Propagation of Singularities of the Linear Stochastic Wave Equation

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Outline

1. Linear stochastic wave equation
2. The simultaneous law of the iterated logarithm
3. Propagation of singularities

9.1. Linear Stochastic Wave Equation

We consider the linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \geq 0,\ x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t} u(0,x) = 0. \end{cases} \tag{1}$$

Here, $\dot{W}$ is a Gaussian noise that is white in time and has a spatially homogeneous covariance given by the Riesz kernel with exponent $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)|x-y|^{-\beta}.$$

If $k = 1 = \beta$, then $\dot{W}$ is the space-time Gaussian white noise.

Dalang (1999) proved that the real-valued process solution of equation (1) is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy), \tag{2}$$

where G is the fundamental solution of the wave equation and W is the martingale measure induced by the noise $\dot{W}$. We only consider the case of $k = 1$. Hence $0 < \beta \le 1$ and

$$G(t,x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}}.$$

The mild solution of (1) is

$$u(t,x) = \frac{1}{2}\int_0^t \int_{\mathbb{R}} \mathbf{1}_{\{|x-y|\le t-s\}}(s,y)\, W(ds\, dy) = \frac{1}{2} W(\Delta(t,x)), \tag{3}$$

where $\Delta(t,x) = \{(s,y) \in \mathbb{R}_+ \times \mathbb{R} : 0 \le s \le t,\ |x-y| \le t-s\}$, see Figure 1.

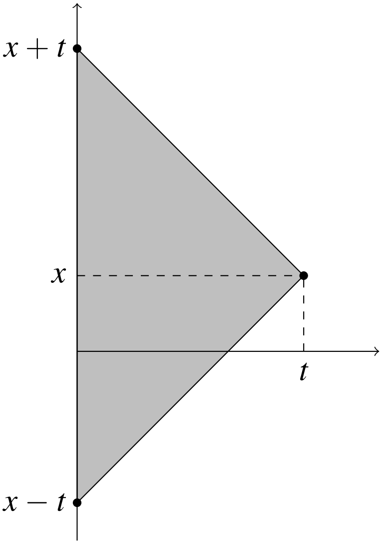

Figure: 1

Consider a new coordinate system (τ, λ) obtained by rotating the (t, x)-coordinates by -45°. In other words,

$$(\tau, \lambda) = \Big(\frac{t - x}{\sqrt{2}}, \frac{t + x}{\sqrt{2}}\Big) \quad \text{and} \quad (t, x) = \Big(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}}\Big).$$

For $\tau \geq 0$, $\lambda \geq 0$, denote

$$\tilde{u}(\tau, \lambda) = u\Big(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}}\Big).$$

We will study the simultaneous LIL and propagation of singularities for $\{\tilde{u}(\tau, \lambda), \tau \geq 0, \lambda \geq 0\}$.

9.2 The simultaneous law of the iterated logarithm and singularities

Recall that, if $B = \{B(t), t \geq 0\}$ is standard Brownian motion, then for every $t \geq 0$, the law of the iterated logarithm states:

$$\limsup_{h \to 0+} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log\log 1/h}} = 1, \qquad \text{a.s.}$$

By Fubini's theorem, we have

$$\mathbb{P}\left(\limsup_{h \to 0+} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log\log 1/h}} = 1 \ \text{ for almost all } t \geq 0 \right) = 1. \qquad (4)$$

In the above, "for almost all $t \geq 0$" can not be strengthened to "for all $t \geq 0$".

In fact, Orey and Taylor (1974) proved that the set

$$\mathscr{S} = \left\{ t \geq 0 : \limsup_{h \to 0+} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log \log 1/h}} = \infty \right\}$$

is dense in $[0, \infty)$, even though it follows from (4) that the Lebesgue measure of $\mathscr{S}$ equals 0.

The points in $\mathscr{S}$ are called singularities of Brownian motion.

