# Lecture 6. Local Times of Gaussian Random Fields

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# Outline

- Local times and their joint continuity
- Hölder conditions for the local times
- Optimal Hölder conditions for the local times.

# 6.1. Local times: existence and joint continuity

Local times of Brownian motion was first studied by P. Lévy (1948), under a different name. The term of "local times" for general Markov processes was introduced by Blumenthal and Getoor (1964).

In late 1960's, Berman started studying local times of Gaussian processes. His work was extended by Pitt (1978) to random fields, and stimulated a lot of works on local times of random fields. More information can be found in Geman and Horowitz (1980), Dozzi (2002), X. (2009), etc.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be an $(N, d)$-random field. For any Borel set $T \subseteq \mathbb{R}^N$, the *occupation measure* of $X$ on $T$ defined by

$$\mu_T(\bullet) = \lambda_N\{t \in T : X(t) \in \bullet\}.$$

If $\mu_T \ll \lambda_d$, then $X$ is said to have a *local time* on $T$, which is defined by

$$L(x, T) = \frac{d\mu_T}{d\lambda_d}(x),$$

where $x$ is the so-called *space variable,* and $T$ is the *time variable.* We write $L(x, t)$ instead of $L(x, [0, t])$.

$L(x, T)$ satisfies the following *occupation density formula*: For every Borel set $T \subseteq \mathbb{R}^N$ and for every measurable function $f : \mathbb{R}^d \to \mathbb{R}_+$,

$$\int_T f(X(t))\, dt = \int_{\mathbb{R}^d} f(x) L(x, T)\, dx. \tag{1}$$

# Joint continuity

Suppose we fix an interval $I = \prod_{\ell=1}^{N}[a_\ell,\, b_\ell]$ in $\mathbb{R}^N$. Let $T = \prod_{\ell=1}^{N}[a_\ell,\, t_\ell] \subset I$. If we can choose a version of the local time, still denoted by $L(x, T)$, such that it is continuous in $(x, t_1, \ldots, t_N) \in \mathbb{R}^d \times I$, then $X$ is said to have a *jointly continuous local time* on $I$.

- The smoother the local time, the rougher the sample path (Berman, 1972).
- When a local time is jointly continuous, $L(x, \bullet)$ can be extended to be a finite Borel measure supported on the level set

$$X^{-1}(x) = \{t \in I : X(t) = x\}$$

[cf. Adler, 1981] and is a useful tool for studying fractal properties of $X^{-1}(x)$.

# Assumptions

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be an $(N, d)$-Gaussian random field defined by

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \tag{2}$$

where $X_1, \ldots, X_d$ are independent copies of a real-valued Gaussian random field $X_0$.

We study the following questions:

- The existence and joint continuity of local times of $X$.
- Hölder conditions for the local times of $X$ and apply these results to study its sample path properties of $X$.

# Existence of local times

The following result [cf. Geman and Horowitz (1980, Theorem 21.9)] is convenient: $X$ has an $L^2(\mathbb{P} \times \lambda_d)$ local time $L(x, T)$ if and only if

$$\int_{\mathbb{R}^d} \int_T \int_T \mathbb{E}\left(e^{i\langle \theta, X(s)-X(t)\rangle}\right) dsdtdx < \infty.$$

In particular, we have

**Theorem 6.1**

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (2) such that

$$\mathbb{E}\left[\left(X_0(s) - X_0(t)\right)^2\right] \asymp \rho(s,t)^2, \quad \text{for } \ s, t \in T. \tag{3}$$

Then $X$ has an $L^2(\mathbb{P} \times \lambda_d)$ local time if and only if $Q > d$.

# Joint continuity of local times

For studying joint continuity of local times, we make use of sectorial local nondeterminism.

## Condition (A4) [sectorial local nondeterminism]

For a constant vector $H = (H_1, \ldots, H_N) \in (0, 1)^N$, there exists a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\Big(X_0(u) \,\Big|\, X_0(t^1), \ldots, X(t^n)\Big) \geq c \sum_{j=1}^{N} \min_{1 \leq k \leq n} \big|u_j - t_j^k\big|^{2H_j}. \tag{4}$$

**Theorem 6.2 [Ayache, Wu and X. (2008), Wu and X. (2011)]**

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (2) such that (3) and (A4) hold. If $Q > d$ then the local time of $X$ is jointly continuous on $T \times \mathbb{R}^d$.

