Lecture 5. Fractal Properties of Gaussian Random Fields

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An introduction to fractal geometry
  - Hausdorff measure and Hausdorff dimension
  - Packing measure and packing dimension

Exact Hausdorff measure functions for the range of fBm

Exact packing measure functions for the range of fBm

Chung’s LIL for fBm and its exceptional sets
Let $X = \{X(t), \ t \in \mathbb{R}^N\}$ be a random field with values in $\mathbb{R}^d$. It generates many random sets, for example,

- **Range** $X([0, 1]^N) = \{X(t) : t \in [0, 1]^N\}$
- **Graph** $\text{Gr}X([0, 1]^N) = \{(t, X(t)) : t \in [0, 1]^N\}$
- **Level set** $X^{-1}(x) = \{t \in \mathbb{R}^N : X(t) = x\}$
- **Excursion set** $X^{-1}(F) = \{t \in \mathbb{R}^N : X(t) \in F\}$, $\forall F \subset \mathbb{R}^d$,
- The set of self-intersections, . . . .

In order to study them, we need some tools such as Hausdorff dimension and packing dimension from fractal geometry.
5.1. Introduction to fractal geometry

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In order to study them, we need some tools such as Hausdorff dimension and packing dimension from fractal geometry.
5.1 Definitions of Hausdorff measure and dimension

Let $\Phi$ be the class of functions $\varphi : (0, \delta) \rightarrow (0, \infty)$ which are right continuous, monotone increasing with $\varphi(0+) = 0$ and such that there exists a finite constant $K > 0$ such that

$$\frac{\varphi(2s)}{\varphi(s)} \leq K \quad \text{for} \quad 0 < s < \frac{1}{2} \delta.$$

A function $\varphi$ in $\Phi$ is often called a measure function or gauge function.

For example, $\varphi(s) = s^\alpha \ (\alpha > 0)$ and $\varphi(s) = s^\alpha \log \log(1/s)$ are measure functions.
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A function $\varphi$ in $\Phi$ is often called a *measure function* or *gauge function*. For example, $\varphi(s) = s^\alpha$ ($\alpha > 0$) and $\varphi(s) = s^\alpha \log \log(1/s)$ are measure functions.
Given \( \varphi \in \Phi \), the \( \varphi \)-Hausdorff measure of \( E \subseteq \mathbb{R}^d \) is defined by

\[
\varphi-m(E) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_i \varphi(2r_i) : E \subseteq \bigcup_{i=1}^{\infty} B(x_i, r_i), \ r_i < \varepsilon \right\},
\]

where \( B(x, r) \) denotes the open ball of radius \( r \) centered at \( x \). The sequence of balls satisfying the two conditions on the right-hand side of (1) is called an \( \varepsilon \)-covering of \( E \).

It can be shown that \( \varphi-m \) is a metric outer measure and all Borel sets in \( \mathbb{R}^d \) is \( \varphi-m \) measurable.

A function \( \varphi \in \Phi \) is called an exact Hausdorff measure function for \( E \) if \( 0 < \varphi-m(E) < \infty \).
If $\varphi(s) = s^\alpha$, we write $\varphi$-m$(E)$ as $\mathcal{H}_\alpha(E)$. The \textit{Hausdorff dimension} of $E$ is defined by

$$\dim_H E = \inf \{ \alpha > 0 : \mathcal{H}_\alpha(E) = 0 \} = \sup \{ \alpha > 0 : \mathcal{H}_\alpha(E) = \infty \},$$

Convention: $\sup \emptyset := 0$.

Hausdorff dimension has the following properties:

1. $E \subseteq F \subseteq \mathbb{R}^d \Rightarrow \dim_H E \leq \dim_H F \leq d$.
2. ($\sigma$-stability):

$$\dim_H \left( \bigcup_{j=1}^{\infty} E_j \right) = \sup_{j \geq 1} \dim_H E_j.$$
An upper density theorem

For any Borel measure $\mu$ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the *upper $\varphi$-density* of $\mu$ at $x \in \mathbb{R}^d$ is defined as

$$
D_{\mu}^\varphi(x) = \limsup_{r \to 0} \frac{\mu(B(x, r))}{\varphi(2r)}.
$$

**Lemma 5.1 [Rogers and Taylor, 1961]**

Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure $\mu$ on $\mathbb{R}^d$ with $0 < \|\mu\| = \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have

$$
K^{-1} \mu(E) \inf_{x \in E} \left\{ D_{\mu}^\varphi(x) \right\}^{-1} \leq \varphi-m(E) \leq K \|\mu\| \sup_{x \in E} \left\{ D_{\mu}^\varphi(x) \right\}^{-1}.
$$
An upper density theorem