Some geometric properties of $\mathscr{S}$ and the λ-fast sets

$$\mathscr{F}(\lambda) = \left\{ t \geq 0 : \limsup_{h \to 0+} \frac{|B(t+h) - B(t)|}{\sqrt{2h \log 1/h}} \geq \lambda \right\}$$

were studied by Orey and Taylor (1974), Khoshnevisan, Peres and X. (2000), among others.

For a random field $X = \{X(t), t \in \mathbb{R}^N\}$, the set of its singularities may have interesting topological and geometric properties.

This was first studied by Walsh (1982) for the Brownian sheet $W = \{W(s,t), s \geq 0, t \geq 0\}$, which is a centered Gaussian random field with covariance function

$$\mathbb{E}\big(W(s_1,t_1)W(s_2,t_2)\big) = (s_1 \wedge s_2)(t_1 \wedge t_2).$$

For each fixed $s > 0$, $\{\frac{1}{\sqrt{s}}W(s,t), t \geq 0\}$ is standard Brownian motion. Hence the LIL states that for every $t \geq 0$,

$$\limsup_{h \to 0+} \frac{|W(s,t+h) - W(s,t)|}{\sqrt{2h \log\log 1/h}} = \sqrt{s}, \qquad \text{a.s.}$$

Zimmerman (1972) proved the following simultaneous LIL: For any $t \geq 0$ fixed,

$$\mathbb{P}\left(\limsup_{h \to 0+} \frac{|W(s, t+h) - W(s,t)|}{\sqrt{2h \log\log 1/h}} = \sqrt{s} \ \text{ for all } s \geq 0 \right) = 1.$$

However, by the result of Orey and Taylor (1974), for any $s > 0$ fixed, there is a random time τ such that

$$\limsup_{h \to 0+} \frac{|W(s, \tau+h) - W(s,\tau)|}{\sqrt{2h \log\log 1/h}} = \infty, \qquad \text{a.s.}$$

In this case, we say that (s, τ) is a singularity in the t-direction.
Similarly, we say that (s, t) is a *singular point* of W in the s-direction if

$$\limsup_{h \to 0+} \frac{|W(s+h, t) - W(s,t)|}{\sqrt{h \log\log(1/h)}} = \infty.$$

Based on the simultaneous LIL of Zimmerman (1972), Walsh (1982) proved the following surprising result.

Let $s_0 > 0$ be fixed. If τ is any positive and finite $\sigma(W(s_0, t) : t \geq 0)$-measurable random variable, then on an event of probability 1, we have

$$\limsup_{h \to 0+} \frac{|W(s_0, \tau + h) - W(s_0, \tau)|}{\sqrt{h \log \log(1/h)}} = \infty$$

$$\iff \limsup_{h \to 0+} \frac{|W(s, \tau + h) - W(s, \tau)|}{\sqrt{h \log \log(1/h)}} = \infty$$

for all $s > s_0$ simultaneously.

The existence of a positive and finite $\sigma(W(s_0, t) : t \geq 0)$-measurable random variable τ is guaranteed by Meyer's section theorem. Walsh's theorem shows that the singularities of the Brownian sheet W propagate in directions parallel to the coordinate axis.

Carmona and Nualart (1988) extended the results of Walsh (1982, 1986) to one-dimensional nonlinear stochastic wave equations driven by the space-time white noise.

The method of Carmona and Nualart (1988) is based on the general theory of semimartingales and two-parameter strong martingales. They showed that, in the white noise case, their solution $X(t,x)$ has the following important properties:

(i). For any $x \in \mathbb{R}$, $\left\{X(\frac{h}{\sqrt{2}}, x+\frac{h}{\sqrt{2}}), h \geq 0\right\}$ is a continuous semimartingale.

(ii). The increments of $X(t,x)$ over a certain class of rectangles form a two-parameter strong martingale.

Carmona and Nualart (1988) proved the law of the iterated logarithm for a semimartingale by the LIL of Brownian motion and a time change.

They also proved that, for a class of two-parameter strong martingales, the law of the iterated logarithm in one variable holds simultaneously for all values of the other variable.

By applying these results and properties (i) and (ii), Carmona and Nualart proved the existence and propagation of singularities of the solution.

