Linear Multifractional Stable Sheets in the Broad Sense: Existence and Joint Continuity of Local Times

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Outline

$(N,1)$-LMSS

LMSS: Linear multifractional stable sheets in the broad sense

- Extends multifractional Brownian sheets and linear fractional stable sheets
- Broad sense: $\alpha \in (0,2]$
  - Multifractional Brownian sheets (Meerschaert, Wu and Xiao, 2008): $\alpha = 2$
  - Linear multifractional stable sheets (Shen, Yu and Li, 2020): $\alpha \in (0,2)$

$(N,1)$-LMSS is defined by

$$X_0^{H(u)}(u) := c \int_{\mathbb{R}^N} \prod_{l=1}^{N} \left[(u_l - v_l)_+^{h_l(u)-1/\alpha} - (-v_l)_+^{h_l(u)-1/\alpha} \right] M_\alpha(\mathrm{d}v), \quad u \in \mathbb{R}_+^N$$

- $\alpha \in (0,2]$
- Hurst index: $H(u) = (h_1(u), \ldots, h_N(u))$
  - $0 < m_l < h_l(u) < M_l < 1$
  - $|h_l(u) - h_l(v)| \leq c' \sum_{l=1}^{N} \min\left\{ |u_l - v_l|^{m_l}, |u_l - v_l|^{M_l} \right\}, u, v \in \mathbb{R}_+^N$
- M_α is symmetric α-stable random measure with Lebesgue control measure
- c is chosen such that $\|X_0^{H(1)}(1)\|_\alpha = 1$.

(N, d)-LMSS

(N, d)-LMSS is defined by

$$X^{H(u)}(u) = \left(X_1^{H(u)}(u), \ldots, X_d^{H(u)}(u)\right)$$

whose components are independent copies of $(N, 1)$-LMSS $X_0^{H(\bullet)}$.

Special cases of (N, d)-LMSS:

References	α	Hurst index
Fractional Brownian sheets (Xiao and Zhang, 2002)	2	Constant
Linear fractional stable sheets (Ayache, Roueff and Xiao, 2007)	$(0, 2]$	Constant
Multifractional Brownian sheets (Meerschaert, Wu and Xiao, 2008)	2	Functional
Linear multifractional stable sheets (Shen, Yu and Li, 2020)	$(0, 2)$	Functional
(N, d)-LMSS (Ding, Peng and Xiao, 2021)	$(0, 2]$	Functional

Local Times

Occupation measure:

- Amount of time a process spends during a time period
- Occupation measure of Y on $\mathcal{I}$: $\mu_{\mathcal{I}}(\bullet) = \lambda_N\{t \in \mathcal{I} : Y(t) \in \bullet\}$
  - $Y : \mathbb{R}^N \to \mathbb{R}^d$ is a Borel vector field
  - $\mathcal{I} \subset \mathbb{R}^N$ is any Borel set
  - λ_N is the Lebesgue measure on $\mathbb{R}^N$

Local time:

- Amount of time a process spends at a level
- Local times $L(\bullet, \mathcal{I})$ of Y on $\mathcal{I}$ is the Radon-Nikodým derivation of $\mu_{\mathcal{I}}$ with respect to λ_d
  - $L(x, \mathcal{I}) = \frac{\mathrm{d}\mu_{\mathcal{I}}}{\mathrm{d}\lambda_d}(x), \quad$ for a.e. $\quad x \in \mathbb{R}^d$
  - $\mu_{\mathcal{I}}$ is absolutely continuous with respect to the Lebesgue measure λ_d in $\mathbb{R}^d$

Main Contributions

Result 1: Sufficient & necessary condition for existence of local times of LMSS

- First time for multifractional processes
- Existence of local times in $\mathcal{L}^2(\lambda_d)$: $\mathbb{E}(L(x,\mathcal{I})^2) < \infty$

Result 2: Sufficient condition for joint continuity of local times of LMSS

- Improve results in the literature via a weaker condition
- Joint continuity of local times proved by estimating the p-th moments: for any $a \in \mathcal{I}$ there exist $c_1, c_2, \beta_1, \beta_2, \beta_3 > 0$ such that for any small δ_1, δ_2,
  - $\mathbb{E}\,|L(x,[a,a+\delta_1])|^p \le c_1\delta_1^{\beta_1}$
  - $\mathbb{E}\,|L(x,[a,a+\delta_1]) - L(x+\delta_2,[a,a+\delta_1])|^p \le c_2\delta_1^{\beta_2}\delta_2^{\beta_3}$

