Multiple Points of Gaussian Random Fields

Yimin Xiao

Michigan State University

CBMS Conference, University of Alabama in Huntsville

August 2–6, 2021

Outline

1. Multiple points of Gaussian random fields
2. Sketch of the proof
3. Applications to stochastic heat and wave equations

8.1. Multiple points of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$ defined by

$$X(t) = (X_1(t), \ldots, X_d(t)), \quad \text{where } X_1, \ldots, X_d \text{ are i.i.d.}$$

Let $m \geq 2$ be fixed. We consider the question: when is

$$\mathbb{P}\Big\{\exists \text{ distinct } t^1, \ldots, t^m \text{ s.t. } X(t^1) = \cdots = X(t^m)\Big\} > 0?$$

Studies on self-intersection have a long history. For Gaussian random fields we mention two approaches:

- Potential-theoretical approach for the Brownian sheet: R. Dalang, D. Khoshnevisan, E. Nualart, D. Wu and X. (2012), Dalang and Mueller (2015).
- Covering argument for fractional Brownian motion: Talagrand (1998).

We apply an approach which is based on Talagrand (1998). Our setting is the same as in Dalang, Mueller and X. (2017). We recall the conditions we will use.

Condition (A1)

Consider a compact interval $T \subset \mathbb{R}^N$. There exist a Gaussian random field $\{v(A,t) : A \in \mathscr{B}(\mathbb{R}_+), t \in T\}$ such that

(a) For all $t \in T$, $A \mapsto v(A,t)$ is an $\mathbb{R}^d$-valued Gaussian noise with i.i.d. components, $v(\mathbb{R}_+, t) = X(t)$, and $v(A, \cdot)$ and $v(B, \cdot)$ are independent whenever A and B are disjoint.

Condition (A1) (continued)

(b) There are constants $a_0 \geq 0$ and $\gamma_j > 0$, $j = 1, \ldots, N$ such that for all $a_0 \leq a \leq b \leq \infty$ and $s, t \in T$,

$$\begin{aligned} &\left\| v([a,b),s) - X(s) - v([a,b),t) + X(t) \right\|_{L^2} \\ &\leq C\Big(\sum_{j=1}^{N} a^{\gamma_j} |s_j - t_j| + b^{-1} \Big), \end{aligned} \tag{1}$$

where $\|Y\|_{L^2} = \left[\mathbb{E}(Y^2)\right]^{1/2}$ for a random variable Y and

$$\left\| v([0,a_0),s) - v([0,a_0),t) \right\|_{L^2} \leq C \sum_{j=1}^{N} |s_j - t_j|. \tag{2}$$

Recall that, under (A1), we have

$$\mathbb{E}\big(|X(s) - X(t)|^2\big) \le c\,\rho(s,t)^2$$

for all for $s, t \in T$, where $\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}$ and where $H_j = (\gamma_j + 1)^{-1}$.

From here, we derive an upper bound for the uniform modulus of continuity for $\{v(x), x \in T\}$.

Moreover, Assumption (A1) can be used to prove the following useful lemma.

Lemma 8.1 [DMX17]

Consider $b > a > 1$ and $r > 0$ small. Set

$$A = \sum_{j=1}^{N} a^{H_j^{-1}-1}\, r^{H_j^{-1}} + b^{-1}.$$

There are constants A_0, K and c such that if $A \le A_0 r$ and

$$u \ge KA \log^{1/2}\left(\frac{r}{A}\right), \tag{3}$$

then for $S(t^0, r) = \{t \in T : \rho(t, t^0) \le r\}$,

$$\mathbb{P}\Big\{ \sup_{t \in S(t^0,r)} |X(t) - X(t^0) - (v([a,b],t) - v([a,b],t^0))| \ge u \Big\}$$
$$\le \exp\Big(- \frac{u^2}{cA^2}\Big).$$

Condition (A5′)

(a) $\|X_1(t)\|_{L^2} \ge c > 0$ for all $t \in T$ and

$$\mathbb{E}\Big[\big(X_1(s) - X_1(t)\big)^2\Big] \ge K\rho(s,t)^2 \quad \text{for all } \ s,t \in T.$$

(b) For $I \subset T$ and $\varepsilon > 0$ small, let I^ε be the ε-neighborhood of I. For every $t \in I$, there is $t' \in \partial I^{(\varepsilon)}$ such that for all $x, \bar{x} \in I$ with $\rho(t,x) \le 2\varepsilon$ and $\rho(t,\bar{x}) \le 2\varepsilon$,

$$\Big|\mathbb{E}\big((X_1(x) - X_1(\bar{x}))X_1(t')\big)\Big| \le C \sum_{j=1}^{N} |x_j - \bar{x}_j|^{\delta_j},$$

where $\delta_j \in (H_j, 1]$, $(j = 1, \ldots, N)$ are constants.