For any Borel measure $\mu$ on $\mathbb{R}^d$ and $\varphi \in \Phi$, the upper $\varphi$-density of $\mu$ at $x \in \mathbb{R}^d$ is defined as

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**Lemma 5.1 [Rogers and Taylor, 1961]**

Given $\varphi \in \Phi$, $\exists K > 0$ such that for any Borel measure $\mu$ on $\mathbb{R}^d$ with $0 < \|\mu\| = \mu(\mathbb{R}^d) < \infty$ and every Borel set $E \subseteq \mathbb{R}^d$, we have

$$K^{-1} \mu(E) \inf_{x \in E} \left\{ \overline{D}_\mu^\varphi(x) \right\}^{-1} \leq \varphi\text{-}m(E) \leq K \|\mu\| \sup_{x \in E} \left\{ \overline{D}_\mu^\varphi(x) \right\}^{-1}.$$
5.2 Packing measure and packing dimension

They were introduced by Tricot (1982), Taylor and Tricot (1985). For any $\varphi \in \Phi$ and $E \subseteq \mathbb{R}^d$, define

$$
\varphi - P(E) = \lim_{\varepsilon \to 0} \sup \left\{ \sum_{i} \varphi(2r_i) : \{B(x_i, r_i)\} \text{ is an } \varepsilon \text{-packing} \right\}.
$$

Here $\varepsilon$-packing means that the balls are disjoint, $x_i \in E$ and $r_i \leq \varepsilon$.

The packing measure $\varphi - p$ of $E$ is defined as:

$$
\varphi - p(E) = \inf \left\{ \sum_{n} \varphi - P(E_n) : E \subseteq \bigcup_{n} E_n \right\}.
$$

A function $\varphi \in \Phi$ is called an exact packing measure function for $E$ if $0 < \varphi - p(E) < \infty$. 
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The packing measure $\varphi-p$ of $E$ is defined as:

$$\varphi-p(E) = \inf \left\{ \sum_n \varphi-P(E_n) : E \subseteq \bigcup_n E_n \right\}.$$ 

A function $\varphi \in \Phi$ is called \textit{an exact packing measure function for $E$} for $E$ if $0 < \varphi-p(E) < \infty$. 
If $\varphi(s) = s^\alpha$, we write $\varphi-p(E)$ as $\mathcal{P}_\alpha(E)$. The **packing dimension** of $E$ is defined as:

$$\dim_p E = \inf\{\alpha > 0 : \mathcal{P}_\alpha(E) = 0\}.$$ 

**Comparison between $\dim_H$ and $\dim_p$:**
For any $\varphi \in \Phi$ and $E \subseteq \mathbb{R}^d$,

$$\varphi-m(E) \leq \varphi-p(E), \quad \dim_H E \leq \dim_p E.$$
A lower density theorem

For any Borel measure \( \mu \) on \( \mathbb{R}^d \) and \( \varphi \in \Phi \), the lower \( \varphi \)-density of \( \mu \) at \( x \in \mathbb{R}^d \) is defined as

\[
D_{\mu}^\varphi(x) = \liminf_{r \to 0} \frac{\mu(B(x, r))}{\varphi(2r)}.
\]

Lemma 5.2 [Taylor and Tricot, 1985]

Given \( \varphi \in \Phi \), \( \exists K > 0 \) such that for any Borel measure \( \mu \) on \( \mathbb{R}^d \) with \( 0 < \| \mu \| = \mu(\mathbb{R}^d) < \infty \) and every Borel set \( E \subseteq \mathbb{R}^d \), we have

\[
K^{-1} \mu(E) \inf_{x \in E} \left\{ D_{\mu}^\varphi(x) \right\}^{-1} \leq \varphi-p(E) \leq K \| \mu \| \sup_{x \in E} \left\{ D_{\mu}^\varphi(x) \right\}^{-1}.
\]
Example: Cantor’s set

Let $C$ denote the standard ternary Cantor set in $[0, 1]$. At the $n$th stage of its construction, $C$ is covered by $2^n$ intervals of length/diameter $3^{-n}$ each. It can be proved that

$$\dim_H C = \dim_P C = \log_3 2.$$  

By using the upper and lower density theorems, one can prove that

$$0 < \mathcal{H}_{\log_3 2}(C) \leq \mathcal{P}_{\log_3 2}(C) < \infty.$$
Example: the range of Brownian motion