In the context of Gaussian random fields, Blath and Martin (2008) extended the result of Walsh (1982) to the semi-fractional Brownian sheets.

Due to the scaling property of the semi-fractional Brownian sheets, Blath and Martin (2008) was able to use the following large deviation result to prove their simultaneous LIL: If $\{Z(t), t \in T\}$ is a continuous centered Gaussian random field which is a.s. bounded, then

$$\lim_{\gamma\to\infty} \frac{1}{\gamma^2} \log \mathbb{P}\left(\sup_{t\in T} Z(t) > \gamma\right) = -\frac{1}{2\sup_{t\in T} \mathbb{E}(Z(t)^2)}. \tag{5}$$

However, this large deviation result is not enough for proving the following analogous LIL for $\{\tilde{u}(s,y), s \geq 0, y \geq 0\}$, where

$$\tilde{u}(s,y) = u\Big(\frac{s+y}{\sqrt{2}}, \frac{-s+y}{\sqrt{2}}\Big).$$

Theorem 9.1 [Lee and X. (2020+)]

For any $y > 0$ fixed, we have

$$\mathbb{P}\left(\limsup_{h \to 0+} \frac{|\tilde{u}(s, y+h) - \tilde{u}(s, y)|}{\sqrt{(s+y)h^{2-\beta} \log\log(1/h)}} = K_\beta \text{ for all } s \in [0, \infty) \right) = 1, \tag{6}$$

where K_β is

$$K_\beta = \left(\frac{2^{(1-\beta)/2}}{(2-\beta)(1-\beta)} \right)^{1/2}.$$

To prove the simultaneous LIL, we make use of more precise results on the tail probability for the supremum of Gaussian random fields based on the metric entropy obtained by Talagrand (1994).

Lemma 9.1 [Talagrand (1994)]

Let $\{Z(t), t \in T\}$ be a mean zero continuous Gaussian process and $\sigma_T^2 = \sup_{t \in T} \mathbb{E}[Z(t)^2]$. Let d_Z be the canonical metric defined by $d_Z(s,t) = \mathbb{E}[(Z(s) - Z(t))^2]^{1/2}$. Assume that for some constant $M > \sigma_T$, $\alpha > 0$ and $0 < \varepsilon_0 \le \sigma_T$,

$$N(T, d_Z, \varepsilon) \le \left(\frac{M}{\varepsilon}\right)^{\alpha} \quad \text{for all } \varepsilon < \varepsilon_0,$$

Then for any $\gamma > \sigma_T^2[(1 + \sqrt{\alpha})/\varepsilon_0]$, we have

$$\mathbb{P}\Big\{\sup_{t \in T} Z(t) \ge \gamma\Big\} \le \left(\frac{KM\gamma}{\sqrt{\alpha}\,\sigma_T^2}\right)^{\alpha} \Phi\left(\frac{\gamma}{\sigma_T}\right), \tag{7}$$

where $\Phi(x) = (2\pi)^{-1/2} \int_x^{\infty} \exp(-z^2/2)\, dz$ and K is a universal constant.

The upper bound in (7) is more precise than (5) if M/σ_T is not too large. However, the upper bound in (7) may not be useful when M/σ_T becomes very large (which will be the case in one part of the proof of Theorem 9.1).

To deal with the latter case, we use the following lemma, which is more efficient if the variance of $Z(t)$ attains its maximum at a unique point because the size of the set T_ρ can be very small.

Lemma 9.2

Let $\{Z(t), t \in T\}$ be a mean zero continuous Gaussian process. For $\rho > 0$, set

$$T_\rho = \{t \in T : \mathbb{E}[Z(t)^2] \geq \sigma_T^2 - \rho^2\}.$$

Assume that there exist constants $v \geq w \geq 1$ such that for all $\rho > 0$, and $0 < \varepsilon \leq \rho(1 + \sqrt{v})/\sqrt{w}$, we have

$$N(T_\rho, d_Z, \varepsilon) \leq A\rho^w \varepsilon^{-v}.$$

Then for any $\gamma > 2\sigma_T\sqrt{w}$, we have

$$\mathbb{P}\Big\{\sup_{t\in T} Z(t) \geq \gamma\Big\} \leq \frac{Aw^{w/2}}{v^{v/2}} K^{v+w}\Big(\frac{\gamma}{\sigma_T^2}\Big)^{v-w} \Phi\Big(\frac{\gamma}{\sigma_T}\Big).$$

We will also need the following estimates on the variance of two types of increments.

