Lecture 5. Fractal Properties of Gaussian Random Fields

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Outline

- An introduction to fractal geometry
  - Hausdorff measure and Hausdorff dimension
  - Packing measure and packing dimension
- Exact Hausdorff measure functions for the range of fBm
- Exact packing measure functions for the range of fBm
- Chung's LIL for fBm and its exceptional sets

5.1. Introduction to fractal geometry

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. It generates many random sets, for example,

- Range $\quad X\left([0,1]^N\right) = \left\{X(t) : t \in [0,1]^N\right\}$
- Graph $\quad \mathrm{Gr}X\left([0,1]^N\right) = \left\{(t, X(t)) : t \in [0,1]^N\right\}$
- Level set $\quad X^{-1}(x) = \left\{t \in \mathbb{R}^N : X(t) = x\right\}$
- Excursion set $X^{-1}(F) = \left\{t \in \mathbb{R}^N : X(t) \in F\right\}$, $\forall F \subseteq \mathbb{R}^d$,
- The set of self-intersections, ...

In order to study them, we need some tools such as Hausdorff dimension and packing dimension from fractal geometry.

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5.1 Definitions of Hausdorff measure and dimension

Let Φ be the class of functions $\varphi : (0, \delta) \to (0, \infty)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ such that

$$\frac{\varphi(2s)}{\varphi(s)} \le K \quad \text{for } 0 < s < \frac{1}{2}\delta.$$

A function φ in Φ is often called a *measure function* or *gauge function*.

For example, $\varphi(s) = s^\alpha$ $(\alpha > 0)$ and $\varphi(s) = s^\alpha \log\log(1/s)$ are measure functions.

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Given $\varphi \in \Phi$, the *φ-Hausdorff measure* of $E \subseteq \mathbb{R}^d$ is defined by

$$\varphi\text{-}m(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i),\ r_i < \varepsilon \right\}, \qquad (1)$$

where $B(x, r)$ denotes the open ball of radius r centered at x. The sequence of balls satisfying the two conditions on the right-hand side of (1) is called an ε-covering of E.

It can be shown that φ-m is a metric outer measure and all Borel sets in $\mathbb{R}^d$ is φ-m measurable.

A function $\varphi \in \Phi$ is called *an exact Hausdorff measure function for E* if $0 < \varphi\text{-}m(E) < \infty$.

If $\varphi(s) = s^\alpha$, we write φ-$m(E)$ as $\mathcal{H}_\alpha(E)$.
The *Hausdorff dimension* of E is defined by

$$\begin{aligned}\dim_{\mathrm{H}} E &= \inf\left\{\alpha > 0 : \ \mathcal{H}_\alpha(E) = 0\right\} \\ &= \sup\left\{\alpha > 0 : \ \mathcal{H}_\alpha(E) = \infty\right\},\end{aligned}$$

Convention: $\sup \varnothing := 0$.

Hausdorff dimension has the following properties:

1. $E \subseteq F \subseteq \mathbb{R}^d \ \Rightarrow \dim_{\mathrm{H}} E \le \dim_{\mathrm{H}} F \le d$.
2. (σ-stability):

$$\dim_{\mathrm{H}}\left(\bigcup_{j=1}^{\infty} E_j\right) = \sup_{j\ge 1} \dim_{\mathrm{H}} E_j.$$

An upper density theorem

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *upper φ-density* of μ at $x \in \mathbb{R}^d$ is defined as

$$\overline{D}_{\mu}^{\varphi}(x) = \limsup_{r \to 0} \frac{\mu(B(x,r))}{\varphi(2r)}.$$

Lemma 5.1 [Rogers and Taylor, 1961]

Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have

$$K^{-1}\mu(E) \inf_{x \in E} \left\{\overline{D}_{\mu}^{\varphi}(x)\right\}^{-1} \le \varphi\text{-}m(E) \le K\|\mu\| \sup_{x \in E} \left\{\overline{D}_{\mu}^{\varphi}(x)\right\}^{-1}.$$

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5.2 Packing measure and packing dimension

They were introduced by Tricot (1982), Taylor and Tricot (1985). For any $\varphi \in \Phi$ and $E \subseteq \mathbb{R}^d$, define

$$\varphi\text{-}P(E) = \lim_{\varepsilon \to 0} \sup \Big\{ \sum_i \varphi(2r_i) : \{\overline{B}(x_i, r_i)\} \text{ is an } \varepsilon\text{-packing} \Big\}.$$

Here ε-packing means that the balls are disjoint, $x_i \in E$ and $r_i \le \varepsilon$.

