2 of 2)

$\tilde{\phi}_1(0) = \pi_{\tilde{G}}\left(\tilde{\psi}_1(0)\right) = \tilde{\psi}_1(0) = -r\cot\alpha\bar{\mathbf{v}}_1$. On the other hand $\tilde{\phi}_2(0) = \pi_{\tilde{G}}(\psi_2(0)) = \mathbf{0}$. Therefore $\tilde{\eta}_1(0) = \tilde{\phi}_1(0) - \tilde{\psi}_1(0) = \mathbf{0}$ and $\tilde{\eta}_2(0) = \tilde{\phi}_2(0) - \tilde{\psi}_2(0) = r\csc\alpha\bar{\mathbf{v}}_2$.

$\tilde{\phi}_1(1) = \mathbf{0}$ and $\tilde{\eta}_1(1) = r\csc\alpha\bar{\mathbf{v}}_1$.

$\tilde{\phi}_2(1) = r\tan\frac{\alpha}{2}\bar{\mathbf{v}}_2$ and $\tilde{\eta}_2(1) = r\csc\alpha\bar{\mathbf{v}}_2$.

Now, $\|\tilde{\eta}_1(0) - \tilde{\eta}_2(0)\| = \|r\csc\alpha\bar{\mathbf{v}}_2\| = r\csc\alpha$ and $\|\tilde{\eta}_1(1) - \tilde{\eta}_2(1)\| = r\|\left(\tan\frac{\alpha}{2}, -1\right)\| = r\sec\frac{\alpha}{2} = r\csc\frac{\bar{\alpha}}{2}$. Thus $\|\tilde{\eta}_1 - \tilde{\eta}_2\| = r\max\left\{\csc\alpha, \csc\frac{\bar{\alpha}}{2}\right\} = r\csc\frac{\bar{\alpha}}{2}$. On the other hand, $\|\tilde{\psi}_1(0) - \tilde{\psi}_2(0)\| = r$ and $\|\tilde{\psi}_1(1) - \tilde{\psi}_2(1)\| = r$. Therefore $\|\tilde{\psi}_1 - \tilde{\psi}_2\| = r$ and so $\|\tilde{\eta}_1 - \tilde{\eta}_2\| = \csc\frac{\bar{\alpha}}{2}\|\tilde{\psi}_1 - \tilde{\psi}_2\|$.

In other words, in this case, $\|\tilde{\eta}_1 - \tilde{\eta}_2\| = L_V\|\tilde{\psi}_1 - \tilde{\psi}_2\|$.

Because $\tilde{G}$ represents an arbitrary quasi-block constraining system in $\mathbb{R}^2$, L_V is a tight Lipschitz constant in $\mathbb{R}^2$.

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Thank you!