Properties of Strong Local Nondeterminism of Gaussian Random Fields

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Lecture 4. Properties of strong local nondeterminism

- Strong local nondeterminism
- Spectral condition for strong local nondeterminism
- A comparison theorem
- Stochastic heat equation
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4.1 Properties of local nondeterminism

The concept of local nondeterminism (LND) of a Gaussian process was first introduced by Berman (1973) for studying local times of Gaussian processes.

A Gaussian process $Y = \{Y(t), t \in \mathbb{R}\}$ is called *locally nondeterministic* on $T \subseteq \mathbb{R}$ if for every integer $m \geq 2$,

$$\lim_{\varepsilon \to 0} \inf_{t_m - t_1 \leq \varepsilon} V_m > 0, \tag{1}$$

where V_m is the relative prediction error:

$$V_m = \frac{\mathrm{Var}\big(Y(t_m) - Y(t_{m-1}) | Y(t_1), \ldots, Y(t_{m-1})\big)}{\mathrm{Var}\big(Y(t_m) - Y(t_{m-1})\big)}$$

and the infimum in (1) is taken over all ordered points $t_1 < t_2 < \cdots < t_m$ in T with $t_m - t_1 \leq \varepsilon$.

(1) is equivalent to the following property: For every integer $m \geq 2$, there exist positive constants $C(m)$ and ε (both may depend on m) such that

$$\begin{aligned} &\mathrm{Var}\Big(\sum_{k=1}^{m} a_k \left(Y(t_k) - Y(t_{k-1})\right)\Big) \\ &\qquad \geq C(m) \sum_{k=1}^{m} a_k^2 \,\mathrm{Var}\big(Y(t_k) - Y(t_{k-1})\big) \end{aligned} \tag{2}$$

for all ordered points $t_1 < t_2 < \cdots < t_m$ in T with $t_m - t_1 < \varepsilon$ and $a_k \in \mathbb{R}$ $(k = 1, \ldots, m)$.

- Pitt (1978) used (2) to define local nondeterminism of a Gaussian random field $X = \{X(t), t \in \mathbb{R}^N\}$ with values in $\mathbb{R}$ by introducing a partial order among $t_1, \ldots, t_m \in \mathbb{R}^N$.
- Pitt (1978) proved that fractional Brownian motion $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ has the following property: For any $u \in \mathbb{R}^N \backslash \{0\}$, and and $r \in (0, |u|)$,

$$\mathrm{Var}\Big(B^H(u) \,\Big|\, B^H(t), \, |t-u| \ge r\Big) = c\, r^{2H},$$

  where $c > 0$ is a constant. This implies that B^H satisfies the strong local nondeterminism on any compact interval $I \subset \mathbb{R}^N \backslash \{0\}$.
- Cuzick and DuPreez (1982) introduced strong local ϕ-nonderterminism and showed its usefulness in studying local times.

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field in $\mathbb{R}$ and let $T \subseteq \mathbb{R}^N$ be a compact interval. In studying precise regularity properties of X, we have made use of the following conditions:

(A4). $\exists$ a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\Big(X(u) \,\Big|\, X(t^1), \ldots, X(t^n)\Big) \geq c \sum_{j=1}^{N} \min_{1 \leq k \leq n} \big|u_j - t_j^k\big|^{2H_j}.$$

Here and below, $H_j \in (0, 1)$ $(j = 1, \ldots, N)$ are constants.

(A4′). ∃ a constant $c > 0$ such that for all $n \ge 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\Big(X(u)\,\Big|\,X(t^1), \ldots, X(t^n)\Big) \ge c \min_{1\le k\le n} \rho(u,\, t^k)^2,$$

where

$$\rho(s,t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}, \qquad \forall\, s, t \in \mathbb{R}^N.$$

These conditions are referred to as properties of strong local nondeterminism (with respect to the metric ρ).

Remarks

- The Brownian sheet W does not satisfy “strong local nondeterminism” with $H_1 = \cdots = H_N = 1/2$. This caused difficulties in studies of some sample path properties of W; cf. Mountford (1989a, 1989b).
- The “sectorial local nondeterminism” was first discovered by Khoshnevisan and X. (2007) for the Brownian sheet; and extended to fractional Brownian sheets by Wu and X. (2007).
- X. (2009), Luan and X. (2012) proved sufficient conditions for “strong local nondeterminism” for a large class of Gaussian fields with stationary increments.