Let $B([0,1])$ be the image of Brownian motion in $\mathbb{R}^d$. Lévy (1948) and Taylor (1953) proved that

$$\dim_H B([0,1]) = \min\{d, 2\} \quad \text{a.s.}$$

Ciesielski and Taylor (1962), Ray and Taylor (1964) proved that

$$0 < \varphi_{d-m}(B([0,1])) < \infty \quad \text{a.s.,}$$

where

$$\begin{align*}
\varphi_1(r) &= r, \\
\varphi_2(r) &= r^2 \log(1/r) \log \log \log (1/r), \\
\varphi_d(r) &= r^2 \log \log (1/r), \quad \text{if } d \geq 3.
\end{align*}$$
Taylor and Tricot (1985) proved that

$$\dim_p B([0, 1]) = \min\{d, 2\}$$

and, if $d \geq 3$, then

$$0 < \psi-p(B([0, 1])) < \infty \quad \text{a.s.,}$$

where $\psi(r) = r^2 / \log \log(1/r)$.

LeGall and Taylor (1986) proved that, if $d = 2$, then for any measure function $\varphi$, either $\varphi-p(B([0, 1])) = 0$ or $\infty$.

**Question:** How to extend the above results to Gaussian random fields?
5.3. Exact Hausdorff and packing measure functions for fractional Brownian motion

For $H \in (0, 1)$, the fBm $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ with index $H$ is a centered $(N, d)$-Gaussian field whose covariance function is

$$\mathbb{E}[B^H_i(s)B^H_j(t)] = \frac{1}{2} \delta_{ij} \left(|s|^{2H} + |t|^{2H} - |s - t|^{2H}\right),$$

where $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.

- When $N = 1$ and $H = 1/2$, $B^H$ is Brownian motion.
- $B^H$ is $H$-self-similar and has stationary increments.

Kahane (1985) proved that

$$\dim_H B^H([0, 1]^N) = \min \left\{d, \frac{N}{H}\right\} \quad \text{a.s.}$$
5.3.1 Exact Hausdorff measure functions for $B^H([0, 1]^N)$ and $\text{Gr}B^H([0, 1]^N)$

**Theorem 5.1 [Talagrand (1995, 1998)]**

Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$.

(i). If $N < Hd$, then

$$K^{-1} \leq \varphi_1-m(B^H([0, 1]^N)) \leq K,$$  a.s.

where $\varphi_1(r) = r^N \log \log(1/r)$.

(ii). If $N = Hd$, then $\varphi_2-m(B^H([0, 1]^N))$ is $\sigma$-finite, where

$$\varphi_2(r) = r^d \log(1/r) \log \log \log(1/r).$$
Theorem 5.2 [X. (1997)]

Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$.

(i). If $N < Hd$, then

$$K^{-1} \leq \varphi_1 - m(\text{Gr}B^H([0, 1]^N)) \leq K, \quad \text{a.s.}$$

where $\varphi_1(r) = r^N H \log \log(1/r)$.

(ii). If $N > Hd$, then

$$K^{-1} \leq \varphi_2 - m(\text{Gr}B^H([0, 1]^N)) \leq K, \quad \text{a.s.},$$

where

$$\varphi_2(r) = r^{N+(1-H)d} \left( \log \log(1/r) \right)^{Hd/N}.$$
5.3.2. Exact packing measure function for $B^H([0, 1]^N)$

**Theorem 5.3 (Xiao, 1996, 2003)**

Let $B^H = \{B^H(t), \ t \in \mathbb{R}^N\}$ be a fBm with values in $\mathbb{R}^d$. If $N < Hd$, then there exists a finite constant $K \geq 1$ such that

$$K^{-1} \leq \varphi_4(p(B^H([0, 1]^N))) \leq K, \quad \text{a.s.}$$

where $\varphi_4(r) = r^{N/H} \left( \log \log (1/r) \right)^{-N/(2H)}$. 
For proving Theorem 5.3, one needs to study the liminf behavior of the sojourn measure

\[ T(r) = \int_{\mathbb{R}^N} 1_{\{|B^H(t)| \leq r\}} dt. \]

A key ingredient is the following small ball probability estimate for \( T(1) \).