The packing measure φ-p of E is defined as:

$$\varphi\text{-}p(E) = \inf\Big\{ \sum_n \varphi\text{-}P(E_n) : \; E \subseteq \bigcup_n E_n \Big\}.$$

A function $\varphi \in \Phi$ is called *an exact packing measure function for E* for E if $0 < \varphi\text{-}p(E) < \infty$.

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If $\varphi(s) = s^{\alpha}$, we write φ-$p(E)$ as $\mathcal{P}_{\alpha}(E)$. The packing dimension of E is defined as:

$$\dim_{P} E = \inf\{\alpha > 0 : \ \mathcal{P}_{\alpha}(E) = 0\}.$$

Comparison between $\dim_{H}$ and $\dim_{P}$:

For any $\varphi \in \Phi$ and $E \subseteq \mathbb{R}^d$,

$$\varphi\text{-}m(E) \le \varphi\text{-}p(E), \qquad \dim_{H} E \le \dim_{P} E.$$

A lower density theorem

For any Borel measure μ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *lower φ-density* of μ at $x \in \mathbb{R}^d$ is defined as

$$\underline{D}_{\mu}^{\varphi}(x) = \liminf_{r \to 0} \frac{\mu(B(x,r))}{\varphi(2r)}.$$

Lemma 5.2 [Taylor and Tricot, 1985]

Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure μ on $\mathbb{R}^d$ with $0 < \|\mu\| \hat{=} \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have

$$K^{-1}\mu(E) \inf_{x \in E} \left\{\underline{D}_{\mu}^{\varphi}(x)\right\}^{-1} \le \varphi\text{-}p(E) \le K\|\mu\| \sup_{x \in E} \left\{\underline{D}_{\mu}^{\varphi}(x)\right\}^{-1}.$$

Example: Cantor's set

Let C denote the standard ternary Cantor set in $[0\,,1]$. At the nth stage of its construction, C is covered by 2^n intervals of length/diameter 3^{-n} each.

It can be proved that

$$\dim_{_{\rm H}} C = \dim_{_{\rm P}} C = \log_3 2.$$

By using the upper and lower density theorems, one can prove that

$$0 < \mathcal{H}_{\log_3 2}(C) \le \mathcal{P}_{\log_3 2}(C) < \infty.$$

Example: the range of Brownian motion

Let $B([0,1])$ be the image of Brownian motion in $\mathbb{R}^d$. Lévy (1948) and Taylor (1953) proved that

$$\dim_{\rm H} B([0,1]) = \min\{d, 2\} \qquad \text{a.s.}$$

Ciesielski and Taylor (1962), Ray and Taylor (1964) proved that

$$0 < \varphi_d\text{-}m\big(B([0,1])\big) < \infty \quad \text{a.s.},$$

where

$$\begin{aligned} \varphi_1(r) &= r, \\ \varphi_2(r) &= r^2 \log(1/r) \log\log\log(1/r), \\ \varphi_d(r) &= r^2 \log\log(1/r), \qquad \text{if } d \geq 3. \end{aligned}$$

Taylor and Tricot (1985) proved that

$$\dim_{\rm P} B([0,1]) = \min\{d, 2\}$$

and, if $d \geq 3$, then

$$0 < \psi\text{-}p\big(B([0,1])\big) < \infty \quad \text{a.s.},$$

where $\psi(r) = r^2 / \log\log(1/r)$.

LeGall and Taylor (1986) proved that, if $d = 2$, then for any measure function φ, either $\varphi\text{-}p\big(B([0,1])\big) = 0$ or ∞.

Question: How to extend the above results to Gaussian random fields?