4.2 Spectral conditions for strong local nondeterminism

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and $X(0) = 0$.
For any $h \in \mathbb{R}^N$ we have

$$\mathbb{E}\big(X(t+h) - X(t)\big)^2 = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle h, \lambda\rangle\big)\, \Delta(d\lambda),$$

where $\Delta(d\lambda)$ is the spectral measure of X, which satisfies

$$\int_{\mathbb{R}^N} \frac{|\lambda|^2}{1+|\lambda|^2}\, \Delta(d\lambda) < \infty.$$

It follows that X has the stochastic integral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^N} \left(e^{i\langle t,\lambda\rangle} - 1\right) \widetilde{W}(d\lambda),$$

where $\widetilde{W}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

Remarks (i). If $Y = \{Y(t), t \in \mathbb{R}^N\}$ is a stationary Gaussian field, let $X(t) = Y(t) - Y(0)$ for all $t \in \mathbb{R}^N$. Then $X = \{X(t), t \in \mathbb{R}^N\}$ has stationary increments and has the same spectral measure as that of Y.
(ii). The spectral measure Δ can be

- absolutely continuous with density $f(\lambda)$, or
- singular with fractal support, or
- singular with a discrete support.

Theorem 4.1 [Xue and X., 2011]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian field with stationary increments and spectral density $f(\lambda)$. If there are constants $H_1, \cdots, H_N \in (0,1]^N$ and $K > 0$ such that

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{2+Q}}, \qquad \lambda \in \mathbb{R}^N, \ |\lambda| \geq K, \tag{3}$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$, then $\exists$ a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in \mathbb{R}^N$,

$$\mathrm{Var}\Big(X(u) \,\Big|\, X(t^1), \ldots, X(t^n)\Big) \geq c \min_{0 \leq k \leq n} \rho(u, t^k)^2, \ \text{ where } t^0 = 0.$$

Observe from (3) that the behavior of $f(\lambda)$ near 0 is not needed.

For proving Theorem 4.1, we need the following lemma.

Lemma 4.1

Assume (3) is satisfied, then for any fixed constant $L > 0$, there exists a positive and finite constant c_1 such that for all functions g of the form

$$g(\lambda) = \sum_{k=1}^{n} a_k \big(e^{i\langle t^k, \lambda\rangle} - 1\big), \tag{4}$$

where $a_k \in \mathbb{R}$ and $t^k \in [-L, L]^N$, we have

$$|g(\lambda)| \le c_1 \, |\lambda| \Big(\int_{\mathbb{R}^N} |g(\xi)|^2 f(\xi) d\xi \Big)^{1/2} \tag{5}$$

for all $\lambda \in \mathbb{R}^N$ that satisfy $|\lambda| \le K$.

Proof of Lemma 4.1. By (3), we can find positive constants C and η, such that

$$f(\lambda) \geq \frac{C}{|\lambda|^\eta}, \quad \forall \lambda \in \mathbb{R}^N \text{ with } |\lambda| \text{ large enough.}$$

Let $\mathcal{G}$ be the collection of the functions $g(z)$ defined by (4) with $a_k \in \mathbb{R}$, $s^k \in [-L, L]^N$ and $z \in \mathbb{C}^N$. Since each $g \in \mathcal{G}$ is an entire function, it follows from Proposition 1 of Pitt (1975) that for any given constant K,

$$c_1 = \sup_{\substack{g\in\mathcal{G} \\ z\in U(0,K)}} \left\{ |g(z)| : \int_{\mathbb{R}^N} |g(\lambda)|^2 f(\lambda)\, d\lambda \leq 1 \right\} < \infty,$$

where $U(0, K) = \{z \in \mathbb{C}^N : |z| < K\}$ is the open ball of radius K in $\mathbb{C}^N$.

Since $g(0) = 0$ and g is analytic in $U(0, K)$, Schwartz's lemma implies

$$\big|g(z)\big| \le c_1 K^{-1}\, |z| \Big(\int_{\mathbb{R}^N} |g(\xi)|^2 f(\xi) d\xi \Big)^{1/2}$$

for all $z \in U(0, K)$. This finishes the proof of Lemma 4.1.

Proof of Theorem 4.2. Denote $r \equiv \min_{0\le k\le n} \rho(u, t^k)$. It is sufficient to prove that for all $a_k \in \mathbb{R}$ $(1 \le k \le n)$,

$$\mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k\, X(t^k)\Big)^2 \ge c\, r^2. \qquad (6)$$

By the stochastic integral representation of X, the left hand side of (6), up to a constant, can be written as

$$\begin{aligned}&\mathbb{E}\Big(X(u)-\sum_{k=1}^{n}a_kX(t^k)\Big)^2\\&=\int_{\mathbb{R}^N}\Big|e^{i\langle u,\lambda\rangle}-1-\sum_{k=1}^{n}a_k\left(e^{i\langle t^k,\,\lambda\rangle}-1\right)\Big|^2 f(\lambda)\,d\lambda.\end{aligned} \tag{7}$$

Hence, we only need to show

$$\int_{\mathbb{R}^N}\Big|e^{i\langle u,\lambda\rangle}-\sum_{k=0}^{n}a_k\,e^{i\langle t^k,\,\lambda\rangle}\Big|^2 f(\lambda)\,d\lambda\geq c\,r^2, \tag{8}$$

where $t^0=0$ and $a_0=-1+\sum_{k=1}^{n}a_k$.