Lemma 9.3

For any $\tau, \lambda, h > 0$,

$$\mathbb{E}[(\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda))^2] = \frac{1}{2} K_\beta^2 \left[(\tau + \lambda)h^{2-\beta} + (3 - \beta)^{-1} h^{3-\beta}\right],$$

where K_β is the constant defined by

$$K_\beta = \left(\frac{2^{(1-\beta)/2}}{(2-\beta)(1-\beta)} \right)^{1/2}. \tag{8}$$

Lemma 9.4

Fix $\lambda \geq 0$. Then, for any $0 \leq \tau \leq \tau'$ and $0 \leq h \leq h'$,

$$\mathbb{E}\Big[\big(\tilde{u}(\tau', \lambda + h') - \tilde{u}(\tau', \lambda + h) - \tilde{u}(\tau, \lambda + h') + \tilde{u}(\tau, \lambda + h)\big)^2\Big]$$
$$= \begin{cases} \frac{1}{2} K_\beta^2 (h' - h)^{2-\beta} \Big[(\tau' - \tau) - \frac{1-\beta}{3-\beta}(h' - h)\Big] & \text{if } h' - h \leq \tau' - \tau, \\ \frac{1}{2} K_\beta^2 (\tau' - \tau)^{2-\beta} \Big[(h' - h) - \frac{1-\beta}{3-\beta}(\tau' - \tau)\Big] & \text{if } h' - h > \tau' - \tau. \end{cases}$$

Proof of the upper bound in Theorem 9.1

First we prove that for any fixed $\lambda > 0$,

$$\mathbb{P}\Big(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta} \log\log(1/h)}} \le K_\beta \text{ for all } \tau \in [0, \infty) \Big) = 1,$$

where K_β is the constant in Lemma 9.3.
It suffices to show that for any $0 \le a < b < \infty$ and any $0 < \varepsilon < 1$,

$$\mathbb{P}\Big(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta} \log\log(1/h)}} \le (1+\varepsilon)K_\beta,\ \forall\, \tau \in [a, b] \Big) = 1. \tag{9}$$

Let $\delta = (c + \lambda)\varepsilon/2$. Since we can cover $[a, b]$ by finitely many intervals $[c, d]$ of length δ, we only need to show that (9) holds for all $\tau \in [c, d]$, where $[c, d] \subset [a, b]$ and $d = c + \delta$.

Choose a real number q such that $1 < q < (1+\varepsilon)^{1/(2-\beta)}$. For every integer $n \geq 1$, consider the event

$$A_n = \left\{ \sup_{\tau \in [c,d]} \sup_{h \in [0,q^{-n}]} \left| \tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda) \right| > \gamma_n \right\}, \qquad (10)$$

where

$$\gamma_n = (1+\varepsilon) K_\beta \sqrt{(c+\lambda)(q^{-n-1})^{2-\beta} \log\log q^n}.$$

To estimate $\mathbb{P}(A_n)$, we will apply Lemma 9.2.
Define $T = [c,d] \times [0, q^{-n}]$ and $Z(\tau, h) = \tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda)$ for $(\tau, h) \in T$. It follows from Lemma 9.3 that $\mathbb{E}\left[Z(\tau,h)^2\right]$ attains its unique maximum σ_T^2 at (d, q^{-n}), where

$$\sigma_T^2 = \frac{1}{2} K_\beta^2 \left[(d+\lambda) q^{-n(2-\beta)} + (3-\beta)^{-1} q^{-n(3-\beta)} \right].$$