Result 3: Local Hölder condition for local times of LMSS

- Improve results in the literature via a weaker condition
- Local Hölder condition: there exists $c > 0$ and a function ϕ_t such that with probability 1,

$$\limsup_{r\to 0} \frac{L(x,U(t,r))}{\phi_t(r)} \le c, \text{ for almost all } t \in \mathcal{I},$$

where $U(t,r)$ is the open ball in $\mathcal{I}$, with center t and radius r.

Main Result 1:
Sufficient & Necessary Condition for Existence of Local Times of LMSS

(N, d)-LMSS admits an $L^2(\lambda_d)$-integrable local time almost surely if and only if

$$0 < \int_{\mathcal{I}} \left(\sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \mathrm{d}v < \infty \qquad (1)$$

- $\alpha \in (0, 2]$
- $\mathcal{I} = [\epsilon, T]^N$, $0 < \epsilon < T$

Sketch of Proof:

Existence of local times

$\Longleftrightarrow$ Existence of the Fourier transform of the occupation measure of LMSS

$\Longleftrightarrow$ $\mathcal{J}(\mathcal{I}) = \int_{\mathcal{I}} \int_{\mathcal{I}} \int_{\mathbb{R}^d} \mathbb{E} \exp(i\langle \xi, X^{H(u)}(u) - X^{H(v)}(v) \rangle) \, \mathrm{d}\xi \, \mathrm{d}u \, \mathrm{d}v < \infty$

$\Longleftrightarrow$ $\int_{\mathcal{I}} \int_{\mathcal{I}} \frac{\mathrm{d}v \, \mathrm{d}u}{\left(\sum_{l=1}^{N} |u_l - v_l|^{h_l(v)} \right)^d} < \infty$

$\Longleftrightarrow$ (1) holds

Main Result 1:
Sufficient & Necessary Condition for Existence of Local Times of LMSS

$$0 < \int_{\mathcal{I}} \left(\sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \mathrm{d}v < \infty$$

$$\Updownarrow$$

$$\begin{cases} \mathcal{C}_1 : d < \inf\limits_{v\in\mathcal{I}} \sum\limits_{l=1}^{N} \frac{1}{h_l(v)} \\ \text{or} \\ \mathcal{C}_2 : d = \inf\limits_{v\in\mathcal{I}} \sum\limits_{l=1}^{N} \frac{1}{h_l(v)}, \quad \int_{\mathcal{I}} \left(\sum\limits_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \mathrm{d}v < \infty \end{cases}$$

Example 1: Let $N = 1$, $\mathcal{I} = [1/3, 1/2]$ and $h_1(v) = \frac{1}{2} - \sqrt{v - \frac{1}{3}}$.

Then if $d = \inf_{v\in\mathcal{I}} 1/h_1(v) = 2$,

we have $\int_{1/3}^{1/2} \left(\frac{1}{h(v)} - d \right)^{-1} \mathrm{d}v \le h(1/3) \int_{1/3}^{1/2} \frac{1}{\sqrt{v-1/3}} \mathrm{d}v = \frac{1}{\sqrt{6}}$.

Therefore $d \le 2$ is sufficient & necessary for the local times to exist.

Example 2: When the Hurst index is constant (e.g. Linear fractional stable sheets), $\mathcal{C}_2$ does not hold and $\mathcal{C}_1$ becomes $d < \sum_{l=1}^{N} 1/h_l$. Therefore $d < \sum_{l=1}^{N} 1/h_l$ is sufficient & necessary for the local times to exist.

