Hitting Probabilities and Polarity of Points for Gaussian Random Fields

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Outline

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7.1 Intersection problems for random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. Various intersection problems can be considered:

(1) For Borel sets $E \subseteq \mathbb{R}^N$ and $F \subseteq \mathbb{R}^d$, when is

$$\mathbb{P}\big(X(E) \cap F \neq \emptyset\big) > 0? \tag{1}$$

(2) [Multiple intersections] Given disjoint sets $E_1, \ldots, E_k \subseteq \mathbb{R}^N$, when does

$$\mathbb{P}\big(X(E_1) \cap \cdots \cap X(E_k) \cap F \neq \emptyset\big) > 0? \tag{2}$$

Question (1) is quite general, which includes intersections of the graph set and level sets:

- Let $\mathrm{Gr}X(E) = \{(t, X(t)) : t \in E\}$ be the graph of X on E. Then (1) is equivalent to

$$\mathbb{P}\big(\mathrm{Gr}X(E) \cap (E \times F) \neq \emptyset\big) > 0.$$

- Take $F = \{0\}$, then (1) is equivalent to

$$\mathbb{P}\big(X^{-1}(0) \cap E \neq \emptyset\big) > 0.$$

The following are some known results about Question (1).

In the case when $E = [a, b]$, $(a, b \in \mathbb{R}^N)$, necessary and sufficient conditions for (1) in terms of certain kind of capacity of F have been established for X being

- Brownian motion Lévy processes
- Some multiparameter Markov processes (Fitzsimmons and Salisbury, 1989)
- The Brownian sheet (Khoshnevisan and Shi, 1999)
- Additive Lévy processes (Khoshnevisan and X., 2002, 2003, 2009)
- Hyperbolic SPDEs (Dalang and Nualart, 2004)

In the special case when $F = \{0\}$, Khoshnevisan and Xiao (2002) for a large class of additive Lévy processes.

For general $E \subseteq \mathbb{R}^N$ and $F \subseteq \mathbb{R}^d$, a necessary and sufficient condition in terms of "thermal capacity" of $E \times F$ was established for Brownian motion B by Watson (1978).

The Hausdorff dimension $B(E) \cap F$ was determined by Khoshnevisan and X. (2015).

For Gaussian random fields and the solutions of some SPDEs, some necessary conditions and sufficient conditions for the hitting probability in **(1)** with $E = [a, b]$, $(a, b \in \mathbb{R}^N)$ have been obtained by Dalang, Khoshnevisan and Nualart (2007, 2009), Biermé, Lacaux and X. (2009), X. (2009), Dalang and Sanz-Solé (2010), Hinojosa-Calleja and Sanz-Solé (2020, 2021).

In Section 7.2, we will work to extend and strengthen the existing results on the hitting probability in **(1)** for Gaussian random fields.

Question (2) is related to existence of self-intersections.

- When $F = \mathbb{R}^d$, then (2) gives existence of k-multiple points.
  - Lévy processes (Khoshnevisan and X., 2005): $F = \mathbb{R}^d$, general $E_1, \ldots, E_k$
  - The Brownian sheet: Dalang et al (2012), Dalang and Mueller (2015), Dalang, Lee, Mueller, and X. (2021): $F = \mathbb{R}^d$, $E_1, \ldots, E_k$ are intervals.
- No results for general $F, E_1, \ldots, E_k$.

In Section 7.2, we will provide some results on the intersection of independent Gaussian random fields, which is technically simpler than Question (2).

7.2 Hitting probabilities of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$ defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad t \in \mathbb{R}^N, \tag{3}$$

where $X_1, \ldots, X_d$ are independent copies of a centered GF X_0.
Given $E \subset \mathbb{R}^N$ and $F \subset \mathbb{R}^d$, in order to provide necessary condition and sufficient condition for

$$\mathbb{P}\{X(E) \cap F \neq \emptyset\} > 0,$$

we recall some concepts on fractals.