5.3. Exact Hausdorff and packing measure functions for fractional Brownian motion

For $H \in (0,1)$, the fBm $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ with index H is a centered (N,d)-Gaussian field whose covariance function is

$$\mathbb{E}\big[B_i^H(s)B_j^H(t)\big] = \frac{1}{2}\,\delta_{ij}\,\left(|s|^{2H} + |t|^{2H} - |s-t|^{2H}\right),$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

- When $N = 1$ and $H = 1/2$, B^H is Brownian motion.
- B^H is H-self-similar and has stationary increments.

Kahane (1985) proved that

$$\dim_{\rm H} B^H([0,1]^N) = \min\left\{d,\ \frac{N}{H}\right\} \qquad \text{a.s.}$$

5.3.1 Exact Hausdorff measure functions for $B^H([0,1]^N)$ and $\mathrm{Gr}B^H([0,1]^N)$

Theorem 5.1 [Talagrand (1995, 1998)]

Let $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$.

(i). If $N < Hd$, then

$$K^{-1} \le \varphi_1\text{-}m\big(B^H([0,1]^N)\big) \le K, \qquad \text{a.s.}$$

where $\varphi_1(r) = r^{\frac{N}{H}} \log\log(1/r)$.

(ii). If $N = Hd$, then $\varphi_2\text{-}m\big(B^H([0,1]^N)\big)$ is σ-finite, where

$$\varphi_2(r) = r^d \log(1/r) \log\log\log(1/r).$$

Theorem 5.2 [X. (1997)]

Let $B^H = \{B^H(t),\ t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$.

(i). If $N < Hd$, then

$$K^{-1} \le \varphi_1\text{-}m\big(\mathrm{Gr}B^H([0,1]^N)\big) \le K, \qquad \text{a.s.}$$

where $\varphi_1(r) = r^{\frac{N}{H}} \log\log(1/r)$.

(ii). If $N > Hd$, then

$$K^{-1} \le \varphi_3\text{-}m\big(\mathrm{Gr}B^H([0,1]^N)\big) \le K, \qquad \text{a.s.,}$$

where

$$\varphi_2(r) = r^{N+(1-H)d}\big(\log\log(1/r)\big)^{Hd/N}.$$

5.3.2. Exact packing measure function for $B^H([0,1]^N)$

Theorem 5.3 (Xiao, 1996, 2003)

Let $B^H = \{B^H(t),\, t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$. If $N < Hd$, then there exists a finite constant $K \geq 1$ such that

$$K^{-1} \leq \varphi_4\text{-}p(B^H([0,1]^N)) \leq K, \quad \text{a.s.}$$

where $\varphi_4(r) = r^{\frac{N}{H}}\big(\log\log(1/r)\big)^{-N/(2H)}$.

For proving Theorem 5.3, one needs to study the liminf behavior of the sojourn measure

$$T(r) = \int_{\mathbb{R}^N} \mathbf{1}_{\{|B^H(t)| \le r\}} dt.$$

A key ingredient is the following small ball probability estimate for $T(1)$.

Lemma 5.4 [Xiao, 1996, 2003]

Assume that $N < Hd$. Then there exists a positive and finite constant $K \ge 1$, depending only on H, N and d such that for any $0 < \varepsilon < 1$,

$$\exp\Big(-\frac{K}{\varepsilon^{2H/N}}\Big) \le \mathbb{P}\{T(1) < \varepsilon\} \le \exp\Big(-\frac{1}{K\varepsilon^{2H/N}}\Big).$$

This leads to the following Chung's LIL for $T(r)$.

Theorem 5.5 (Xiao, 1996, 2003)

If $N < Hd$, then with probability one,

$$\liminf_{r \to 0} \frac{T(r)}{\varphi_4(r)} = K, \tag{2}$$

where $0 < K < \infty$ is a constant depending on H, N and d only.

By the stationarity of increments of B^H and the lower density theorem, we derive the lower bound in Theorem 5.3.

The proof of upper bound in Theorem 5.3 requires a different argument.

5.4 Exact Hausdorff measure function for the ranges of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$:

$$X(t) = \big(X_1(t), \ldots, X_d(t)\big), \qquad t \in \mathbb{R}^N, \tag{3}$$

where $X_1, \ldots, X_d$ are independent copies of a centered Gaussian field X_0. We assume that X_0 satisfies the following conditions from Lecture 3.