**Lemma 5.4 [Xiao, 1996, 2003]**

Assume that \( N < Hd \). Then there exists a positive and finite constant \( K \geq 1 \), depending only on \( H, N \) and \( d \) such that for any \( 0 < \varepsilon < 1 \),

\[
\exp \left( -\frac{K}{\varepsilon^{2H/N}} \right) \leq \mathbb{P}\{ T(1) < \varepsilon \} \leq \exp \left( -\frac{1}{K\varepsilon^{2H/N}} \right).
\]
This leads to the following Chung’s LIL for $T(r)$.

**Theorem 5.5 (Xiao, 1996, 2003)**

If $N < Hd$, then with probability one,

$$
\liminf_{r \to 0} \frac{T(r)}{\varphi_4(r)} = K,
$$

(2)

where $0 < K < \infty$ is a constant depending on $H, N$ and $d$ only.

By the stationarity of increments of $B^H$ and the lower density theorem, we derive the lower bound in Theorem 5.3.

The proof of upper bound in Theorem 5.3 requires a different argument.
5.4 Exact Hausdorff measure function for the ranges of Gaussian random fields

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field in $\mathbb{R}^d$:

$$X(t) = (X_1(t), \ldots, X_d(t)), \quad t \in \mathbb{R}^N,$$

where $X_1, \ldots, X_d$ are independent copies of a centered Gaussian field $X_0$. We assume that $X_0$ satisfies the following conditions from Lecture 3.

**Assumption (A1)**

Consider a compact interval $T \subset \mathbb{R}^N$. There exists a Gaussian random field $\{v(A, t) : A \in \mathcal{B}(\mathbb{R}_+), t \in T\}$ such that

(a) For all $t \in T$, $A \mapsto v(A, t)$ is a real-valued Gaussian noise, $v(\mathbb{R}_+, t) = X_0(t)$, and $v(A, \cdot)$ and $v(B, \cdot)$ are independent whenever $A$ and $B$ are disjoint.
Assumption (A1) (continued)

(b) There are constants \(a_0 \geq 0\) and \(\gamma_j > 0\), \(j = 1, \ldots, N\) such that for all \(a_0 \leq a \leq b \leq \infty\) and \(s = (s_1, \ldots, s_N), t = (t_1, \ldots, t_N) \in T\),

\[
\| v([a, b), s) - X_0(s) - v([a, b), t) + X_0(t) \|_{L^2} \leq C \left( \sum_{j=1}^{N} a^{\gamma_j} |s_j - t_j| + b^{-1} \right),
\]

\(4\)

where \(\| Y \|_{L^2} = \left[ \mathbb{E}(Y^2) \right]^{1/2}\) for a random variable \(Y\) and

\[
\| v([0, a_0), s) - v([0, a_0), t) \|_{L^2} \leq C \sum_{j=1}^{N} |s_j - t_j|.
\]

\(5\)
Condition (A4') [strong local nondeterminism]

There exists a constant $c > 0$ such that $\forall \ n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\text{Var}(X_0(u) \mid X_0(t^1), \ldots, X_0(t^n)) \geq c \ \min_{1 \leq k \leq n} \ \rho(u, t^k)^2,$$

(6)

where $\rho(s, t)$ is the metric on $\mathbb{R}^N$ defined by

$$\rho(s, t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j},$$

and where $H_j = (\gamma_j + 1)^{-1}$ ($j = 1, \ldots, N$).

These conditions are weaker than those in Luan and X. (2012).
Theorem 5.6

Let \( X = \{X(t), \ t \in \mathbb{R}^N\} \) be a centered Gaussian field with values in \( \mathbb{R}^d \) such that \( X_0 \) satisfies (A1) and (A4').

(i). If \( Q = \sum_{j=1}^{N} H_j^{-1} < d \), then

\[
K^{-1} \leq \varphi_5-m(X([0,1]^N)) \leq K, \quad \text{a.s.},
\]

where \( \varphi_5(r) = r^Q \log \log(1/r) \).

(ii). If \( Q > d \), then \( X([0,1]^N) \) has positive \( d \)-dimensional Lebesgue measure a.s.

The problem to determine the exact Hausdorff measure function for \( X([0,1]^N) \) in the “critical case” \( Q = d \) is open.
Proof of Theorem 5.6

The lower bound in (7) is proved by using the upper density theorem in Lemma 5.1. A natural measure on $X([0, 1]^N)$ is the sojourn measure

$$\mu(B) = \lambda_N \{ t \in [0, 1]^N : X(t) \in B \}, \quad \forall B \in \mathcal{B}(\mathbb{R}^d),$$

where $\lambda_N$ denotes the Lebesgue measure on $\mathbb{R}^N$.