We also need an additional assumption.

Assumption (A6)

For any m distinct points $t^1, \ldots, t^m \in T$, $X_1(t^1), \ldots, X_1(t^m)$ are linearly independent random variables, or equivalently, the Gaussian vector $(X_1(t^1), \ldots, X_1(t^m))$ is non-degenerate.

This is equivalent to $\det \mathrm{Cov}(X_1(t^1), \ldots, X_1(t^m)) > 0$, which holds if $\{X_1(t), t \in \mathbb{R}^N\}$ has a property of local nondeterminism: for all $n \le m$ and all $t, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X_1(t) \big| X_1(t^1), \ldots, X_1(t^n)\big) \ge c \min\{\rho(t, t^j)^2 : 1 \le j \le n\}.$$

Hence (A6) is weaker than the property of strong local nondeterminism.

Main result

Theorem 8.1 [Dalang, Lee, Mueller and X. (2021)]

Let $T \subset \mathbb{R}^N$ (or, say $(0,\infty)^N$) be a compact interval such that (A1), (A5$'$) and (A6) hold. If $mQ \le (m-1)d$, then $\{X(t), t \in T\}$ has no m-multiple points almost surely.

Remarks:
(i). The proof for $mQ < (m-1)d$ is easy. The case when $mQ = (m-1)d$, which is called the critical (dimension) case, is more difficult.
(ii). If $mQ > (m-1)d$, then we can show that $v(x)$ has m-multiple points with positive probability.

(iii) Theorem 8.1 is applicable to

- the Brownian sheet
- fractional Brownian sheets
- Solutions of systems of stochastic heat and wave equations indicated earlier.
- ...

8.2. Sketch of the proof of Theorem 8.1

Consider m distinct points $t^1, \ldots, t^m \in T$ such that

$$\rho(t^i, t^j) \geq \eta \quad \text{for} \quad i \neq j,$$

where $\eta > 0$. For $\varepsilon > 0$ small (say $\varepsilon < \eta/4$), let

$$B^i_\varepsilon = \left(\prod_{j=1}^{N} [t^i_j - \varepsilon^{1/H_j}, t^i_j + \varepsilon^{1/H_j}] \right) \cap T.$$

Consider the random set $M_\varepsilon := M_{t^1,\ldots,t^m;\varepsilon}$ defined by

$$M_\varepsilon = \Big\{ z \in \mathbb{R}^d : \exists\, (t^1, \ldots, t^m) \in B^1_\varepsilon \times \cdots \times B^m_\varepsilon \\ \text{such that } z = X(t^1) = \cdots = X(t^m) \Big\}.$$

We will prove that under the conditions of Theorem 8.1, $M_\varepsilon = \emptyset$ a.s.

By applying the small ball probability estimate due to Talagrand (1993) [cf. Ledoux, 1994], we have

Lemma 8.2

There exist constants K and $0 < \delta_0 < 1$ such that for all $(s^1, \ldots, s^m) \in B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}$, $0 < a < b$, and $0 < u < r < \delta_0$, we have

$$\mathbb{P}\Big(\sup_{1\le i\le m} \sup_{x^i \in S(s^i,r)} |v([a,b),x^i) - v([a,b),s^i)| \le u \Big) \geq \exp\Big(- K\frac{r^Q}{u^Q}\Big).$$

Recall that $S(s,r) = \{x \in T : \rho(x,s) \le r\}$.

The key component for proving Theorem 8.1 is the following:

Proposition 8.1

Suppose (A1) holds for $B^1_{2\rho}, \ldots, B^m_{2\rho}$. Then there are constants K and $0 < \delta < 1$ such that for all $0 < r_0 < \delta$ and $(s^1, \ldots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho}$, we have

$$\mathbb{P}\Big(\exists\, r \in [r_0^2, r_0],\ \sup_{1\le i\le m}\ \sup_{x^i\in S(s^i,r)} |X(x^i) - X(s^i)| \le Kr\big(\log\log 1/r\big)^{-1/Q}\Big)$$
$$\ge 1 - \exp\Big(-\big(\log 1/r_0\big)^{1/2}\Big).$$

Proof of Proposition 8.1 Let $1 < a < b$, $r > 0$ small and

$$A = \sum_{j=1}^{N} a^{1/H_j - 1} r^{1/H_j} + b^{-1}.$$

Lemma 8.1 and (A1) imply that there are constants A_0, K and c such that for all $(s^1, \ldots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho}$ if $A \le A_0 r$ and $u \ge K A (\log(r/A))^{1/2}$, then

$$\mathbb{P}\Big(\sup_{1 \le i \le m} \sup_{x^i \in S(s^i, r)} \big|(X(x^i) - X(s^i)) - (v([a,b), x^i) - v([a,b), s^i))\big| \ge u \Big)$$

$$\le m \exp\Big(-\frac{u^2}{cA^2} \Big).$$

Applying Lemma 8.2 to $\{v([a,b), x)\}$, we can prove the proposition in a similar way to the proof of Proposition 7.1 in Lecture 7. The details are omitted here.