Assumption (A1)

Consider a compact interval $T \subset \mathbb{R}^N$. There exists a Gaussian random field $\{v(A,t) : A \in \mathscr{B}(\mathbb{R}_+),\ t \in T\}$ such that
(a) For all $t \in T$, $A \mapsto v(A,t)$ is a real-valued Gaussian noise, $v(\mathbb{R}_+, t) = X_0(t)$, and $v(A, \cdot)$ and $v(B, \cdot)$ are independent whenever A and B are disjoint.

Assumption (A1) (continued)

(b) There are constants $a_0 \geq 0$ and $\gamma_j > 0$, $j = 1, \ldots, N$ such that for all $a_0 \leq a \leq b \leq \infty$ and $s = (s_1, \ldots, s_N)$, $t = (t_1, \ldots, t_N) \in T$,

$$\Big\| v([a,b),s) - X_0(s) - v([a,b),t) + X_0(t) \Big\|_{L^2} \leq C\Big(\sum_{j=1}^{N} a^{\gamma_j} |s_j - t_j| + b^{-1} \Big), \tag{4}$$

where $\|Y\|_{L^2} = \big[\mathbb{E}(Y^2)\big]^{1/2}$ for a random variable Y and

$$\Big\| v([0,a_0),s) - v([0,a_0),t) \Big\|_{L^2} \leq C \sum_{j=1}^{N} |s_j - t_j|. \tag{5}$$

Condition (A4′) [strong local nondeterminism]

There exists a constant $c > 0$ such that $\forall\, n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X_0(u) \,\big|\, X_0(t^1), \ldots, X_0(t^n)\big) \geq c \min_{1\leq k\leq n} \rho(u,\, t^k)^2, \tag{6}$$

where $\rho(s,t)$ is the metric on $\mathbb{R}^N$ defined by

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j},$$

and where $H_j = (\gamma_j + 1)^{-1}$ $(j = 1, \ldots, N)$.

These conditions are weaker than those in Luan and X. (2012).

Theorem 5.6

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian field with values in $\mathbb{R}^d$ such that X_0 satisfies (A1) and (A4′).

(i). If $Q = \sum_{j=1}^{N} H_j^{-1} < d$, then

$$K^{-1} \le \varphi_5\text{-}m\big(X([0,1]^N)\big) \le K, \qquad \text{a.s.}, \tag{7}$$

where $\varphi_5(r) = r^Q \log\log(1/r)$.

(ii). If $Q > d$, then $X([0,1]^N)$ has positive d-dimensional Lebesgue measure a.s.

The problem to determine the exact Hausdorff measure function for $X([0,1]^N)$ in the "critical case" $Q = d$ is open.

Proof of Theorem 5.6

The lower bound in (7) is proved by using the upper density theorem in Lemma 5.1. A natural measure on $X([0,1]^N)$ is the sojourn measure

$$\mu(B) = \lambda_N\big\{t \in [0,1]^N : X(t) \in B\big\}, \quad \forall\, B \in \mathcal{B}(R^d),$$

where λ_N denotes the Lebesgue measure on $\mathbb{R}^N$.
For any $0 < r < 1$ and $t^0 \in [0,1]^N := I$, we consider

$$\mu\big(B(X(t^0), r)\big) = \int_I \mathbf{1}_{\{|X(t)-X(t^0)|\le r\}}\, dt,$$

which is the sojourn time of X in the ball $B(X(t^0), r)$.

The following moment estimate is essential for determining the asymptotic behavior of $\mu\big(B(X(t^0), r)\big)$ as $r \to 0$.