Main Result 1: Comparison to Existing Results on Existence of Local Times

References	α	Type	Condition
Fractional Brownian sheets Xiao and Zhang (2002)	2	Sufficient	$d < \sum_{l=1}^{N} \frac{1}{h_l}$
Linear fractional stable sheets Ayache, Roueff and Xiao (2007)	$(0, 2]$	Sufficient & necessary	$d < \sum_{l=1}^{N} \frac{1}{h_l}$
Multifractional Brownian sheets Meerschaert, Wu and Xiao (2008)	2	Sufficient	$d < \sum_{l=1}^{N} \frac{1}{h_l(v)}$ for all $v \in \mathcal{I}$
Linear multifractional stable sheets Shen, Yu and Li (2020)	$(0, 2)$	Sufficient	$d < \sum_{l=1}^{N} \frac{1}{\sup_{v\in\mathcal{I}} h_l(v)}$
(N, d)-LMSS Ding, Peng and Xiao (2021)	$(0, 2]$	Sufficient & necessary	$\mathcal{C}_1 : d < \inf_{v\in\mathcal{I}} \sum_{l=1}^{N} \frac{1}{h_l(v)}$ or $\mathcal{C}_2 : \begin{cases} d = \inf_{v\in\mathcal{I}} \sum_{l=1}^{N} \frac{1}{h_l(v)} \\ \int_{\mathcal{I}} \left(\sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \mathrm{d}v < \infty \end{cases}$

Main Result 1:
Existence of Local Times of Multifractional Brownian Sheets

Multifractional Brownian sheets Meerschaert, Wu and Xiao (2008)	$\alpha = 2$	Sufficient	$d < \sum_{l=1}^{N} \frac{1}{h_l(v)}$ for all $v \in \mathcal{I}$
(N, d)-LMSS Ding, Peng and Xiao (2021)	$\alpha \in (0, 2]$	Sufficient & necessary	$\mathcal{C}_1 : d < \inf_{v \in \mathcal{I}} \sum_{l=1}^{N} \frac{1}{h_l(v)}$ or $\mathcal{C}_2 : \begin{cases} d = \inf_{v \in \mathcal{I}} \sum_{l=1}^{N} \frac{1}{h_l(v)} \\ \int_{\mathcal{I}} \left(\sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \mathrm{d}v < \infty \end{cases}$

In Meerschaert, Wu and Xiao (2008):

• "If $d > \sum_{l=1}^{N} \frac{1}{h_l(t)}$ for some $t \in \mathcal{I}$ then it can be proved using Theorem 21.9 of Geman and Horowitz (1980) that multifractional Brownian sheet has no $L^2(\lambda_d)$-integrable local times on $\mathcal{I}$."

Verified by $\mathcal{C}_1$ and $\mathcal{C}_2$

• "If $d = \sum_{l=1}^{N} \frac{1}{h_l(t)}$ for some $t \in \mathcal{I}$, the equality only holds for t in a set of Lebesgue measure 0, the existence of local times is rather subtle, and requires imposing further assumption on $(h_1(t), \ldots, h_N(t))$. Hence it will not be discussed here."

According to $\mathcal{C}_2$, the "further assumption on $(h_1(t), \ldots, h_N(t))$ " could be $\int_{\mathcal{I}} \left(\sum_{l=1}^{N} \frac{1}{h_l(v)} - d \right)^{-1} \mathrm{d}v < \infty$, this means $v \mapsto \sum_{l=1}^{N} \frac{1}{h_l(v)}$ reaches its infimum in $\mathcal{I}$ "slowly".

Main Result 2: Sufficient Condition for Joint Continuity of Local Times of LMSS

(N, d)-LMSS $X^{H(\bullet)}$ has a jointly continuous local time on $\mathcal{I} = [\epsilon, T]^N$ if $d < \inf_{v \in \mathcal{I}} \sum_{l=1}^{N} \frac{1}{h_l(v)}$.

Sketch of Proof: Assume $d < \inf_{v \in \mathcal{I}} \sum_{l=1}^{N} 1/h_l(v)$, there exists $c_1, c_2 > 0$ such that

1. $\mathbb{E}[L(x, I_{a,\delta})^n] \leq c_1 \, (n!)^{N-\overline{\beta}} \, \delta^{n\overline{\beta}}, \quad x \in \mathbb{R}^d, n \geq 1$