Let $\delta(\cdot) : \mathbb{R}^N \to [0,1]$ be a function in $C^\infty(\mathbb{R}^N)$ such that $\delta(0) = 1$ and it vanishes outside the open ball $B_\rho(0,1)$.

Denote by $\widehat{\delta}$ the Fourier transform of δ. Then $\widehat{\delta}(\cdot) \in C^\infty(\mathbb{R}^N)$ and decays rapidly as $|\lambda| \to \infty$.
Let A be the diagonal matrix with $H_1^{-1}, \ldots, H_N^{-1}$ on its diagonal and let $\delta_r(t) = r^{-Q}\delta(r^{-A}t)$. By the inverse Fourier transform,

$$\delta_r(t) = (2\pi)^{-N} \int_{\mathbb{R}^N} e^{-i\langle t,\lambda\rangle}\, \widehat{\delta}(r^A\lambda)\, d\lambda.$$

Since $\min\{\rho(u,t^k) : 0 \le k \le n\} = r$, we have

$$\delta_r(u - t^k) = 0 \quad \text{for} \quad k = 0, 1, \ldots, n.$$

Hence,

$$\begin{aligned} I &= \int_{\mathbb{R}^N} \left(e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle t^k,\lambda\rangle} \right) e^{-i\langle u,\lambda\rangle}\, \widehat{\delta}(r^A\lambda)\, d\lambda \\ &= (2\pi)^N \left(\delta_r(0) - \sum_{k=0}^{n} a_k\, \delta_r(u-t^k) \right) \\ &= (2\pi)^N\, r^{-Q}. \end{aligned} \tag{9}$$

We split the integral in (9) over $\{\lambda : |\lambda| < K\}$ and $\{\lambda : |\lambda| \geq K\}$ and denote the two integrals by I_1 and I_2, respectively. It follows from Lemma 4.1 that

$$\begin{aligned} I_1 &\leq \int_{|\lambda|<K} \Big| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k e^{i\langle t^k,\lambda\rangle} \Big| |\hat{\delta}(r^A\lambda)| d\lambda \\ &\leq c_1 \left[\int_{\mathbb{R}^N} \Big| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k e^{i\langle t^k,\lambda\rangle} \Big|^2 f(\lambda)\, d\lambda \right]^{1/2} \\ &\qquad\qquad \times \int_{|\lambda|<K} |\lambda| |\hat{\delta}(r^A\lambda)| d\lambda \\ &\leq c_2 \left[\mathbb{E} \Big(X(u) - \sum_{k=0}^{n} a_k X(t^k) \Big)^2 \right]^{1/2}. \end{aligned} \tag{10}$$

On the other hand, the Cauchy-Schwarz inequality gives

$$
\begin{aligned}
I^2 \le & \int_{|\lambda|\ge K} \Big| e^{i\langle u,\lambda\rangle} - \sum_{k=0}^{n} a_k\, e^{i\langle t^k,\lambda\rangle}\Big|^2 f(\lambda)\, d\lambda \\
& \qquad \times \int_{|\lambda|\ge K} \frac{\big|\widehat{\delta}(r^A\lambda)\big|^2}{f(\lambda)}\, d\lambda \\
\le\ & \mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2 \cdot r^{-Q} \int_{\mathbb{R}^N} \frac{\big|\widehat{\delta}(\lambda)\big|^2}{f(r^{-A}\,\lambda)}\, d\lambda \\
\le\ & c\,\mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2 \cdot r^{-2Q-2}.
\end{aligned}
$$

We square both sides of (9) and use the above to obtain

$$(2\pi)^{2N} r^{-2Q} \le c\, r^{-2Q-2}\, \mathbb{E}\Big(X(u) - \sum_{k=1}^{n} a_k X(t^k)\Big)^2.$$

This proves (8) and hence the theorem.

Remarks

- This method can be modified to prove sectorial local nondeterminism (by choosing appropriate function $\delta(\cdot) : \mathbb{R}^N \to [0, 1]$.
- The method is applied in Lan, Marinucci and X. (2018) to prove strong local nondeterminism for isotropic Gaussian random fields on the sphere $\mathbb{S}^2$.