Lemma 5.5

If $d > Q$, then there is a finite constant C such that for every $t^0 \in I$ and all integers $n \geq 1$,

$$\mathbb{E}\left[\mu\big(B(X(t^0), r)\big)^n\right] \leq C^n n!\, r^{Qn}.$$

Proof. For $n = 1$, by Fubini's theorem we have

$$\begin{aligned}
\mathbb{E}\big[\mu\big(B(X(t^0), r)\big)\big] &= \int_I \mathbb{P}\left\{|X(t) - X(t^0)| < r\right\} dt \\
&\leq \int_I \min\left\{1, c\left(\frac{r}{\rho(t, t^0)}\right)^d\right\} dt \\
&= \int_{\{t:\rho(t,t^0)\leq cr\}\cap I} dt + c \int_{\{t:\rho(t,t^0)>cr\}\cap I} \left(\frac{r}{\rho(t, t^0)}\right)^d dt.
\end{aligned}$$

It is elementary to verify that

$$\mathbb{E}\big[\mu\big(B(X(t^0),r)\big)\big] \le c r^Q.$$

For $n \ge 2$,

$$\mathbb{E}\big[\mu\big(B(X(t^0),r)\big)^n\big] = \int_{I^n} \mathbb{P}\big\{\big|X(t^j)-X(t^0)\big| < r, 1 \le j \le n\big\} dt^1 \cdots dt^n.$$

It is sufficient to consider $t^1, \cdots, t^n \in I$ that satisfy

$$t^j \ne t^0, \quad \text{for } j = 1, \cdots, n \quad \text{and} \quad t^j \ne t^k \quad \text{for} \quad j \ne k.$$

By Condition (A4′), we have

$$\begin{aligned} &\mathrm{Var}\big(X_0(t^n) - X_0(t^0)\big|X_0(t^1) - X_0(t^0), \cdots, X_0(t^{n-1}) - X_0(t^0)\big) \\ &\ge \mathrm{Var}\big(X_0(t^n)\big|X_0(t^0), X_0(t^1), \cdots, X_0(t^{n-1})\big) \\ &\ge c \min_{0 \le k \le n-1} \rho(t^n, t^k)^2. \end{aligned} \tag{8}$$

Since conditional distributions in Gaussian processes are still Gaussian, it follows from Anderson's inequality and (8) that

$$\int_I \mathbb{P}\Big\{\big|X(t^n) - X(t^0)\big| < r \big| X(t^1) - X(t^0), \cdots, X(t^{n-1}) - X(t^0)\Big\} dt^n$$

$$\leq c \int_I \sum_{k=0}^{n-1} \min\left\{1, c\left(\frac{r}{\rho(t^n, t^k)}\right)^d\right\} dt^n$$

$$\leq c\, n \int_I \min\left\{1, c\left(\frac{r}{\rho(t^n, 0)}\right)^d\right\} dt^n$$

$$\leq c\, n r^Q.$$

Iterating the procedure proves Lemma 5.5.

From Lemma 5.5 and the Borel-Cantelli lemma, we can prove the following law of the iterated logarithm for the sojourn measure of X.

Proposition 5.1

For every $t^0 \in I$, we have

$$\limsup_{r \to 0} \frac{\mu\big(B(X(t^0), r)\big)}{\varphi_5(r)} \le C < \infty, \quad a.s.$$

This and Fubini's theorem yield: a.s.

$$\limsup_{r \to 0} \frac{\mu\big(B(X(t^0), r)\big)}{\varphi_5(r)} \le C \quad a.e. \ t^0 \in I.$$

Hence, the lower bound in (7) follows from Lemma 5.1.

For proving the upper bound in (7), we need the following small ball probability estimates.

Lemma 5.6 [X. (2009)]

Under the conditions of Theorem 5.6, There exist constants c and c' such that for all $t^0 \in I = [0, 1]^N$ and $0 < \varepsilon < r$,

$$\exp\left(-c'\left(\frac{r}{\varepsilon}\right)^Q\right) \le \mathbb{P}\Big\{\sup_{t\in I:\rho(t,t^0)\le r} |X(t)-X(t_0)| \le \varepsilon\Big\} \le \exp\left(-c\left(\frac{r}{\varepsilon}\right)^Q\right).$$

The main estimate is given in the following lemma.