For any $0 < r < 1$ and $t^0 \in [0, 1]^N := I$, we consider

$$\mu(B(X(t^0), r)) = \int_I 1_{\{ |X(t) - X(t^0)| \leq r \}} \, dt,$$

which is the sojourn time of $X$ in the ball $B(X(t^0), r)$. 
The following moment estimate is essential for determining the asymptotic behavior of $\mu(B(X(t^0), r))$ as $r \to 0$.

**Lemma 5.5**

If $d > Q$, then there is a finite constant $C$ such that for every $t^0 \in I$ and all integers $n \geq 1$,

$$
\mathbb{E} \left[ \mu(B(X(t^0), r))^n \right] \leq C^n n! r^{Qn}.
$$

**Proof.** For $n = 1$, by Fubini’s theorem we have

$$
\mathbb{E}[\mu(B(X(t^0), r))] = \int_I \mathbb{P}\{ |X(t) - X(t^0)| < r \} \, dt
$$

$$
\leq \int_I \min \left\{ 1, c \left( \frac{r}{\rho(t, t^0)} \right)^d \right\} \, dt
$$

$$
= \int_{\{t: \rho(t, t^0) \leq cr\} \cap I} \, dt + c \int_{\{t: \rho(t, t^0) > cr\} \cap I} \left( \frac{r}{\rho(t, t^0)} \right)^d \, dt.
$$
It is elementary to verify that
\[ \mathbb{E} \left[ \mu(B(X(t^0), r)) \right] \leq cr^Q. \]

For \( n \geq 2 \),
\[ \mathbb{E} \left[ \mu(B(X(t^0), r))^n \right] = \int_{I^n} \mathbb{P} \{ |X(t^j) - X(t^0)| < r, 1 \leq j \leq n \} \, dt^1 \cdots dt^n. \]

It is sufficient to consider \( t^1, \ldots, t^n \in I \) that satisfy
\[ t^j \neq t^0, \quad \text{for } j = 1, \ldots, n \quad \text{and} \quad t^j \neq t^k \quad \text{for } j \neq k. \]

By Condition (A4'), we have
\[
\begin{align*}
\text{Var}(X_0(t^n) - X_0(t^0)|X_0(t^1) - X_0(t^0), \ldots, X_0(t^{n-1}) - X_0(t^0)) & \\
\quad & \geq \text{Var}(X_0(t^n)|X_0(t^0), X_0(t^1), \ldots, X_0(t^{n-1})) \\
\quad & \geq c \min_{0 \leq k \leq n-1} \rho(t^n, t^k)^2.
\end{align*}
\]
Since conditional distributions in Gaussian processes are still Gaussian, it follows from Anderson’s inequality and (8) that

\[
\int_I \mathbb{P}\left\{ |X(t^n) - X(t^0)| < r |X(t^1) - X(t^0), \ldots, X(t^{n-1}) - X(t^0)| \right\} dt^n 
\leq c \int_I \sum_{k=0}^{n-1} \min \left\{ 1, c \left( \frac{r}{\rho(t^n, t^k)} \right)^d \right\} dt^n 
\leq c n \int_I \min \left\{ 1, c \left( \frac{r}{\rho(t^n, 0)} \right)^d \right\} dt^n 
\leq c nr^{Q}.
\]

Iterating the procedure proves Lemma 5.5.
From Lemma 5.5 and the Borel-Cantelli lemma, we can prove the following law of the iterated logarithm for the sojourn measure of $X$.

**Proposition 5.1**

For every $t^0 \in I$, we have

$$\limsup_{r \to 0} \frac{\mu\left(B(X(t^0), r)\right)}{\varphi_5(r)} \leq C < \infty, \quad \text{a.s.}$$

This and Fubini’s theorem yield: a.s.

$$\limsup_{r \to 0} \frac{\mu\left(B(X(t^0), r)\right)}{\varphi_5(r)} \leq C \quad \text{a.e. } t^0 \in I.$$

Hence, the lower bound in (7) follows from Lemma 5.1.
For proving the upper bound in (7), we need the following small ball probability estimates.