For any $(\tau, h), (\tau', h') \in T$, without loss of generality, we may assume that $\tau \le \tau'$. Then by Lemma 9.3 and 9.4, we have

$$\begin{aligned} & d_Z((\tau, h), (\tau', h')) \\ & \le \mathbb{E}\big[(Z(\tau, h) - Z(\tau, h'))^2\big]^{1/2} + \mathbb{E}\big[(Z(\tau', h') - Z(\tau, h'))^2\big]^{1/2} \\ & = \mathbb{E}\big[(\tilde{u}(\tau, \lambda + h) - \tilde{u}(\tau, \lambda + h'))^2\big]^{1/2} \\ & \quad + \mathbb{E}\big[(\tilde{u}(\tau', \lambda + h') - \tilde{u}(\tau', \lambda) - \tilde{u}(\tau, \lambda + h') + \tilde{u}(\tau, \lambda))^2\big]^{1/2} \\ & \le C(q^{-n(2-\beta)/2}|\tau - \tau'|^{1/2} + |h - h'|^{(2-\beta)/2}). \end{aligned} \tag{11}$$

Next, in order to apply Lemma 9.2, we estimate $N(T_\rho, d_Z, \varepsilon)$, where

$$T_\rho = \{(\tau, h) \in T : \sigma_T^2 - \mathbb{E}[Z(\tau, h)^2] \le \rho^2\}.$$

It can be shown that

$$T_\rho \subset [d - C_1 q^{n(2-\beta)}\rho^2, d] \times [q^{-n} - C_2 \rho^{2/(2-\beta)}, q^{-n}]$$

for some constants C_1 and C_2. This and (11) imply that

$$N(T_\rho, d_Z, \varepsilon) \le C_0 (\rho/\varepsilon)^{2+\frac{2}{2-\beta}}.$$

By Lemma 9.2 with $v = w = 2 + \frac{2}{2-\beta}$, we have

$$\mathbb{P}(A_n) \le C \exp\Big(- \frac{\gamma_n^2}{2\sigma_T^2} \Big) = (n \log q)^{-p_n},$$

where

$$p_n = \frac{(1+\varepsilon)^2}{q^{2-\beta}\left[\frac{d+\lambda}{c+\lambda} + (3-\beta)^{-1}(c+\lambda)^{-1} q^{-n}\right]}$$

which is eventually bigger than 1. Hence $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. This is enough for proving (**??**).

Proof of the lower bound in Theorem 9.1

Next, we prove the corresponding lower bound: For any $\lambda > 0$,

$$\mathbb{P}\Big(\limsup_{h\to 0+} \frac{|\tilde{u}(\tau,\lambda+h) - \tilde{u}(\tau,\lambda)|}{\sqrt{(\tau+\lambda)h^{2-\beta}\log\log(1/h)}} \geq K_\beta,\ \forall \tau \in [a,b]\Big) = 1, \tag{12}$$

where K_β is the constant in (8).

Similarly to the previous section, we only need to show that (12) holds for all $\tau \in [c,d]$, where $[c,d] \subset [a,b]$ and $d = c + \delta$.

The following are the main ingredients.

Lemma 9.5

Let $\tau > 0$, $\lambda > 0$ and $q > 1$. Then for all $0 < \varepsilon < 1$,

$$\mathbb{P}\Big(\frac{\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda + q^{-n-1})}{\tilde{\sigma}[(\tau, \lambda + q^{-n}), (\tau, \lambda + q^{-n-1})]} \geq (1-\varepsilon)\sqrt{2 \log \log q^n} \text{ i.o.}\Big) = 1,$$

where

$$\tilde{\sigma}[(\tau, \lambda), (\tau', \lambda')] = \mathbb{E}[(\tilde{u}(\tau, \lambda) - \tilde{u}(\tau', \lambda'))^2]^{1/2}.$$

This is proved by an extended Borel-Cantelli lemma.