Hausdorff dimension and Capacity

For any metric $\widetilde{\rho}$ on $\mathbb{R}^p$, any $\beta > 0$ and $E \subseteq \mathbb{R}^p$, the β-dimensional Hausdorff measure in the metric $\widetilde{\rho}$ of E is defined by

$$\mathcal{H}_{\widetilde{\rho}}^{\beta}(E) = \lim_{\delta \to 0} \inf \left\{ \sum_{n=1}^{\infty} (2r_n)^{\beta} : E \subseteq \bigcup_{n=1}^{\infty} B_{\widetilde{\rho}}(r_n), \ r_n \leq \delta \right\},$$

where $B_{\widetilde{\rho}}(r)$ denotes an open ball of radius r in the metric space $(\mathbb{R}^p, \widetilde{\rho})$.
The corresponding Hausdorff dimension of E is defined by

$$\dim_{\mathrm{H}}^{\widetilde{\rho}} E = \inf \left\{ \beta > 0 : \ \mathcal{H}_{\widetilde{\rho}}^{\beta}(E) = 0 \right\}.$$

$\widetilde{\rho}$ will be omitted if it is the Euclidean metric.

The Bessel-Riesz type capacity of order α on the metric space $(\mathbb{R}^p, \widetilde{\rho})$ is defined by

$$\mathcal{C}^{\alpha}_{\widetilde{\rho}}(E) = \left[\inf_{\mu \in \mathcal{P}(E)} \int \int f_\alpha\big(\widetilde{\rho}(u,v)\big)\mu(du)\mu(dv) \right]^{-1},$$

where $\mathcal{P}(E)$ is the family of probability measures carried by E and the function $f_\alpha : (0,\infty) \to (0,\infty)$ is defined by

$$f_\alpha(r) = \begin{cases} r^{-\alpha} & \text{if } \alpha > 0, \\ \log\left(\frac{e}{r \wedge 1}\right) & \text{if } \alpha = 0, \\ 1 & \text{if } \alpha < 0. \end{cases} \tag{4}$$

The dimension p and metric $\widetilde{\rho}$ can be chosen appropriately based on the hitting probability problem, as we will show below.

We start by stating the following result which was motivated by Dalang, Khoshnevisan and Nualart (2007).

Theorem 7.1 [Biermé, Lacaux and X. (2009)]

If X is defined by (3) such that X_0 satisfies:

$$\mathbb{E}\Big[\big(X_0(s) - X_0(t)\big)^2\Big] \asymp \sum_{j=1}^{N} |s_j - t_j|^{2H_j} \ \text{ for all } \ s, t \in I(= [\varepsilon, 1]^N), \quad (5)$$

where $0 < H_j \leq 1$ $(1 \leq j \leq d)$ are constants, and $\exists$ $c_{7,1} > 0$ such that for all $s, t \in I$,

$$\mathrm{Var}\,\big(X_0(t) \big| X_0(s)\big) \geq c_{7,1} \sum_{j=1}^{N} |s_j - t_j|^{2H_j}. \qquad (6)$$

Theorem 7.1 (continued)

Then $\forall$ Borel set $F \subset \mathbb{R}^d$,

$$c_{7,2}\, \mathcal{C}^{d-Q}(F) \le \mathbb{P}\Big\{X(I) \cap F \ne \emptyset\Big\} \le c_{7,3}\, \mathcal{H}^{d-Q}(F),$$

where $Q = \sum_{j=1}^{N} \frac{1}{H_j}$, $\mathcal{C}^{d-Q}$ is $(d-Q)$-dimensional Riesz capacity and $\mathcal{H}^{d-Q}$ is $(d-Q)$-dimensional Hausdorff measure.

It is an open problem if $\mathcal{H}^{d-Q}(F)$ in the above can be replaced by $\mathcal{C}^{d-Q}(F)$.

Recently, Dalang, Mueller and X. (2017) proved that, if $d = Q$, then for every $x \in \mathbb{R}^d$,

$$\mathbb{P}\big\{X(I) \cap \{x\} \ne \emptyset\big\} = \mathbb{P}\big\{\exists t \in I : X(t) = x\big\} = 0.$$

We will discuss this result in Section 7.3 below.