For any integer $p \geq 1$, let

$$R_p = \Big\{(s^1, \ldots, s^m) \in B^1_{2\rho} \times \cdots \times B^m_{2\rho} : \exists\, r \in [2^{-2p}, 2^{-p}] \ \text{ s. t.}$$
$$\sup_{1 \leq i \leq m} \sup_{x^i \in S(s^i, r)} |X(x^i) - X(s^i)| \leq Kr\Big(\log\log\frac{1}{r}\Big)^{-1/Q}\Big\}.$$

Proposition 8.1 can be re-stated as $\forall\, (s^1, \ldots, s^m) \in B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}$,

$$\mathbb{P}\big\{(s^1, \ldots, s^m) \in R_p\big\} \geq 1 - \exp\big(-\sqrt{p}/2\big).$$

This and Fubini's theorem imply that with probability 1,

$$\lambda(R_p) \geq \lambda\big(B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}\big)\Big(1 - \exp\big(-\sqrt{p}/4\big)\Big) \tag{4}$$

for all p large enough, where λ denotes Lebesgue measure.

This means that for most of points $(s^1, \ldots, s^m) \in B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}$, the oscillations of $X(x^i)$ in $S(s^i, r)$ $(i = 1, \ldots, m)$ are characterized by $r\big(\log\log \frac{1}{r}\big)^{-1/Q}$ along a sequence of $r_p \to 0$.

The points in $B^1_{2\varepsilon} \times \cdots \times B^m_{2\varepsilon}$ where the oscillation is large can be covered by much fewer balls of ρ-radius r.
The largest such oscillation can be bounded by the uniform modulus of continuity of X on T.
The effect of these points can be shown to be negligible and so we will focus on dealing with the points in R_p that satisfies (4).

Suppose that for each small $\varepsilon > 0$, (A5$'$) holds for the rectangles $B^1_{2\varepsilon}, \ldots, B^m_{2\varepsilon}$ and there are $(\hat{t}^1, \ldots, \hat{t}^m)$ on the boundary of $B^1_{3\varepsilon} \times \cdots \times B^m_{3\varepsilon}$ such that for every $i = 1, \ldots, m$ and all $x, y \in B^i_{2\varepsilon}$,

$$\left|\mathbb{E}\big[(X(x) - X(y))X(\hat{t}^i)\big]\right| \le C \sum_{j=1}^{N} |x_j - y_j|^{\delta_j}.$$

Let Σ_2 denote the σ-field generated by $X(\hat{t}^1), \ldots, X(\hat{t}^m)$. Define

$$X^2(x) = \mathbb{E}(X(x)|\Sigma_2), \quad X^1(x) = X(x) - X^2(x).$$

The processes X^1 and X^2 are independent.

Lemma 8.3

There is a constant K depending on $\hat{t}^1, \ldots, \hat{t}^m$ such that for all $i = 1, \ldots, m$, for all $x, y \in B^i_{2\varepsilon}$,

$$\left|X^2(x) - X^2(y)\right| \le K \sum_{j=1}^{N} |x_j - y_j|^{\delta_j} \max_{1\le i\le m} \left|X(\hat{t}^i)\right|.$$

This shows that the process X^1 is a small perturbation of X, provided $\max_{1\le i\le m} \left|v(\hat{t}^i)\right|$ is not too big. (We can control is easily using Condition (A2) (a).)

Lemma 8.4

Suppose (A6) is satisfied. There exists a constant K (depending on $t^1, \ldots, t^m$) such that for all ε small, $a_2, \ldots, a_m \in \mathbb{R}^d$, $r > 0$, and $(x^1, \ldots, x^m) \in B^1_\varepsilon \times \cdots \times B^m_\varepsilon$,

$$\mathbb{P}\left\{ \sup_{2 \le i \le m} |X^2(x^1) - X^2(x^i) - a_i| \le r \right\} \le K r^{(m-1)d}.$$

This is proved by showing $X^2(x^1), \ldots, X^2(x^m)$ are linearly dependent.

Construction of a covering of M_ε

Denote by $\mathscr{C}$ the family of "generalized dyadic cubes" of the form $C = I_{q,1} \times \cdots \times I_{q,m}$ of order q.

We say that such a cube C is *good* if

$$\sup_{1 \le i \le m} \sup_{x,y \in I_{q,i}} |X^1(x) - X^1(y)| \le d_q,$$

where $d_q = K 2^{-q} (\log\log 2^q)^{-1/Q}$.