The proof of this theorem relies on Kolmogorov's continuity theorem and the moment estimates for $L(t, D)$ and $L(x, D) - L(y, D)$.

The moment estimates in Lemmas 6.1 and 6.2 are more precise than what are needed for proving joint continuity.

# Notations

We assume $0 < H_1 \leq \ldots \leq H_N < 1$.
Under the condition that

$$Q = \sum_{\ell=1}^{N} \frac{1}{H_\ell} > d,$$

there exists a unique $\tau \in \{1, \ldots, N\}$ such that

$$\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} \leq d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell}.$$

We will distinguish three cases:

$$\text{Case 1:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} < d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell},$$

$$\text{Case 2:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} = d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell} \quad \text{and } H_{\tau-1} = H_\tau,$$

$$\text{Case 3:} \quad \sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell} = d < \sum_{\ell=1}^{\tau} \frac{1}{H_\ell} \quad \text{and } H_{\tau-1} < H_\tau.$$

## Lemma 6.1

For any $D := \overline{B}_\rho(a, r) \subseteq T$, we have

$$\mathbb{E}\left[L(x, D)^n\right] \le \begin{cases} c_{6,1}^n \,(n!)^{\eta_\tau}\, r^{n\alpha} & \text{Cases 1 \& 2,} \\ c_{6,1}^n \,(n!)^{\eta_\tau}\, r^{n\alpha} \log^n\left(1+n\right) & \text{Case 3,} \end{cases} \tag{5}$$

where

$$\begin{aligned} \alpha &= Q - d, \\ \eta_\tau &= \tau + H_\tau d - \sum_{\ell=1}^{\tau} \frac{H_\tau}{H_\ell}. \end{aligned} \tag{6}$$

## Lemma 6.2

For $\gamma \in (0, 1)$ small and all even number $n \geq 2$, we have

$$\mathbb{E}\left[\Big(L(x, D) - L(y, D)\Big)^n\right] \leq \begin{cases} c_{6,2}^n\,(n!)^{\eta_\tau + (2H_\tau+1)\gamma}\, r^{n(\alpha-\gamma)} & \text{Cases 1 \& 2,} \\ c_{6,2}^n\,(n!)^{\eta_\tau + (2H_\tau+1)\gamma}\, r^{n(\alpha-\gamma)}\,\log^n(e+n) & \text{Case 3.} \end{cases} \quad (7)$$

Proof of Theorem 6.2 follows from Lemmas 6.1, 6.2, and a multiparameter version of Kolmogorov's continuity theorem.

# Proof of Lemma 6.1

We will need the following lemma.

## Lemma 6.3

Let $\beta$, $\gamma$ and $p$ be positive constants such that $\gamma\beta > p$. There exists a constant $C > 0$ such that for all $A \in (0, 1)$, $r > 0$, $u^* \in \mathbb{R}^p$, all integers $n \geq 1$ and distinct $u_1, \ldots, u_n \in O_p(u^*, r)$ we have

$$\int_{O_p(u^*,r)} \frac{du}{\left[A + \min\limits_{1\leq j\leq n} |u - u_j|^\gamma\right]^\beta} \leq C\, n\, A^{\frac{p}{\gamma}-\beta}, \tag{8}$$

where $O_p(u^*, r) \subset \mathbb{R}^p$ denotes the Euclidean ball.

Proof of Lemma 6.1 Recall that [e.g., Geman and Horowitz (1980)] for all $x \in \mathbb{R}^d$, any Borel set $D \subseteq \mathbb{R}^N$ and integer $n \geq 1$,

$$\mathbb{E}\Big[L(x,D)^n\Big] = (2\pi)^{-nd} \int_{D^n} \int_{\mathbb{R}^{nd}} \exp\bigg(-i \sum_{j=1}^{n} \langle u^j, x\rangle\bigg) \times \mathbb{E} \exp\bigg(i \sum_{j=1}^{n} \langle u^j, X(t^j)\rangle\bigg)\, d\bar{u}\, d\bar{t},$$

where

$$\bar{u} = (u^1, \ldots, u^n) \in \mathbb{R}^{nd}, \quad \bar{t} = (t^1, \ldots, t^n) \in D^n.$$