For any Borel set $F \subseteq \mathbb{R}^d$, consider the inverse image

$$X^{-1}(F) = \{t \in \mathbb{R}^N : X(t) \in F\}.$$

Theorem 7.2 [Biermé, Lacaux and X. (2009)]

Let X be as in Theorem 7.1 and let $F \subseteq \mathbb{R}^d$ be a Borel set such that $\sum_{j=1}^N \frac{1}{H_j} > d - \dim_{\rm H} F$. Then with positive probability,

$$\begin{aligned} &\dim_{\rm H}\big(X^{-1}(F) \cap I\big) \\ &= \min_{1\le k\le N}\left\{\sum_{j=1}^{k} \frac{H_k}{H_j} + N - k - H_k(d - \dim_{\rm H} F)\right\}. \end{aligned}$$

The following extension of Theorem 7.1 is useful.

Theorem 7.3 [Z. Chen and X. (2012)]

Assume that (5) and (6) hold. Then for all compact sets $E \subseteq I$ and $F \subseteq \mathbb{R}^d$,

$$c_{7,4}\, \mathcal{C}^d_{\rho_1}(E \times F) \le \mathbb{P}\Big\{X(E) \cap F \ne \emptyset\Big\} \le c_{7,5}\, \mathcal{H}^d_{\rho_1}(E \times F),$$

where $\mathcal{C}^d_{\rho_1}$ and $\mathcal{H}^d_{\rho_1}$ denote respectively the d-dimensional Riesz capacity and d-dimensional Hausdorff measure in the metric space $(\mathbb{R}^{N+d}, \rho_1)$, and where

$$\rho_1\big((s,x),(t,y)\big) = \max\Big\{\sum_{j=1}^N |s_j - t_j|^{H_j},\ \|x-y\|\Big\}.$$

Theorem 7.3 implies the following result on hitting probability of $X^{-1}(\{a\})$:
For every $a \in \mathbb{R}^d$ and Borel set $E \subseteq I$, $\exists\, c_{7,6} \geq 1$, s.t.

$$c_{7,6}^{-1}\, \mathcal{C}_\rho^d(E) \leq \mathbb{P}\big\{X^{-1}(\{a\}) \cap E \neq \varnothing\big\} \leq c_{7,6}\, \mathcal{H}_\rho^d(E).$$

In the above, $\mathcal{C}_\rho^d$ is the Bessel-Riesz capacity of order d in the metric ρ, and $\mathcal{H}_\rho^d(E)$ is the d-dimensional Hausdorff measure of E in the metric ρ defined as before by

$$\rho(s,t) = \sum_{j=1}^N |s_j - t_j|^{H_j}.$$

Proof of Theorem 7.3

The proof makes use of the following two lemmas.

Lemma 7.1 [Biermé, Lacaux and X. (2009)]

Assume the conditions of Theorem 7.1 hold. For any constant $M > 0$, there exist positive constants c and δ_0 such that for all $r \in (0, \delta_0)$, $t \in I$ and all $x \in [-M, M]^d$,

$$\mathbb{P}\left\{ \inf_{s \in B_\rho(t,r) \cap I} \|X(s) - x\| \le r \right\} \le c\, r^d. \tag{7}$$

In the above $B_\rho(t, r) = \{s \in \mathbb{R}^N : \rho(s, t) \le r\}$ denotes the closed ball of radius r in the metric ρ in $\mathbb{R}^N$.

Lemma 7.2 [Biermé, Lacaux and X. (2009)]

There exists a positive and finite constant c such that for all $\varepsilon \in (0, 1)$, $s, t \in I$ and $x, y \in \mathbb{R}^d$, we have

$$\int_{\mathbb{R}^{2d}} \exp\left(-\frac{1}{2}(\xi,\, \eta)\left(\varepsilon I_{2d} + \mathrm{Cov}\big(X(s), X(t)\big)\right)(\xi,\, \eta)^T \right) e^{-i(\langle \xi, x\rangle + \langle \eta, y\rangle)} d\xi d\eta \;\le \frac{c}{\rho_1\big((s,x),(t,y)\big)^d}.$$

In the above, I_{2d} denotes the identity matrix of order $2d$, $\mathrm{Cov}\big(X(s), X(t)\big)$ denotes the covariance matrix of the random vector $(X(s), X(t))$, and $(\xi, \eta)^T$ is the transpose of the row vector (ξ, η).