For each $x \in R_p$, we can find a good dyadic cube C containing x of smallest order q, where $p \le q \le 2p$.

We obtain a family $\mathscr{G}_p^1$ of disjoint good dyadic cubes of order between p and $2p$ that meet R_p.

Let $\mathscr{G}_p^2$ be the family of dyadic cubes of order $2p$ that meet $B_\varepsilon^1 \times \cdots \times B_\varepsilon^m$ but are not contained in any cube of $\mathscr{G}_p^1$. (These are the *bad* cubes.)

Let $\mathscr{G}_p = \mathscr{G}_p^1 \cup \mathscr{G}_p^2$, which covers $B_{2\varepsilon}^1 \times \cdots \times B_{2\varepsilon}^m$.

Note that for each $C \in \mathscr{C}$, the events $\{C \in \mathscr{G}_p^1\}$ and $\{C \in \mathscr{G}_p^2\}$ are in the σ-field $\Sigma_1 := \sigma(v^1(x) : x \in T)$.

For each $p \geq 1$, we construct a family $\mathscr{F}_p$ of balls in $\mathbb{R}^d$ (depending on ω).
For each $C \in \mathscr{C}$, we choose a distinguished point $x_C = (x_C^1, \ldots, x_C^m)$ in $C \cap (B_{2\varepsilon}^1 \times \cdots \times B_{2\varepsilon}^m)$. Let the ball $B_{p,C}$ be defined as follows:

(i) If $C \in \mathscr{G}_p^1$, take $B_{p,C}$ as the ball of center $X^1(x_C^1)$ of radius $r_{p,C} = 4d_q$.

(ii) If $C \in \mathscr{G}_p^2$, take $B_{p,C}$ as the ball of center $X^1(x_C^1)$ of radius $r_{p,C} = K2^{-2p}p^{1/2}$.

(iii) Otherwise, take $B_{p,C} = \varnothing$ and $r_{p,C} = 0$.

Note that for each $p \geq 1, C \in \mathscr{C}$, the random variable $r_{p,C}$ is Σ_1-measurable. Consider the event

$$\Omega_{p,C} = \left\{ \omega \in \Omega : \sup_{2 \leq i \leq m} |X(x_C^1, \omega) - X(x_C^i, \omega)| \leq r_{p,C}(\omega) \right\}.$$

Define $\mathscr{F}_p(\omega) = \{B_{p,C} : C \in \mathscr{G}_p(\omega), \omega \in \Omega_{p,C}\}$.

Claim 1: There is an event Ω^* of probability one such that for all p large enough and $\omega \in \Omega^* \cap \Omega_{p,C}$, the family $\mathscr{F}_p(\omega)$ covers M_ε.

This is proved by making use of Lemma 8.3.

Claim 2: $\mathbb{P}\{\Omega_{p,C} \,|\Sigma_1\} \le K\, r_{p,C}^{(m-1)d}$.

This is proved by making use of Lemma 8.4.

To finish the proof, we make use of an argument of geometric flavor. Let

$$\phi(r) = r^{mQ-(m-1)d}(\log\log(1/r))^m.$$

We consider the following quantity related to M_ε:

$$\phi\text{-}m\big(M_\varepsilon\big) = \liminf_{p\to\infty} \sum_{B_{p,C}\in\mathscr{F}_p} \phi(r_{p,C}).$$

Notice that, if $mQ-(m-1)d > 0$, then ϕ-$m(M_\varepsilon)$ gives an upper bound for the ϕ-Hausdorff measure of M_ε. However, ϕ-$m(M_\varepsilon)$ is well-defined even if $mQ-(m-1)d \le 0$.

By applying Fatou's lemma, Claims 1 and 2, we derive

$$\begin{aligned}
\mathbb{E}\big[\phi\text{-}m(M_\varepsilon)\big] &\le \liminf_{p\to\infty} \mathbb{E}\Big\{ \sum_{B_{p,C}\in\mathscr{F}_p} \phi(r_{p,C}) \Big\} \\
&= \liminf_{p\to\infty} \mathbb{E}\Big\{ \sum_{C\in\mathscr{G}_p} \phi(r_{p,C}) \mathbf{1}_{\Omega_{p,C}} \Big\} \\
&= \liminf_{p\to\infty} \mathbb{E}\Big\{ \mathbb{E}\Big[\sum_{B_{p,C}\in\mathscr{G}_p} \phi(r_{p,C}) \mathbf{1}_{\Omega_{p,C}} \Big| \Sigma_1 \Big] \Big\} \\
&\le K\rho^{mQ}.
\end{aligned}$$

This implies that $\phi\text{-}m(M_\varepsilon) < \infty$ a.s.

However, if $mQ \le (m-1)d$, then $\phi(r) \to \infty$ as $r \to 0$. This implies that M_ε is empty.

Thank you!