Proposition 5.2

Assume that the conditions of Theorem 5.6 hold. There exist positive constants δ_0 and C such that for any $t^0 \in I$ and $0 < r_0 \leq \delta_0$, we have

$$\mathbb{P}\Big\{\exists\, r \in [r_0^2, r_0],\ \sup_{t \in I: \rho(t,t^0) \leq r} |X(t) - X(t^0)| \leq Cr\Big(\log\log(1/r)\Big)^{-1/Q}\Big\} \geq 1 - \exp\Big(-\Big(\log(1/r_0)\Big)^{1/2}\Big).$$

Proof. The method of proof comes form Talagrand (1995). We provide the main steps. Let $U > 1$ be a number whose value will be determined later. For $k \geq 0$, let $r_k = r_0 U^{-2k}$. Consider the largest integer k_0 such that

$$k_0 \leq \frac{\log(1/r_0)}{2\log U}.$$

Thus, for $k \le k_0$ we have $r_0^2 \le r_k \le r_0$. It thereby suffices to prove that

$$\mathbb{P}\Big\{ \exists k \le k_0, \sup_{t\in I:\rho(t,t^0)\le r_k} |X(t) - X(t^0)| \le c\, r_k \Big(\log\log \frac{1}{r_k}\Big)^{-1/Q} \Big\} \ge 1 - \exp\bigg(- \Big(\log \frac{1}{r_0}\Big)^{1/2} \bigg). \tag{9}$$

Let $a_k = r_0^{-1} U^{2k-1}$ and we define for $k = 0, 1, \cdots$

$$X_{0,k}(t) = v([a_k, a_{k+1}),\ t)$$

and

$$\widehat{X}_k(t) = \big(X_{1,k}(t), \cdots, X_{d,k}(t)\big),$$

where $X_{1,k}(t), \cdots, X_{d,k}(t)$ are independent copies of $X_{0,k}(t)$. It follows that $X_1 - X_{1,k}, \cdots, X_d - X_{d,k}$ are independent copies of $X_0 - X_{0,k}$.

The Gaussian random fields $\widehat{X}_0, \widehat{X}_1, \cdots$ are independent. By Lemma 5.6 we can find a constant $c > 0$ such that, if r_0 is small enough, then for each $k \ge 0$

$$\begin{aligned} &\mathbb{P}\Big\{ \sup_{t \in I: \rho(t,t^0) \le r_k} \big|\widehat{X}_k(t) - \widehat{X}_k(t^0)\big| \le c\, r_k \big(\log\log(1/r_k)\big)^{-1/Q} \Big\} \\ &\ge \exp\Big(- \frac{1}{4} \log\log(1/r_k)\Big) = \frac{1}{(\log 1/r_k)^{1/4}} \\ &\ge \big(2 \log 1/r_0\big)^{-1/4}. \end{aligned}$$

By the independence,

$$\begin{aligned} &\mathbb{P}\Big\{ \exists k \le k_0, \sup_{t \in I: \rho(t,t^0) \le r_k} \big|\widehat{X}_k(t) - \widehat{X}_k(t^0)\big| \le c\, r_k \Big(\log\log(1/r_k)\Big)^{-1/Q} \Big\} \\ &\ge 1 - \Big(1 - \frac{1}{(2 \log 1/r_0)^{1/4}}\Big)^{k_0} \ge 1 - \exp\Big(- \frac{k_0}{(2\log 1/r_0)^{1/4}}\Big), \end{aligned} \tag{10}$$

where the last inequality follows from $1 - x \le e^{-x}$ for all $x \ge 0$.

To deal with $\{X(t) - \widehat{X}_k(t)\}$, we claim that for any $u \geq c r_k U^{-\beta} \sqrt{\log U}$, where $\beta = \min\{{H_N}^{-1} - 1,\, 1\}$,

$$\mathbb{P}\Big\{ \sup_{t \in I: \rho(t,t^0) \leq r_k} \big| X(t) - \widehat{X}_k(t) - (X(t^0) - \widehat{X}_k(t^0)) \big| \geq u \Big\} \leq \exp\Big(- \frac{u^2}{c r_k^2 U^{-2\beta}} \Big). \tag{11}$$

To see this, it's enough to prove that (11) holds for X_0, by applying Lemma 3.3.
Consider $S = \{t \in I : \rho(t, t^0) \leq r_k\}$ and on S the distance

$$d(s,t) = \big\| X_0(s) - X_{0,k}(s) - (X_0(t) - X_{0,k}(t)) \big\|_{L^2}.$$

Then $d(s,t) \leq c \sum_{i=1}^N |s_i - t_i|^{H_i}$ and $N(S, d, \varepsilon) \leq c \big(r_k / \varepsilon \big)^Q$.