Proof of Theorem 7.3 The upper bound in (14) can be proved by a covering argument using Lemma 7.1.

The the lower bound in (14) can be proved by using Lemma 7.2 and a capacity argument.

We omit the details.

Intersections of independent Gaussian fields

Let $X^H = \{X^H(s), s \in \mathbb{R}^{N_1}\}$ and $X^K = \{X^K(t), t \in \mathbb{R}^{N_2}\}$ be two independent Gaussian fields with values in $\mathbb{R}^d$ such that the associate random fields X_0^H and X_0^K satisfy (5) and (6) respectively on $I_1 \subseteq \mathbb{R}^{N_1}$ with $H = (H_1, \ldots, H_{N_1})$ and on $I_2 \subseteq \mathbb{R}^{N_2}$ with $K = (K_1, \ldots, K_{N_2})$.

Theorem 7.4 [Z. Chen and X. (2012)]

There exists a constant $C \geq 1$ such that

$$C^{-1}\, \mathcal{C}^d_{\rho_2}(E_1 \times E_2) \leq \mathbb{P}\big\{X^H(E_1) \cap X^K(E_2) \neq \varnothing\big\} \leq C\, \mathcal{H}^d_{\rho_2}(E_1 \times E_2),$$

where

$$\rho_2\big((s,t),(s',t')\big) = \sum_{i=1}^{N_1} |s_i - s_i'|^{H_i} + \sum_{j=1}^{N_2} |t_j - t_j'|^{K_j}.$$

When $E_1 = I_1$ and $E_2 = I_2$ are two intervals, Theorem 7.4 implies that

(i) If $d > \sum_{j=1}^{N_1} \frac{1}{H_j} + \sum_{j=1}^{N_2} \frac{1}{K_j}$, then

$$\mathbb{P}\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\} = 0.$$

(ii) If $d < \sum_{j=1}^{N_1} \frac{1}{H_j} + \sum_{j=1}^{N_2} \frac{1}{K_j}$, then

$$\mathbb{P}\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\} > 0.$$

- What happens in the critical case of

$$d = \sum_{j=1}^{N_1} \frac{1}{H_j} + \sum_{j=1}^{N_2} \frac{1}{K_j}? \tag{8}$$

Theorem 7.5 [Z. Chen and X. (2012)]

If X^H (or X^K) satisfies the conditions of Theorem 5.6, then, in the critical case (8), $\mathbb{P}\{X^H(I_1) \cap X^K(I_2) \neq \varnothing\} = 0.$

Proof By Theorem 5.6, the exact Hausdorff measure function for $X^H(I_1)$ is

$$\varphi(r) = r^{\sum_{j=1}^{N_1} \frac{1}{H_j}} \log\log \frac{1}{r}.$$

This implies that

$$\mathcal{H}_{d-\sum_{j=1}^{N_2} \frac{1}{K_j}}(X^H(I_1)) = 0 \qquad \text{a.s.}$$

Therefore, the conclusion follows from Theorem 7.1.

7.3 Polarity of points (the critical case)

In Dalang, Mueller and X. (2017), the following assumptions are made.

Condition (A1)

Consider a compact interval $T \subset \mathbb{R}^N$. There exists a Gaussian random field $\{v(A,t) : A \in \mathscr{B}(\mathbb{R}_+),\ t \in T\}$ such that

(a) For all $t \in T$, $A \mapsto v(A,t)$ is a real-valued Gaussian noise, $v(\mathbb{R}_+, t) = X_1(t)$, and $v(A,\cdot)$ and $v(B,\cdot)$ are independent whenever A and B are disjoint.

Condition (A1) (continued)

(b) There are constants $a_0 \geq 0$ and $\gamma_j > 0$, $j = 1, \ldots, N$ such that for all $a_0 \leq a \leq b \leq \infty$ and $s, t \in T$,

$$\begin{aligned} &\big\| v([a,b),s) - X_1(s) - v([a,b),t) + X_1(t) \big\|_{L^2} \\ &\leq C\Big(\sum_{j=1}^{N} a^{\gamma_j} |s_j - t_j| + b^{-1} \Big), \end{aligned} \tag{9}$$

where $\|Y\|_{L^2} = \big[\mathbb{E}(Y^2)\big]^{1/2}$ for a random variable Y and

$$\big\| v([0,a_0),s) - v([0,a_0),t) \big\|_{L^2} \leq C \sum_{j=1}^{N} |s_j - t_j|. \tag{10}$$

Recall that $\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}$, where $H_j = (\gamma_j + 1)^{-1}$.