For all $\tau \in [c, d]$ we write

$$\begin{aligned}\tilde{u}(\tau, \lambda + q^{-n}) - \tilde{u}(\tau, \lambda) = \tilde{u}(d, \lambda + q^{-n}) - \tilde{u}(d, \lambda + q^{-n-1}) \\ + \tilde{u}(\tau, \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda) \\ - \Delta\tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}]),\end{aligned}$$

where the last term is the the increment of $\tilde{u}$ over the rectangle $(\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}]$.
The first difference in the right hand side of (27) is dealt by Lemma 9.5.
For the second difference, (9) says that for all $\tau \in [c, d]$ simultaneously,

$$\begin{aligned}&|\tilde{u}(\tau, \lambda + q^{-n-1}) - \tilde{u}(\tau, \lambda)| \\ &\leq K_\beta \sqrt{(\tau + \lambda + q^{-n-1})(q^{-n-1})^{2-\beta} \log\log q^n}.\end{aligned}$$

eventually for all large n.

To derive a bound for the term $\Delta\tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])$, we consider the event

$$A_n = \left\{ \sup_{\tau\in[c,d]} |\Delta\tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| > \gamma_n \right\},$$

where

$$\gamma_n = K_\beta \, \phi_n(d) \sqrt{(q^{-n})^{2-\beta} \log\log q^n}$$

and

$$\begin{aligned} \phi_n(\tau) = (1 - \varepsilon/4) \left(\frac{q-1}{q} \right)^{\frac{2-\beta}{2}} (d + \lambda)^{1/2} \\ - q^{-\frac{2-\beta}{2}} (\tau + \lambda + q^{-n-1})^{1/2} - (1 - \varepsilon)(\tau + \lambda)^{1/2}. \end{aligned}$$

Consider n large enough such that $q^{-n} - q^{-n-1} \leq d - c$. Then

$$\mathbb{P}(A_n) \leq \mathbb{P}(A_n^1) + \mathbb{P}(A_n^2),$$

where

$$A_n^1 = \left\{ \sup_{\tau \in [c, d-(q^{-n}-q^{-n-1})]} |\Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| > \gamma_n \right\},$$

$$A_n^2 = \left\{ \sup_{\tau \in [d-(q^{-n}-q^{-n-1}), d]} |\Delta \tilde{u}((\tau, d] \times (\lambda + q^{-n-1}, \lambda + q^{-n}])| > \gamma_n \right\}.$$

By Lemma 9.2,

$$\mathbb{P}(A_n^1) \leq C \exp\left(-\frac{\gamma_n^2}{2\sigma_T^2} \right) \leq (n \log q)^{-p_n},$$

where

$$p_n = \frac{1}{d-c} \left(\frac{q}{q-1} \right)^{2-\beta} \phi_n(d)^2.$$

We can check that $\sum_{n=1}^{\infty} \mathbb{P}(A_n^1) < \infty$.

Since the size of $[d-(q^{-n}-q^{-n-1}),d]$ is small, we can apply Lemma 9.1 to see that for n large,

$$\mathbb{P}(A_n^2) \le C\phi_n(d)^2(q^n \log n)\exp(-C'\phi_n(d)^2 q^n \log n)$$

which also yields $\sum_{n=1}^{\infty} \mathbb{P}(A_n^2) < \infty$.
Combing the three parts, we derive that for all $\tau \in [c,d]$ simultaneously,

$$\begin{aligned}
&\left|\tilde{u}(\tau,\lambda+q^{-n}) - \tilde{u}(\tau,\lambda)\right| \\
&\ge \left|\tilde{u}(d,\lambda+q^{-n}) - \tilde{u}(d,\lambda+q^{-n-1})\right| \\
&\qquad - \left|\tilde{u}(\tau,\lambda+q^{-n-1}) - \tilde{u}(\tau,\lambda)\right| \\
&\qquad - \left|\Delta\tilde{u}((\tau,d]\times(\lambda+q^{-n-1},\lambda+q^{-n}])\right| \\
&\ge (1-\varepsilon)K_\beta\sqrt{(\tau+\lambda)(q^{-n})^{2-\beta}\log\log q^n},
\end{aligned}$$

where the last inequality holds infinitely often in n. This concludes the proof.