Now we estimate the d-diameter D of S. By Condition (A1), we have for any $s, t \in S$,

$$
\begin{aligned}
&\|X_0(s) - X_{0,k}(s) - (X_0(t) - X_{0,k}(t))\|_{L^2} \\
&\le C\left(\sum_{j=1}^{N} a_k^{H_j^{-1}-1}|s_j - t_j| + a_{k+1}^{-1}\right) \le C r_k U^{-\beta},
\end{aligned}
$$

where $\beta = \min\{H_N{}^{-1} - 1,\, 1\}$. Therefore, $D \le C r_k U^{-\beta}$. Notice that

$$
\begin{aligned}
\int_0^D \sqrt{\log N(S, d, \varepsilon)}\, d\varepsilon &\le c \int_0^{C r_k U^{-\beta}} \sqrt{\log r_k/\varepsilon}\, d\varepsilon \\
&\le c r_k \int_0^{C U^{-\beta}} \sqrt{\log 1/u}\, du \le c r_k U^{-\beta} \sqrt{\log U}.
\end{aligned}
$$

Hence (11) follows from Lemma 3.3.

Let $U = (\log 1/r_0)^{1/\beta}$. Then for $r_0 > 0$ small

$$U^\beta\,(\log U)^{-1/2} \ge \left(\log\log\frac{1}{r_0}\right)^{1/Q}.$$

Take $u = cr_k(\log\log 1/r_0)^{-1/Q}$. It follows from (11) that

$$\mathbb{P}\Big\{\sup_{t\in I:\rho(t,t^0)\le r_k}\big|X(t)-\widehat{X}_k(t)-\big(X(t^0)-\widehat{X}_k(t^0)\big)\big| \ge c\,r_k\big(\log\log\frac{1}{r_0}\big)^{-1/Q}\Big\}$$
$$\le \exp\Big(-\frac{cU^\beta}{(\log\log 1/r_0)^{2/Q}}\Big).$$

Combining this with (10), we get

$$\mathbb{P}\Big\{\exists k \le k_0, \sup_{\rho(t,t^0)\le r_k} \big|X(t) - X(t^0)\big| \le c\, r_k\big(\log\log(1/r_k)\big)^{-1/Q}\Big\}$$
$$\ge 1 - \exp\Big(-\frac{k_0}{(2\log 1/r_0)^{1/4}}\Big) - k_0 \exp\Big(-\frac{cU^\beta}{(\log\log 1/r_0)^{2/Q}}\Big).$$

This proves (9) and Proposition 5.2.

With Proposition 5.2 in hand, we proceed to construction of an economic covering for $X([0,1]^N)$.

For $k \geq 1$, consider the set

$$R_k = \Big\{ t \in [0,1]^N : \exists\, r \in [2^{-2k}, 2^{-k}] \text{ such that } \sup_{s \in I : \rho(s,t) \leq r} \big| X(s) - X(t) \big| \leq c\, r (\log\log \frac{1}{r})^{-1/Q} \Big\}.$$

By Lemma 5.7 we have that for every $t \in [0,1]^N$,

$$\mathbb{P}\{t \in R_k\} \geq 1 - \exp(-\sqrt{k/2}).$$

This and Fubini's theorem imply that

$$\mathbb{E}[\lambda_N(R_k)] \geq 1 - \exp(-\sqrt{k/2}).$$

Or

$$\mathbb{E}[\lambda_N(I \backslash R_k)] \leq \exp(-\sqrt{k/2}).$$

By Markov's inequality, we have

$$\begin{aligned}\mathbb{P}\Big\{\lambda_N(R_k) < 1-\exp(-\sqrt{k}/2)\Big\} &= \mathbb{P}\Big\{\lambda_N(I\backslash R_k) > \exp(-\sqrt{k}/2)\Big\}\\ &\le \frac{\mathbb{E}[\lambda_N(I\backslash R_k)]}{\exp(-\sqrt{k}/2)}\\ &\le \exp\bigg(-\Big(\frac{1}{\sqrt{2}}-\frac{1}{2}\Big)\sqrt{k}\bigg).\end{aligned}$$