Condition (A5)

(a). There is a constant $c > 0$ such that $\|X_1(t)\|_{L^2} \geq c$ for all $t \in T$.

(b). For $I \subset T$ and $\varepsilon > 0$ small, let I^ε be the ε-neighborhood of I. For every $t \in I$, there is $t' \in \partial I^{(\varepsilon)}$ such that for all $x, \bar{x} \in I$ with $\rho(t, x) \leq 2\varepsilon$ and $\rho(t, \bar{x}) \leq 2\varepsilon$,

$$\left|\mathbb{E}((X_1(x) - X_1(\bar{x}))X_1(t'))\right| \leq C \sum_{j=1}^{N} |x_j - \bar{x}_j|^{\delta_j},$$

where $\delta_j \in (H_j, 1]$, $(j = 1, \ldots, N)$ are constants.

The following is the main result of Dalang, Mueller and X. (2017).

Theorem 7.6

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field that satisfies Conditions (A1) and (A5). Assume that $Q = d$. Then for every $z \in \mathbb{R}^d$, $\mathbb{P}\{\exists t \in T : X(t) = z\} = 0$.

Theorem 7.6 is proved by constructing an economic covering for the image $X\big(B_\varepsilon(t^0)\big)$ (where $\varepsilon > 0$ and $t^0 \in T$ are fixed) by using the method of Talagrand (1998).

See also X. (1997), where the exact Hausdorff measure of the level set $L_z = \{t \in T : X(t) = z\}$ was determined.

The main ingredient for proving Theorem 7.6 is the following proposition, which was proved as Proposition 5.2 in Lecture 5.

Proposition 7.1

Let Assumption (A1) hold. Then there are constants K_1 and δ_0 such that for every $0 < r_0 < \delta_0$ and $t^0 \in T$, we have

$$\mathbb{P}\left\{\exists r \in [r_0^2, r_0]: \sup_{t:\rho(t,t^0)<r} |X(t) - X(t^0)| \le K_1 \frac{r}{(\log\log\frac{1}{r})^{1/Q}}\right\}$$
$$\ge 1 - \exp\left[-\left(\log\frac{1}{r_0}\right)^{\frac{1}{2}}\right].$$

For $t^0 \in T$ and $\varepsilon > 0$, set

$$B_\varepsilon(t^0) = \{t \in T : \rho(t, t^0) \le \varepsilon\},$$
$$B'_\varepsilon(t^0) = \{t \in T : \rho(t, t^0) \le 2\varepsilon\}.$$

For proving Theorem 7.6, it is sufficient to prove the following

Proposition 7.2

Assume that (A1) holds and $Q = d$. Fix $t^0 \in T$, and consider the following (random) subset of $\mathbb{R}^d$:

$$M(\varepsilon, t^0) = X\big(B_\varepsilon(t^0)\big),$$

which is the image of $B_\varepsilon(t^0)$ under the mapping $t \mapsto X(t)$. Then for every $z \in \mathbb{R}^d$,

$$\mathbb{P}\{z \in M(\varepsilon, t^0)\} = \mathbb{P}\big\{\exists t \in B_\varepsilon(t^0) : X(t) = z\big\} = 0.$$

We work to prove Proposition 7.2.

Let $t' \in B_\varepsilon(t^0)$ be given by (A5) (b). We define two $\mathbb{R}^d$-valued Gaussian random fields

$$X^2(t) = \mathbb{E}(X(t) \big| X(t')), \qquad X^1(t) = X(t) - X^2(t).$$

Notice that the random fields X^1 and X^2 are independent. Further, X^1 is independent of the random vector $X(t')$. The following lemma shows that X^2 can be viewed as a perturbation part.