Lemma 5.6 [X. (2009)]
Under the conditions of Theorem 5.6, there exist constants $c$ and $c'$ such that for all $t^0 \in I = [0, 1]^N$ and $0 < \varepsilon < r$,

$$\exp \left( -c' \left( \frac{r}{\varepsilon} \right)^Q \right) \leq \mathbb{P} \left\{ \sup_{t \in I : \rho(t, t^0) \leq r} |X(t) - X(t^0)| \leq \varepsilon \right\} \leq \exp \left( -c \left( \frac{r}{\varepsilon} \right)^Q \right).$$

The main estimate is given in the following lemma.
Proposition 5.2

Assume that the conditions of Theorem 5.6 hold. There exist positive constants $\delta_0$ and $C$ such that for any $t^0 \in I$ and $0 < r_0 \leq \delta_0$, we have

$$\mathbb{P}\left\{ \exists r \in [r_0^2, r_0], \sup_{t \in I: \rho(t, t^0) \leq r} |X(t) - X(t^0)| \leq Cr \left( \log \log \left( \frac{1}{r} \right) \right)^{-1/Q} \right\} \geq 1 - \exp \left(- \left( \log \left( \frac{1}{r_0} \right) \right)^{1/2} \right).$$

Proof. The method of proof comes from Talagrand (1995). We provide the main steps. Let $U > 1$ be a number whose value will be determined later. For $k \geq 0$, let $r_k = r_0 U^{-2^k}$. Consider the largest integer $k_0$ such that

$$k_0 \leq \frac{\log (1/r_0)}{2 \log U}.$$
Thus, for $k \leq k_0$ we have $r_0^2 \leq r_k \leq r_0$. It thereby suffices to prove that

$$\mathbb{P}\left\{ \exists k \leq k_0, \sup_{t \in I: \rho(t,t^0) \leq r_k} |X(t) - X(t^0)| \leq c r_k \left( \log \log \frac{1}{r_k} \right)^{-1/2} \right\}$$

(9)

$$\geq 1 - \exp \left( - \left( \log \frac{1}{r_0} \right)^{1/2} \right).$$

Let $a_k = r_0^{-1} U^{2k-1}$ and we define for $k = 0, 1, \cdots$

$$X_{0,k}(t) = \nu([a_k, a_{k+1}), t)$$

and

$$\hat{X}_k(t) = (X_{1,k}(t), \cdots, X_{d,k}(t)),$$

where $X_{1,k}(t), \cdots, X_{d,k}(t)$ are independent copies of $X_{0,k}(t)$. It follows that $X_1 - X_{1,k}, \cdots, X_d - X_{d,k}$ are independent copies of $X_0 - X_{0,k}$. 
The Gaussian random fields $\hat{X}_0, \hat{X}_1, \cdots$ are independent. By Lemma 5.6 we can find a constant $c > 0$ such that, if $r_0$ is small enough, then for each $k \geq 0$

$$
P\left\{ \sup_{t \in I: \rho(t,t^0) \leq r_k} |\hat{X}_k(t) - \hat{X}_k(t^0)| \leq c \, r_k \left( \log \log (1/r_k) \right)^{-1/Q} \right\}
$$

$$
\geq \exp \left( - \frac{1}{4} \log \log (1/r_k) \right) = \frac{1}{(\log 1/r_k)^{1/4}}
$$

$$
\geq (2 \log 1/r_0)^{-1/4}.
$$

By the independence,

$$
P\left\{ \exists k \leq k_0, \sup_{t \in I: \rho(t,t^0) \leq r_k} |\hat{X}_k(t) - \hat{X}_k(t^0)| \leq c \, r_k \left( \log \log (1/r_k) \right)^{-1/Q} \right\}
$$

$$
\geq 1 - \left( 1 - \frac{1}{(2 \log 1/r_0)^{1/4}} \right)^{k_0} \geq 1 - \exp \left( - \frac{k_0}{(2 \log 1/r_0)^{1/4}} \right), \tag{10}
$$

where the last inequality follows from $1 - x \leq e^{-x}$ for all $x > 0$. 
To deal with \( \{X(t) - \tilde{X}_k(t)\} \), we claim that for any \( u \geq cr_kU^{-\beta} \sqrt{\log U} \), where \( \beta = \min\{H_N^{-1} - 1, 1\} \),

\[
P\left\{ \sup_{t \in I: \rho(t,t^0) \leq r_k} \left| X(t) - \tilde{X}_k(t) - (X(t^0) - \tilde{X}_k(t^0)) \right| \geq u \right\} \leq \exp\left( -\frac{u^2}{cr_k^2U^{-2\beta}} \right). \tag{11}
\]