We see that $\mathbb{E}\big[L(x,D)^n\big]$ is at most

$$\int_{D^n}\prod_{k=1}^{d}\bigg\{\int_{\mathbb{R}^n}\exp\bigg[-\frac{1}{2}\mathrm{Var}\bigg(\sum_{j=1}^{n}u_k^j X_0(t^j)\bigg)\bigg]\,d\overline{u}_k\bigg\}\,d\overline{t}$$
$$=\int_{D^n}\Big[\mathrm{detCov}\left(X_0(t^1),\ldots,X_0(t^n)\right)\Big]^{-\frac{d}{2}}\,d\overline{t},$$

where $\overline{u}_k=(u_k^1,\ldots,u_k^n)\in\mathbb{R}^n$, $\overline{t}=(t^1,\ldots,t^n)$ and the equality follows from the fact that for any positive definite $n\times n$ matrix $\Gamma$,

$$\int_{\mathbb{R}^n}\frac{[\det(\Gamma)]^{1/2}}{(2\pi)^{n/2}}\exp\Big(-\frac{1}{2}x'\Gamma x\Big)\,dx=1.$$

By using the fact that for any Gaussian vector $(Z_1, \ldots, Z_n)$

$$\det \mathrm{Cov}(Z_1, \ldots, Z_n) = \mathrm{Var}(Z_1) \prod_{j=1}^{n} \mathrm{Var}(Z_j | Z_1, \ldots, Z_{j-1}).$$

and Condition (A4) we derive

$$\begin{aligned}
\mathbb{E}\big[L(x, D)^n\big] &\le c^n \int_{D^n} \prod_{j=1}^{n} \Bigg[ \sum_{\ell=1}^{N} \min_{0 \le s \le j-1} |t_\ell^j - t_\ell^s|^{2H_\ell} \Bigg]^{-\frac{d}{2}} d\bar{t} \\
&\le c^n \int_{D^n} \prod_{j=1}^{n} \Bigg[ \sum_{\ell=1}^{\tau} \min_{0 \le s \le j-1} |t_\ell^j - t_\ell^s|^{2H_\ell} \Bigg]^{-\frac{d}{2}} d\bar{t}.
\end{aligned}$$

To estimate the last integral, we will integrate in the order of $dt_1^n, \ldots, dt_N^n, \ldots, dt_1^1, \ldots, dt_N^1$. In Case 1, if $\tau = 1$, which implies that $H_1 d < 1$, we apply Lemma 6.3 and Lemma 2.3 in Xiao (1997) to derive

$$
\begin{aligned}
&\int_D \frac{dt_1^n \cdots dt_N^n}{\Big(\min\limits_{0 \le s \le n-1} |t_1^n - t_1^s|^{2H_1}\Big)^{d/2}} \\
&= (2r)^{\sum_{\ell=2}^N \frac{1}{H_\ell}} \int_{a_1 - r^{\frac{1}{H_1}}}^{a_1 + r^{\frac{1}{H_1}}} \frac{dt_1^n}{\min\limits_{0 \le s \le n-1} |t_1^n - t_1^s|^{H_1 d}} \\
&\le c\, n^{H_1 d}\, r^{\sum_{\ell=1}^N \frac{1}{H_\ell} - d} \\
&= c\, n^{\eta_1}\, r^{\alpha}.
\end{aligned}
$$

If $\tau > 1$, since $H_1 d > 1$, we apply Lemma 6.3 with $A = \sum_{\ell=2}^{\tau} \min_{0\le s\le n-1} |t_\ell^n - t_\ell^s|^{2H_\ell}$ and $p = 1$ at first to derive

$$\int_{a_1 - r^{\frac{1}{H_1}}}^{a_1 + r^{\frac{1}{H_1}}} \frac{dt_1^n}{\Big(\min\limits_{0\le s\le n-1} |t_1^n - t_1^s|^{2H_1} + \sum\limits_{\ell=2}^{\tau} \min\limits_{0\le s\le n-1} |t_\ell^n - t_\ell^s|^{2H_\ell}\Big)^{d/2}} \le \frac{c\, n}{\Big(\sum\limits_{\ell=2}^{\tau} \min\limits_{0\le s\le n-1} |t_\ell^n - t_\ell^s|^{H_\ell}\Big)^{d - \frac{1}{H_1}}}.$$