Lemma 7.3

There is a finite constant C such that for $t, \bar{t} \in B'_\varepsilon(t^0)$,

$$|X^2(t) - X^2(\bar{t})| \le C|X(t')| \sum_{j=1}^N |t_j - \bar{t}_j|^{\delta_j}.$$

For $p \geq 1$, consider the random set

$$R_p = \Big\{ x \in B'_\varepsilon(t^0) : \exists r \in [2^{-2p}, 2^{-p}) \text{ with}$$

$$\sup_{\bar{x}:\, \rho(\bar{x},x) \leq r} |X(\bar{x}) - X(x)| \leq \frac{K_1\, r}{(\log\log \frac{1}{r})^{\frac{1}{Q}}} \Big\},$$

and the event

$$\Omega_{p,1} = \Big\{ \lambda_N(R_p) \geq \lambda(B'_\varepsilon(t^0))\Big(1 - \exp\big(-\sqrt{p}/4\big)\Big) \Big\}.$$

which can be described as the event "a large portion of $B'_\varepsilon(t^0)$ consists of points at which X has minimal oscillation." As in the proof of the upper bound in Theorem 5.6, we have

$$\mathbb{P}\big(\Omega^c_{p,1}\big) \leq \frac{\mathbb{E}(\lambda_N(B'_\varepsilon(t^0) \setminus R_p))}{\lambda_N(B'_\varepsilon(t^0)) \exp\big(-\sqrt{p}/4\big)} \leq \exp\Big(-\frac{3}{4}\sqrt{p}\Big). \qquad (11)$$

This gives

$$\sum_{p=1}^{\infty} \mathbb{P}(\Omega_{p,1}^c) < +\infty. \tag{12}$$

Fix $\beta \in]0, \min(\min_{j=1,\ldots,N}(\delta_j H_j^{-1} - 1), 1)[$ (which is possible since $\delta_j > H_j, j = 1, \ldots, N$) and set

$$\Omega_{p,2} = \{|X(t')| \le 2^{\beta p}\}.$$

Then $\sum_{p\ge 1} \mathbb{P}(\Omega_{p,2}^c) < +\infty$.
By Lemma 7.3, we have that, on the event $\Omega_{p,2}$,

$$|X^2(x) - X^2(\bar{x})| \le C 2^{\beta p} \sum_{j=1}^{N} |x_j - \bar{x}_j|^{\delta_j} \le \tilde{C}\, 2^{\beta p} \sum_{j=1}^{N} r^{\delta_j H_j^{-1}}$$

for all for $x, \bar{x} \in B'_\varepsilon(t^0)$ that satisfy $\rho(x, \bar{x}) \le cr$.

Therefore, there is a constant $K_2 > K_1$ such that on the event $\Omega_{p,3} \stackrel{\text{def}}{=} \Omega_{p,1} \cap \Omega_{p,2}$, for each $x \in R_p$, there exists $r \in [2^{-2p}, 2^{-p}]$ such that

$$\sup_{\bar{x}:\, \rho(\bar{x},x) \le r} |X^1(\bar{x}) - X^1(x)| \le K_2 \frac{r}{(\log\log \frac{1}{r})^{1/Q}}. \tag{13}$$

An "anisotropic dyadic cubes" of order ℓ in $\mathbb{R}^N$ is of the form

$$\prod_{j=1}^{N} \Big[\frac{m_j}{2^{\ell H_j^{-1}}}, \frac{m_j + 1}{2^{\ell H_j^{-1}}}\Big],$$

where $m_j \in \mathbb{N}$. For $x \in \mathbb{R}^N$, let $C_\ell(x)$ denote the anisotropic dyadic cube of order ℓ that contains x.

The cube $C_\ell(x)$ is called "good" if

$$\sup_{\bar{x}\in C_\ell(x)\cap B_\varepsilon(t^0)} |X^1(y) - X^1(\bar{x})| \le d_\ell, \tag{14}$$

where

$$d_\ell = K_2 \, \frac{2^{-\ell}}{(\log\log 2^\ell)^{1/Q}}$$

By (13), when $\Omega_{p,3}$ occurs, we can find a family $\mathcal{H}_{1,p}$ of non-overlapping good anisotropic dyadic cubes (they may have intersecting boundaries) of order $\ell \in [p, 2p]$ that covers R_p. This family only depends on the random field X^1.