2. For $x, y \in \mathbb{R}^d, |x-y| \leq 1,$ even integer $n \geq 2,$

$$\mathbb{E}\left[(L(x, I_{a,\delta}) - L(y, I_{a,\delta}))^n\right] \leq c_2 \inf_{v \in I_{a,\delta}, \sigma \in \mathcal{S}(N)} \left\{ |x-y|^{n\kappa(\sigma(H(v)))} \delta^{n(\beta(\sigma(H(v)) - (n-1) h_{\gamma(\sigma(H(v)))}(v) \kappa(\sigma(H(v))))} \right\}$$

- For any $I_{a,\delta} = \prod_{l=1}^{N} [a_l, a_l + \delta] \subset \mathcal{I}$ with $\delta \in (0, 1]$ sufficiently small
- $\overline{\beta} = \sup_{v \in I_{a,\delta}, \sigma \in \mathcal{S}(N)} \beta(\sigma(H(v)))$ with $\beta(\sigma(H(v))) := N - \gamma(\sigma(H(v))) + h_{\gamma(\sigma(H(v)))}(v) \left(\sum_{l=1}^{\gamma(\sigma(H(v)))} \frac{1}{h_{\sigma(l)}(v)} - d \right)$
- $\mathcal{S}(N)$ is the group of permutations of $\{1, \ldots, N\}$
- For each $\sigma \in \mathcal{S}(N)$, let $\sigma(H(v)) := \left(h_{\sigma(1)}(v), \ldots, h_{\sigma(N)}(v) \right), \quad v \in \mathcal{I}$
- $\gamma(H(v)) := \min \left\{ m \in \{1, \ldots, N\} : \; d < \sum_{l=1}^{m} \frac{1}{h_l(v)} \right\}$
- For each $v \in I_{a,\delta}$, $\kappa(H(v))$ is real such that $n\kappa(H(v)) \in \left(0, 1 \wedge \frac{\alpha(H(v))}{2\gamma(H(v))} \right)$ with $\alpha(H(v)) := \sum_{l=1}^{\gamma(H(v))} \frac{1}{h_l(v)} - d$

Main Result 2: Comparison to Existing Results on Joint Continuity of Local Times

References	α	Type	Condition
Fractional Brownian sheets Xiao and Zhang (2002)	2	Sufficient	$d < \min\left\{\frac{1}{h_l}; l \in \{1, \ldots, N\}\right\}$
Linear fractional stable sheets Ayache, Roueff and Xiao (2007)	$\left(\frac{1}{\min\limits_{l\in\{1,\ldots,N\}} h_l}, 2\right]$	Sufficient	$d < \sum\limits_{l=1}^{N} \frac{1}{h_l}$
Multifractional Brownian sheets Meerschaert, Wu and Xiao (2008)	2	Sufficient	$d < \sum\limits_{l=1}^{N} \frac{1}{h_l(v)}$ for all $v \in \mathcal{I}$
Linear multifractional stable sheets Shen, Yu and Li (2020)	$(0, 2)$	Sufficient	$d < \sum\limits_{l=1}^{N} \frac{1}{\sup\limits_{v\in\mathcal{I}} h_l(v)}$
(N, d)-LMSS Ding, Peng and Xiao (2021)	$(0, 2]$	Sufficient	$d < \inf\limits_{v\in\mathcal{I}} \sum\limits_{l=1}^{N} \frac{1}{h_l(v)}$

Main Result 3: Local Hölder Condition for Local Times of LMSS

Assume $d < \inf_{v\in\mathcal{I}} \sum_{l=1}^{N} 1/h_l(v)$, there exists $c > 0$ such that with probability 1,

$$\limsup_{r\to 0} \frac{L(x, U(t,r))}{\varphi_t(r)} \leq c, \quad \text{for } L(x,\bullet)\text{-almost all } t \in \mathcal{I}$$

- $\alpha \in (0, 2]$
- $\mathcal{I} = [\epsilon, T]^N$
- $U(t, r)$ is the open ball with center t and radius r
- Scaling function: $\varphi_t(r) := r^{\beta(H(t))}\big(\log(\log(r^{-1}))\big)^{N-\beta(H(t))}, \quad \text{for} \quad r < e^{-1}$
- $\beta(H(t)) := N - \gamma(H(t)) + h_{\gamma(H(t))}(v)\left(\sum_{l=1}^{\gamma(H(t))} \frac{1}{h_l(v)} - d\right)$

References

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