Since $H_{\tau-1}\big(d - \sum_{\ell=1}^{\tau-2} \frac{1}{H_\ell}\big) > 1$, we can apply Lemma 6.3 repeatedly for $\tau - 1$ many times to get

$$\int_D \frac{dt_1^n \cdots dt_N^n}{\left(\sum_{\ell=1}^{\tau} \min_{0\le s\le n-1} |t_\ell^n - t_\ell^s|^{2H_\ell}\right)^{d/2}} \le c\, n^{\tau-1} r^{\sum_{\ell=\tau+1}^{N} \frac{1}{H_\ell}}$$

$$\times \int_{a_\tau - r^{\frac{1}{H_\tau}}}^{a_\tau + r^{\frac{1}{H_\tau}}} \frac{dt_\tau^n}{\left(\min_{0\le s\le n-1} |t_\tau^n - t_\tau^s|^{H_\tau}\right)^{d-\sum_{\ell=1}^{\tau-1} \frac{1}{H_\ell}}}.$$

Notice that $H_\tau\left(d-\sum_{\ell=1}^{\tau-1}\frac{1}{H_\ell}\right) < 1$, by applying Lemma 2.3 in Xiao (1997), we derive

$$\int_D \frac{dt_1^n\cdots dt_N^n}{\left(\sum_{\ell=1}^{\tau}\min_{0\le s\le n-1}|t_\ell^n - t_\ell^s|^{2H_\ell}\right)^{d/2}}$$
$$\le c\, n^{\tau-1+H_\tau\left(d-\sum_{\ell=1}^{\tau-1}\frac{1}{H_\ell}\right)}\, r^{\sum_{\ell=1}^{N}\frac{1}{H_\ell}-d} = c\, n^{\eta_\tau} r^{\alpha}.$$

By iterating the procedure for integrating $dt_1^{n-1},\ldots,dt_N^{n-1}$ and so on, we obtain (5) for Case 1.

The rest of the proof (Cases 2 & 3) of Lemma 6.1 is similar and thus omitted.

The proof of Lemma 6.2 is more complicated, and is omitted here (please see Wu and X. 2011).

# 6.2. Hölder conditions for local times

Lemmas 6.1 and 6.2 can be applied to derive local and uniform Hölder conditions for the maximum local time $L^*(D) = \sup_{x \in \mathbb{R}^d} L(x, D)$.

**Theorem 6.3 [Wu and X., 2011]**

There exists a constant $c_{6,3} > 0$ such that for every $a \in T$,

$$\limsup_{r \to 0} \frac{L^*(\overline{B}_\rho(a, r))}{\varphi_1^\rho(r)} \le c_{6,3}, \qquad \text{a.s.} \quad \text{Cases 1 \& 2,}$$

$$\limsup_{r \to 0} \frac{L^*(\overline{B}_\rho(a, r))}{\varphi_2^\rho(r)} \le c_{6,3}, \qquad \text{a.s.} \quad \text{Case 3,}$$

where $\overline{B}_\rho(a, r) \subset I$ is the $\rho$-ball and where

$$\varphi_1^\rho(r) = r^\alpha \left( \log\log(1/r)\right)^{\eta_\tau},$$
$$\varphi_2^\rho(r) = r^\alpha \left( \log\log(1/r)\right)^{\tau-1} \log\log\log\left(1/r\right).$$

To state the uniform Hölder condition, let

$$\Phi_1^\rho(r) = r^\alpha \left( \log(1/r)\right)^{\eta_\tau},$$
$$\Phi_2^\rho(r) = r^\alpha \left( \log(1/r)\right)^{\tau-1} \log\log\left(1/r\right).$$

## Theorem 6.4 [Wu and X., 2011]

$$\limsup_{r\to 0}\sup_{a\in T}\frac{L^*(\overline{B}_\rho(a,r))}{\Phi_1^\rho(r)}\le c_{6,4}, \qquad \text{a.s.} \quad \text{Cases 1 \& 2,}$$

$$\limsup_{r\to 0}\sup_{a\in T}\frac{L^*(\overline{B}_\rho(a,r))}{\Phi_2^\rho(r)}\le c_{6,4}, \qquad \text{a.s.} \quad \text{Case 3}$$

Theorems 6.3 and 6.4 can be applied to derive lower bounds for Chung-type LIL and modulus of non-differentiability for $X$.

Unless $H_1 = \cdots = H_N$, it is not known whether the Hölder conditions for the local times are optimal (even though we believe they are, at lease in Cases 1 & 2).

# 6.3. Optimal Hölder conditions

We can establish optimal Hölder conditions for the local times under strong local nondeterminism. This is done in Khoshnevisan, Lee, and X. (2021).