Let $\mathcal{H}_{2,p}$ be the family of non-overlapping dyadic cubes of order $2p$ that meet $B_\varepsilon(t^0)$ but are not contained in any cube of $\mathcal{H}_{1,p}$. For p large enough, these cubes are contained in $B'_\varepsilon(t^0)$, hence in $B'_\varepsilon(t^0) \setminus R_p$.
Therefore, when $\Omega_{p,3}$ occurs, the number of cubes in $\mathcal{H}_{2,p}$ is at most N_p, where

$$N_p \, 2^{-2pQ} \le \lambda_N(B'_\varepsilon(t^0)) \exp\big(- \sqrt{p}/4 \big),$$

so

$$N_p \le K \, 2^{2pQ} \exp\left(-\sqrt{p}/4\right), \tag{15}$$

where K does not depend on p.

Let $\Omega_{p,4}$ be the event "the inequality

$$\sup_{x,\bar{x}\in C} |X(x) - X(\bar{x})| \le K_3\, 2^{-2p}\sqrt{p} \tag{16}$$

holds for each dyadic cube C of order $2p$ of $\mathbb{R}^N$ that meets $B_\varepsilon(t^0)$."

We choose K_3 large enough so that $\sum_{p\ge 1} \mathbb{P}\big(\Omega^c_{p,4}\big) < +\infty$. This is possible by Lemma 3.3 in Lecture 3 [it is Lemma 2.1 from Talagrand (1995)].

Set $\mathcal{H}_p = \mathcal{H}_{1,p} \cup \mathcal{H}_{2,p}$. This family is well-defined for all $p \ge 1$, and it is a non-overlapping cover of $B_\varepsilon(t^0)$.

Set

$$r_A = 4d_\ell = 4K_2 2^{-\ell}(\log \ell)^{-1/Q} \quad \text{if } A \in \mathcal{H}_{1,p} \text{ and } A \text{ is of order } \ell \in [p, 2p],$$

$$r_A = K_3 2^{-2p}\sqrt{p} \quad \text{if } A \in \mathcal{H}_{2,p}.$$

Let $f(r) = r^d \log\log \frac{1}{r}$. If $\Omega_{p,3} \cap \Omega_{p,4}$ occurs, then we can verify that for p large enough,

$$\sum_{A \in \mathcal{H}_p} f(r_A) \le K\lambda_N(B_\varepsilon(t^0)). \tag{17}$$

For each $A \in \mathcal{H}_p$, we pick a distinguished point p_A in A (say the lower left corner). Let B_A be the Euclidean ball in $\mathbb{R}^d$ centered at $X(p_A)$ with radius r_A.

Let $\mathcal{F}_p$ be the family of balls $\{B_A,\ A \in \mathcal{H}_p\}$. For p large enough, on $\Omega_{p,3} \cap \Omega_{p,4}$, $\mathcal{F}_p$ covers $M(\varepsilon, t^0)$.

Since $f(r)/r^d \to 0$ as $r \to 0+$, it follows from (17) that

$$\lambda_d\big(M(\varepsilon, t^0)\big) = 0 \text{ a.s.}$$

This and Fubini's theorem imply that for a.e. $z \in \mathbb{R}^d$, $\mathbb{P}\big(z \in M(\varepsilon, t^0)\big) = 0$.

To prove that for every $z \in \mathbb{R}^d$, $\mathbb{P}\big(z \in M(\varepsilon, t^0)\big) = 0$, we introduce the random field X^3 defined by

$$X^3(t) = \frac{1}{\alpha(t)}(z - X^1(t)), \quad \forall\, t \in \mathbb{R}^N,$$

where

$$\alpha(t) = \frac{\mathbb{E}[X_1(t)X_1(t')]}{\mathbb{E}[X_1(t')^2]}.$$

Notice that $\mathbb{E}[X(t)|X(t')] = \alpha(t)X(t')$.
It can be verified that $1/2 \le \alpha(t) \le 3/2$ for all $t \in B_\varepsilon(t^0)$ when ε is small enough. Moreover, the function $t \mapsto \alpha(t)$ is Hölder continuous by Condition (A5)(b).
For any $z \in \mathbb{R}^d$, by the decomposition