Hence, by the Borel-Cantelli lemma, we have $\mathbb{P}(\Omega_1) = 1$, where

$$\Omega_1 = \Big\{\omega : \ \lambda_N(R_k) \ge 1-\exp(-\sqrt{k}/2) \text{ for all } k \text{ large enough}\Big\}.$$

On the other hand, by Lemma 3.3, we have $\mathbb{P}(\Omega_2) = 1$, where Ω_2 it the event that for every rectangle I_n of side-lengths $2^{-n/H_i}(i = 1, \cdots, N)$ that meets $[0, 1]^N$, we have

$$\sup_{s,t \in I_n} \left| X(t) - X(s) \right| \le C 2^{-n} \sqrt{n},$$

where $C > 0$ is a constant.
Now we show that for every $\omega \in \Omega_1 \cap \Omega_2$, we have

$$\varphi_5\text{-}m(X([0, 1]^N)) \le K < \infty, \qquad \text{a.s.}$$

For any $n \ge 1$, we divide $[0, 1]^N$ into 2^{nQ} disjoint (half open and half closed) rectangles of side-lengths $2^{-n/H_i}$ $(i = 1, \cdots, N)$. Denote by $I_n(x)$ the unique rectangle of side-lengths $2^{-n/H_i}(i = 1, \cdots, N)$ containing x.

Consider $k \geq 1$ such that

$$\lambda_N(R_k) \geq 1 - \exp(-\sqrt{k}/2).$$

For any $x \in R_k$ we can find the smallest integer n with $k \leq n \leq 2k$ such that

$$\sup_{s,t \in I_n(x)} \left| X(t) - X(s) \right| \leq c\, 2^{-n} (\log\log 2^n)^{-1/Q}. \tag{12}$$

Thus we have

$$R_k \subseteq V = \bigcup_{n=k}^{2k} V_n$$

and each V_n is a union of rectangles $I_n(x)$ satisfying (12). Notice that $X(I_n(x))$ can be covered by a ball of radius $r_n = c2^{-n}(\log\log 2^n)^{-1/Q}$.

Since $\varphi_5(2r_n) \le c 2^{-nQ} = c\lambda_N(I_n)$, we obtain

$$\sum_{n=k}^{2k} \sum_{I_n \in V_n} \varphi_5(2r_n) \le \sum_{n} \sum_{I_n \in V_n} c\lambda_N(I_n) = C\lambda_N(V) \le C. \qquad (13)$$

Thus $X(V)$ is contained in the union of a family of balls B_n of radius r_n with $\sum_n \varphi_5(2r_n) \le C$.

On the other hand, $[0,1]^N \backslash V$ is contained in a union of rectangles of side-lengths $2^{-q/H_i}(i = 1, \cdots, N)$ where $q = 2k+1$, none of which meets R_k. There can be at most

$$2^{Qq} \lambda_N([0,1]^N \backslash V) \le c 2^{Qq} \exp(-\sqrt{k}/2)$$

such rectangles.

Since $\omega \in \Omega_2$, for each of these rectangles I_q, $X(I_q)$ is contained in a ball of radius $c2^{-q}\sqrt{q}$.

Thus $X([0,1]^N \setminus V)$ can be covered by a sequence $\{B_n\}$ of balls of radius $r_n = c2^{-q}\sqrt{q}$ such that

$$\sum_n \varphi_5(2r_n) \le \Big(c2^{Qq} \exp(-\sqrt{k}/2)\Big)\Big(c2^{-qQ} q^{Q/2} \log\log(c2^q/\sqrt{q})\Big) \le 1 \tag{14}$$

for all k large enough. Since k can be arbitrarily large, it follows from (13) and (14) that

$$\varphi_5\text{-}m\big(X([0,1]^N)\big) \le K, \quad \text{a.s.}$$

This finishes the proof of Part (i) of Theorem 5.6.

Part (ii) is related to the existence of local times. A proof based on Fourier analysis will be given in Lecture 6.

If Condition (A4$'$) in Theorem 5.6 is replaced by (A4), then the exact Hausdorff measure function for $X([0, 1]^N)$ is different. See the recent paper of Lee (2021).

Thank you