**Condition (A4′) [strong local nondeterminism]**

There exists a constant $c > 0$ such that $\forall\ n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\Big(X_0(u) \,\Big|\, X_0(t^1), \ldots, X_0(t^n)\Big) \geq c \min_{1 \leq k \leq n} \rho(u,\, t^k)^2, \tag{9}$$

where $\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}$.

## Lemma 6.4

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (2) such that (3) and (A4′) hold. If $Q > d$, then there exists a finite constant $C$ such that for all Borel subsets $S$ of $T$, for all $x \in \mathbb{R}^d$ and all integers $n \geq 1$, we have

$$\mathbb{E}\big[L(x, D)^n\big] \leq C^n (n!)^{d/Q} \lambda_N(D)^{n(1-d/Q)}.$$

In particular, for all $a \in T$ and $r \in (0, 1)$ with $B_\rho(a, r) \subset T$, we have

$$\mathbb{E}\big[L(x, B_\rho(a, r))^n\big] \leq C^n (n!)^{d/Q} r^{n(Q-d)}.$$

## Lemma 6.5

Under the conditions of Lemma 6.4, there exist constants $C$ and $K$ such that for all $\gamma \in (0,1)$ small enough, for all Borel sets $D \subseteq T$, for all $x, y \in \mathbb{R}^d$, for all even integers $n \geq 2$, we have

$$\mathbb{E}\Big[(L(x,S) - L(y,S))^n\Big] \leq C^n |x-y|^{n\gamma} (n!)^{d/Q+K\gamma} \lambda_N(D)^{n(1-(d+\gamma)/Q)}.$$

In particular, for all $a \in T$, $0 < r < 1$ with $B_\rho(a,r) \subset T$, we have

$$\mathbb{E}\Big[(L(x,B_\rho(a,r)) - L(y,B_\rho(a,r)))^n\Big] \leq C^n |x-y|^{n\gamma} (n!)^{d/Q+K\gamma} r^{n(Q-d-\gamma)}.$$

Conditions (A4) and (A4$'$) have different effects on the Hölder conditions for the local times of $X$. We can compare the following Hölder conditions with Theorem 6.4.

## Theorem 6.5 [Khoshnevisan, Lee, and X. (2021)]

Under the conditions of Lemma 6.4, there exist finite constants $C$ and $C'$ such that for any $t \in T$,

$$\limsup_{r \to 0} \frac{L^*(B_\rho(t, r))}{\varphi_3^\rho(r)} \le C \quad \text{a.s.} \tag{10}$$

and

$$\limsup_{r \to 0} \sup_{t \in T} \frac{L^*(B_\rho(t, r))}{\Phi_3^\rho(r)} \le C' \quad \text{a.s.} \tag{11}$$

where $\varphi_3^\rho(r) = r^\alpha (\log\log(1/r))^{d/Q}$ and

$$\Phi_3^\rho(r) = r^\alpha (\log(1/r))^{d/Q}.$$

Recall that, under Conditions (A1) and (A4′), Chung's LIL for $X$ holds at $t \in T$ and $X$ has an exact modulus of non-differentiability.
By using these results and the following inequality: For any $t \in T$,

$$\begin{aligned}\lambda_N\big(B_\rho(t,r)\big) &= \int_{\overline{X\left(B_\rho(t,r)\right)}} L(x, B_\rho(t,r))\,dx \\ &\le L^*(B_\rho(t,r)) \cdot \Big(\sup_{s,t\in B_\rho(t,r)} |X(s)-X(t)|\Big)^d,\end{aligned}$$

we derive that the Hölder conditions in Theorem 6.5 are optimal.

## Theorem 6.6 [Khoshnevisan, Lee, and X. (2021)]

Let $X = \{X(t),\, t \in \mathbb{R}^N\}$ be a centered Gaussian random field defined by (2) such that $X_0$ satisfies Conditions (A1) and (A4′). There exist positive constants $K$ and $K'$ such that for any $t \in T$,

$$\limsup_{r\to 0} \frac{L^*(B_\rho(t,r))}{\varphi_3^\rho(r)} \ge K \quad \text{a.s.} \tag{12}$$

$$\limsup_{r\to 0} \sup_{t\in T} \frac{L^*(B_\rho(t,r))}{\Phi_3^\rho(r)} \ge K' \quad \text{a.s.} \tag{13}$$

The results in this lecture can be conveniently applied to the solutions of stochastic heat and wave equations with the Gaussian noise that is white in time and colored in space, as well as some fractional-colored noises.

Thank you!