$$X(x) = X^1(x) + \alpha(x)X(t'),$$

we have

$$X(x) = z \qquad \Longleftrightarrow \qquad X^3(x) = X(t'). \tag{18}$$

Denote by $g_{X(t')}(w)$ the density function of $X(t')$. By the independence of $X^1(x)$ and $X(t')$, we have

$$\begin{aligned}\mathbb{P}\big\{z \in M(\varepsilon, t^0)\big\} &= \mathbb{P}\big\{\exists x \in B_\varepsilon(t^0) : X^3(x) = X(t')\big\} \\ &= \int_{\mathbb{R}^d} dw\, g_{X(t')}(w)\mathbb{P}\{\exists x \in B_\varepsilon(t^0) : X^3(x) = w\}.\end{aligned} \tag{19}$$

It can be proved as on the previous page that $\lambda_d\big[X^3(B_\varepsilon(t^0))\big] = 0$ a.s. This implies that for a.e. $w \in \mathbb{R}^d$,

$$\mathbb{P}\{\exists x \in B_\varepsilon(t^0) : X^3(x) = w\} = 0.$$

Therefore, (19) yields $\mathbb{P}\big\{z \in M(\varepsilon, t^0)\big\} = 0$.

This proves Proposition 7.2 and thus Theorem 7.6.

7.4 Polarity of points for systems of linear stochastic heat and wave equations

Let $\hat{u} = \{\hat{u}(t,x),\ t \in \mathbb{R}_+,\ x \in \mathbb{R}^k\}$ be the mild solution of a linear system of d uncoupled heat equations:

$$\begin{cases} \frac{\partial}{\partial t}\hat{u}_j(t,x) = \Delta \hat{u}_j(t,x) + \dot{W}_j(t,x), \qquad j = 1, \ldots, d, \\ u(0,x) = 0, \qquad x \in \mathbb{R}^k. \end{cases} \tag{20}$$

Here, $\hat{u}(t,x) = (\hat{u}_1(t,x), \ldots, \hat{u}_d(t,x))$ and Δ is the Laplacian in the spatial variables. The Gaussian noise $\dot{W}$ is white in time and has a spatially homogeneous covariancegiven by the Riesz kernel with exponent $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}_j(t,x)\dot{W}_j(s,y)) = \delta(t-s)|x-y|^{-\beta}.$$

If $k = 1 = \beta$, then $\dot{W}$ is the space-time white noise.

Theorem 7.7

Suppose $(4+2k)/(2-\beta) = d$. Then d is the critical dimension for hitting points and points are polar for $\hat{u}$. That is, for all $z \in \mathbb{R}^{(4+2k)/(2-\beta)}$,

$$\mathbb{P}\{\exists(t,x) \in (0,+\infty) \times \mathbb{R}^k : \hat{u}(t,x) = z\} = 0.$$

In particular, in the case when $\hat{W}$ is the space-time white noise and $d = 6$, all points are polar for $\hat{u}$.

Now let $\hat{v}$ be the solution of the stochastic wave equation in spatial dimension k driven by W with $\beta \geq 1$.

$$\begin{cases} \frac{\partial^2}{\partial t^2}\hat{v}_j(t,x) &= \Delta \hat{v}_j(t,x) + \dot{W}_j(t,x), \qquad j = 1, \ldots, d, \\ \hat{v}(0,x) &= 0, \qquad \frac{\partial}{\partial t}\hat{v}(0,x) = 0, \qquad x \in \mathbb{R}^k. \end{cases}$$

Theorem 7.8

Suppose $k = 1 = \beta$ or $1 < \beta < k \wedge 2$, and $d = \frac{2(k+1)}{2-\beta}$. Then d is the critical dimension for hitting points and points are polar for $\hat{v}$, that is, for all $z \in \mathbb{R}^d$,

$$\mathbb{P}\{\exists (t,x) \in (0,+\infty) \times \mathbb{R}^k : \hat{v}(t,x) = z\} = 0.$$

In particular, in the case when W is the space-time white noise and $d = 4$, all points are polar for $\hat{v}$.

Thank you!