4.2 A comparison theorem

Now we consider the case where the spectral measure Δ may be singular.

For any $\lambda \in \mathbb{R}^N$ and $h > 0$, denote by $C(\lambda, h)$ the cube with side-length $2h$ and center λ, i.e.,

$$C(\lambda, h) = \{x \in \mathbb{R}^N : |x_j - \lambda_j| \le h, j = 1, \cdots, N\}.$$

Let $L^2(C(0, L))$ be the subspace of $g \in L^2(\mathbb{R}^N)$ whose support is contained in $C(0, L)$.

Theorem 4.2 [Luan and X., 2012]

Let $\{Y(t), t \in \mathbb{R}^N\}$ be a real, centered Gaussian field with stationary increments and $Y(0) = 0$. If for some $h > 0$ the spectral measure Δ of Y satisfies

$$\begin{aligned} 0 < \liminf_{|\lambda|\to\infty} \rho(0, \lambda)^{Q+2}\Delta(C(\lambda, h)) \\ \le \limsup_{|\lambda|\to\infty} \rho(0, \lambda)^{Q+2}\Delta(C(\lambda, h)) < \infty, \end{aligned} \tag{11}$$

then for any $L > 0$ such that $LhN < \log 2$, for all $u, t^1, \ldots, t^n \in C(0, L)$,

$$\operatorname{Var}\Big(Y(u) \,\Big|\, Y(t^1), \ldots, Y(t^n)\Big) \ge c \min_{0\le k\le n} \rho(u,\, t^k)^2.$$

Proof of Theorem 4.2

Lemma 4.2 (Pitt, 1975)

Let $\widetilde{\Delta}(d\lambda)$ be a positive measure on $\mathbb{R}^N$. If, for some constant $h > 0$, $\widetilde{\Delta}(d\lambda)$ satisfies

$$0 < \liminf_{|\lambda|\to\infty} \widetilde{\Delta}(C(\lambda, h)) \leq \limsup_{|\lambda|\to\infty} \widetilde{\Delta}(C(\lambda, h)) < \infty.$$

Then for every $L > 0$ satisfying $LhN < \log 2$, we have

$$\int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 \widetilde{\Delta}(d\lambda) \asymp \int_{\mathbb{R}^N} |\widehat{\psi}(\lambda)|^2 d\lambda$$

for all $\psi \in L^2(C(0, L))$.

Lemma 4.3 (Luan and X. 2012)

Let $\Delta_1(d\lambda)$ be a measure on $\mathbb{R}^N$ such that for some $h > 0$,

$$0 < \liminf_{|\lambda|\to\infty} \rho(0,\lambda)^{Q+2}\Delta_1(C(\lambda,h)) \le \limsup_{|\lambda|\to\infty} \rho(0,\lambda)^{Q+2}\Delta_1(C(\lambda,h)) < \infty.$$

Then for any $L > 0$ with $LhN < \log 2$, $\exists$ constants c_3 and c_4 such that

$$\int_{\mathbb{R}^N} |g(\lambda)|^2 \Delta_1(d\lambda) \asymp \int_{\mathbb{R}^N} \frac{|g(\lambda)|^2}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{Q+2}}\, d\lambda$$

for all $g(\lambda)$ as in Lemma 4.1.

Theorem 4.2 follows from Lemma 4.3 and Theorem 4.2.

Examples

Example 4.1. Let $\{\xi_n, n \in \mathbb{Z}^N\}$ and $\{\eta_n, n \in \mathbb{Z}^N\}$ be two independent sequences of i.i.d. $N(0, 1)$ random variables. Let

$$Z(t) = \sum_{n \in \mathbb{Z}^N} a_n \big(\xi_n \cos \langle n, t\rangle + \eta_n \sin \langle n, t\rangle \big), \quad t \in \mathbb{R}^N,$$

where $\{a_n, n \in \mathbb{Z}^N\}$ is a sequence of real numbers such that

$$a_n^2 \asymp \frac{1}{\big(\sum_{j=1}^N |n_j|^{H_j}\big)^{Q+2}}.$$

By Theorem 4.2, the Gaussian field $Y(t) = Z(t) - Z(0)$ has the property of strong local nondeterminism.

Example 4.2. Let μ be the measure on $\mathbb{R}$ obtained by "patching" fractal probability measures on $[n, n+1]$, and let the spectral measure Δ be given by

$$\frac{d\mu(\lambda)}{|\lambda|^{1+2H}},$$

then Theorem 4.2 implies that any Gaussian process X with spectral measure Δ has the property of SLND which is similar to that of fBm B^H.