To see this, it’s enough to prove that (11) holds for \( X_0 \), by applying Lemma 3.3.
Consider \( S = \{t \in I : \rho(t, t^0) \leq r_k\} \) and on \( S \) the distance

\[
d(s, t) = \left\| X_0(s) - X_{0,k}(s) - (X_0(t) - X_{0,k}(t)) \right\|_{L^2}.
\]

Then \( d(s, t) \leq c \sum_{i=1}^N |s_i - t_i|^{H_i} \) and \( N(S, d, \varepsilon) \leq c \left( \frac{r_k}{\varepsilon} \right)^Q \).
Now we estimate the $d$-diameter $D$ of $S$. By Condition (A1), we have for any $s, t \in S$,

$$
\|X_0(s) - X_{0,k}(s) - (X_0(t) - X_{0,k}(t))\|_{L^2} \\
\leq C \left( \sum_{j=1}^{N} a_k^{H_j-1} |s_j - t_j| + a_{k+1}^{-1} \right) \leq Cr_k U^{-\beta},
$$

where $\beta = \min\{H_N^{-1} - 1, 1\}$. Therefore, $D \leq Cr_k U^{-\beta}$.

Notice that

$$
\int_0^{D} \sqrt{\log N(S, d, \varepsilon)} d\varepsilon \leq c \int_0^{Cr_k U^{-\beta}} \sqrt{\log r_k / \varepsilon} d\varepsilon \\
\leq cr_k \int_0^{C U^{-\beta}} \sqrt{\log 1/u} du \leq cr_k U^{-\beta} \sqrt{\log U}.
$$

Hence (11) follows from Lemma 3.3.
Let $U = (\log 1/r_0)^{1/\beta}$. Then for $r_0 > 0$ small

$$U^\beta (\log U)^{-1/2} \geq \left(\log \log \frac{1}{r_0}\right)^{1/Q}.$$ 

Take $u = c r_k (\log \log 1/r_0)^{-1/Q}$. It follows from (11) that

$$\mathbb{P}\left\{ \sup_{t \in I: \rho(t, t^0) \leq r_k} |X(t) - \hat{X}_k(t) - (X(t^0) - \hat{X}_k(t^0))| \geq c r_k \left(\log \log \frac{1}{r_0}\right)^{-1/Q} \right\} \leq \exp \left(- \frac{c U^\beta}{(\log \log 1/r_0)^{2/Q}} \right).$$
Combining this with (10), we get

\[ \mathbb{P}\left\{ \exists k \leq k_0, \sup_{\rho(t, t^0) \leq r_k} |X(t) - X(t^0)| \leq c r_k \left( \log \log \left( \frac{1}{r_k} \right) \right)^{-1/Q} \right\} \geq 1 - \exp\left( - \frac{k_0}{(2 \log 1/r_0)^{1/4}} \right) - k_0 \exp\left( - \frac{c U^\beta}{(\log \log 1/r_0)^{2/Q}} \right). \]

This proves (9) and Proposition 5.2.

With Proposition 5.2 in hand, we proceed to construction of an economic covering for \(X([0, 1]^N)\).
For $k \geq 1$, consider the set

$$R_k = \left\{ t \in [0, 1]^N : \exists r \in [2^{-2k}, 2^{-k}] \text{ such that} \right\}
\sup_{s \in I : \varrho(s, t) \leq r} |X(s) - X(t)| \leq c r (\log \log \frac{1}{r})^{-1/Q} \right\}.$$

By Lemma 5.7 we have that for every $t \in [0, 1]^N$,

$$\mathbb{P}\{t \in R_k\} \geq 1 - \exp(-\sqrt{k/2}).$$

This and Fubini’s theorem imply that

$$\mathbb{E}[\lambda_N(R_k)] \geq 1 - \exp(-\sqrt{k/2}).$$

Or

$$\mathbb{E}[\lambda_N(I \setminus R_k)] \leq \exp(-\sqrt{k/2}).$$
By Markov’s inequality, we have

\[ P \left\{ \lambda_N(R_k) < 1 - \exp(-\sqrt{k}/2) \right\} = P \left\{ \lambda_N(I \setminus R_k) > \exp(-\sqrt{k}/2) \right\} \leq \frac{E[\lambda_N(I \setminus R_k)]}{\exp(-\sqrt{k}/2)} \leq \exp \left( - \left( \frac{1}{\sqrt{2}} - \frac{1}{2} \right) \sqrt{k} \right). \]