9.3 Propagation of singularities

For $s_0 > 0$, denote by $\mathscr{F}_{s_0}$ be the σ-field generated by $\{W\big(B \cap \Pi(s_0)\big) : B \in \mathscr{B}(\mathbb{R}^2)\}$ and the $\mathbb{P}$-null sets, where $\Pi(s_0) = \{(s, y) : 0 \le s < s_0/\sqrt{2}, y \in \mathbb{R}\}$.

The following theorem shows that the singularities of $u(t, x)$ propagate along the straight lines curves $s + y = c$ and $s - y = -c$.

Theorem 9.2 [Lee and X. (2020+)]

Let $s_0 > 0$. The following statements hold.

(i) There exists a positive and finite $\mathscr{F}_{s_0}$-measurable r.v. Λ such that

$$\limsup_{h \to 0+} \frac{|\tilde{u}(s_0, \Lambda + h) - \tilde{u}(s_0, \Lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty \quad \text{a.s.}$$

Theorem 9.2 [continued]

(ii) For any positive and finite $\mathscr{F}_{s_0}$-measurable r.v. Λ, with probability 1,

$$\limsup_{h\to 0+} \frac{|\tilde{u}(s_0, \Lambda+h) - \tilde{u}(s_0, \Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty$$

$$\iff \limsup_{h\to 0+} \frac{|\tilde{u}(s, \Lambda+h) - \tilde{u}(s, \Lambda)|}{\sqrt{h^{2-\beta}\log\log(1/h)}} = \infty$$

for all $s > s_0$ simultaneously.

Part (i) of Theorem 9.2 is proved by using Meyer's section theorem [Dellacherie (1972, p.18)]:

Let $(\Omega, \mathscr{G}, \mathbb{P})$ be a complete probability space and S be a $\mathscr{B}(\mathbb{R}_+) \times \mathscr{G}$-measurable subset of $\mathbb{R}_+ \times \Omega$. Then there exists a $\mathscr{G}$-measurable random variable Λ with values in $(0, \infty]$ such that

(a) the graph of Λ, denoted by $[\Lambda] := \{(t, \omega) \in \mathbb{R}_+ \times \Omega : \Lambda(\omega) = t\}$, is contained in S;

(b) $\{\Lambda < \infty\}$ is equal to the projection $\pi(S)$ of S onto Ω.

For fixed $s_0 > 0$, we decompose $\tilde{u}$ into $\tilde{u}_1 + \tilde{u}_2$, where

$$\tilde{u}_i(\tau, \lambda) = u_i\Big(\frac{\tau + \lambda}{\sqrt{2}}, \frac{-\tau + \lambda}{\sqrt{2}}\Big), \quad i = 1, 2,$$

and

$$u_1(t, x) = \frac{1}{2} W\big(\Delta(t, x) \cap \Pi(s_0)\big),$$
$$u_2(t, x) = \frac{1}{2} W\big(\Delta(t, x) \cap \Pi(s_0)^c\big).$$

It can be proven that there exists a positive, finite, $\mathscr{F}_{\tau_0}$-measurable random variable Λ such that

$$\limsup_{h \to 0+} \frac{|\tilde{u}_1(s_0, \Lambda + h) - \tilde{u}_1(s_0, \Lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty \quad \text{a.s.}$$

This is proved by taking

$$S = \left\{ (\lambda, \omega) : \limsup_{h \to 0+} \frac{|\tilde{u}_1(s_0, \lambda + h)(\omega) - \tilde{u}_1(s_0, \lambda)(\omega)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = \infty \right\}$$

and applying Meyer's section theorem.

Moreover, for $\lambda > 0$,

$$\mathbb{P}\left(\limsup_{h \to 0+} \frac{|\tilde{u}_2(\tau, \lambda + h) - \tilde{u}_2(\tau, \lambda)|}{\sqrt{h^{2-\beta} \log\log(1/h)}} = K_\beta (\tau - s_0 + \lambda)^{1/2} \text{ for all } \tau \geq s_0 \right)$$
$$= 1.$$

Combining the above ingredients yields Theorem 9.2.

Thank you!