Hence, by the Borel-Cantelli lemma, we have \( P(\Omega_1) = 1 \), where

\[ \Omega_1 = \left\{ \omega : \lambda_N(R_k) \geq 1 - \exp(-\sqrt{k}/2) \text{ for all } k \text{ large enough} \right\}. \]
On the other hand, by Lemma 3.3, we have $P(\Omega_2) = 1$, where $\Omega_2$ is the event that for every rectangle $I_n$ of side-lengths $2^{-n/H_i} (i = 1, \cdots, N)$ that meets $[0, 1]^N$, we have

$$\sup_{s,t \in I_n} |X(t) - X(s)| \leq C 2^{-n} \sqrt{n},$$

where $C > 0$ is a constant.

Now we show that for every $\omega \in \Omega_1 \cap \Omega_2$, we have

$$\varphi_5-m(X([0, 1]^N)) \leq K < \infty, \quad \text{a.s.}$$

For any $n \geq 1$, we divide $[0, 1]^N$ into $2^{nQ}$ disjoint (half open and half closed) rectangles of side-lengths $2^{-n/H_i} (i = 1, \cdots, N)$. Denote by $I_n(x)$ the unique rectangle of side-lengths $2^{-n/H_i} (i = 1, \cdots, N)$ containing $x$. 
Consider \( k \geq 1 \) such that

\[
\lambda_N(R_k) \geq 1 - \exp(-\sqrt{k}/2).
\]

For any \( x \in R_k \) we can find the smallest integer \( n \) with \( k \leq n \leq 2k \) such that

\[
\sup_{s,t \in I_n(x)} |X(t) - X(s)| \leq c 2^{-n} (\log \log 2^n)^{-1/Q}.
\]

(12)

Thus we have

\[
R_k \subseteq V = \bigcup_{n=k}^{2k} V_n
\]

and each \( V_n \) is a union of rectangles \( I_n(x) \) satisfying (12). Notice that \( X(I_n(x)) \) can be covered by a ball of radius \( r_n = c2^{-n} (\log \log 2^n)^{-1/Q} \).
Since $\varphi_5(2r_n) \leq c2^{-nQ} = c\lambda_N(I_n)$, we obtain

$$
\sum_{n=k}^{2k} \sum_{I_n \in V_n} \varphi_5(2r_n) \leq \sum_{n} \sum_{I_n \in V_n} c\lambda_N(I_n) = C\lambda_N(V) \leq C. \quad (13)
$$

Thus $X(V)$ is contained in the union of a family of balls $B_n$ of radius $r_n$ with $\sum_n \varphi_5(2r_n) \leq C$.

On the other hand, $[0, 1]^N \setminus V$ is contained in a union of rectangles of side-lengths $2^{-q/H_i}(i = 1, \cdots, N)$ where $q = 2k + 1$, none of which meets $R_k$. There can be at most

$$
2^{Qq}\lambda_N([0, 1]^N \setminus V) \leq c2^{Qq}\exp(-\sqrt{k}/2)
$$

such rectangles.
Since $\omega \in \Omega_2$, for each of these rectangles $I_q$, $X(I_q)$ is contained in a ball of radius $c2^{-q}\sqrt{q}$.
Thus $X([0, 1]^N \setminus V)$ can be covered by a sequence $\{B_n\}$ of balls of radius $r_n = c2^{-q}\sqrt{q}$ such that

$$
\sum_n \varphi_5(2r_n) \leq \left(c2^{Oq} \exp(-\sqrt{k}/2)\right) \left(c2^{-qO} q^{O/2} \log \log(c2^q / \sqrt{q})\right) \\
\leq 1
$$

(14)
for all $k$ large enough. Since $k$ can be arbitrarily large, it follows from (13) and (14) that

$$
\varphi_5-m(X([0, 1]^N)) \leq K, \quad \text{a.s.}
$$

This finishes the proof of Part (i) of Theorem 5.6. Part (ii) is related to the existence of local times. A proof based on Fourier analysis will be given in Lecture 6.
If Condition (A4') in Theorem 5.6 is replaced by (A4), then the exact Hausdorff measure function for $X([0, 1]^N)$ is different. See the recent paper of Lee (2021).
Thank you