4.3 SLND of linear SHE

Consider the linear stochastic heat equation

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) = \frac{1}{2}\Delta u(t,x) + \sigma \dot{W}, \quad t \geq 0,\ x \in \mathbb{R}^k, \\ &u(0,x) \equiv 0, \end{aligned} \tag{12}$$

where Δ is the Laplacian operator, σ is a constant or a deterministic function, and $\dot{W}$ is a Gaussian noise that is white in time and has a spatially homogeneous covariance [Dalang (1999)] given by the Riesz kernel with exponent β if $k \geq 1$ and $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)|x-y|^{-\beta}.$$

If $k = 1 = \beta$, then $\dot{W}$ is the space-time Gaussian white noise considered by Walsh (1986).

It follows from Dalang (1999) that the mild solution of (12) is the mean zero Gaussian random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}\}$ defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \widetilde{G}_{t-r}(x-y)\, \sigma W(drdy), \quad t \geq 0, x \in \mathbb{R},$$

where $\widetilde{G}_t(x)$ is the Green kernel given by

$$\widetilde{G}_t(x) = (2\pi t)^{-1/2} \exp\Big(- \frac{|x|^2}{2t}\Big), \qquad \forall\, t > 0, x \in \mathbb{R}^k.$$

Dalang, Khoshnevisan, and Nualart (2007) that for any $0 < a < b < \infty$,

$$\mathbb{E}\Big(|u(t,x) - u(s,y)|^2\Big) \asymp \rho((t,x),(s,y))^2 \tag{13}$$

for all $(t,x),(s,y) \in [a,b] \times [-b,b]^k$, where

$$\rho((t,x),(s,y)) = |t-s|^{\frac{2-\beta}{4}} + |x-y|^{\frac{2-\beta}{2}}.$$

Even though the solution $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ is not stationary nor has stationary increments, by using the following representation in Dalang, Mueller and X. (2017):

$$u(t,x) = \int_{\mathbb{R}} \int_{\mathbb{R}^k} e^{-i\xi x} \frac{e^{-i\tau t} - e^{-t\xi^2}}{|\xi|^2 - i\tau} |\xi|^{(\beta-k)/2} W(d\tau, d\xi),$$

we can prove

Theorem 4.3 [Khoshnevisan, Lee, and X. (2021)]

For any $0 < a < b < \infty$, there exists a constant $C > 0$ such that for all integers $n \geq 1$, for all $(t,x), (t^1,x^1), \ldots, (t^n,x^n) \in [a,b] \times [-b,b]^k$,

$$\mathrm{Var}\big(u_1(t,x) | u_1(t^1,x^1), \ldots, u_1(t^n,x^n)\big) \geq C \min_{1 \leq i \leq n} \rho((t,x),(t^i,x^i))^2.$$

Consequently, many regularity properties of $\{u(t,x), t \geq 0, x \in \mathbb{R}\}$ can be derived.

4.4 SLND of linear stochastic wave equation

The linear stochastic wave equation

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \geq 0,\ x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, \end{cases} \tag{14}$$

where $\dot{W}$ is a Gaussian noise as in the previous section with exponent β if $k \geq 1$ and $\beta \in (0, k \wedge 2)$.

The existence of real-valued process solution to (14) was studied by Walsh (1986) for the space-time while noise and by Dalang (1999) in the more general setting.

Recall that the fundamental solution of the wave equation G is

$$G(t,x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}} \quad \text{if } \ k=1;$$

$$G(t,x) = c_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(k-2)/2} (t^2 - |x|^2)_+^{-1/2}, \quad \text{if } k \geq 2 \text{ is even;}$$

and

$$G(t,x) = c_k \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{(k-3)/2} \frac{\sigma_t^k(dx)}{t}, \quad \text{if } \ k \geq 3 \text{ is odd,}$$

where σ_t^k is the uniform surface measure on the sphere $\{x \in \mathbb{R}^k : |x| = t\}$.

For any dimension $k \geq 1$, the Fourier transform of G in variable x is given by

$$\mathscr{F}(G(t,\cdot))(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0,\ \xi \in \mathbb{R}^k. \tag{15}$$

Dalang (1999) proved that the real-valued process solution of equation (14) is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy), \tag{16}$$

where W is the martingale measure induced by the noise $\dot{W}$. The range of β has been chosen so that the stochastic integral exists.

Recall from Dalang (1999) that

$$\mathbb{E}\Big[\Big(\int_0^t \int_{\mathbb{R}^k} H(s,y)W(ds\,dy)\Big)^2\Big] = c\int_0^t ds \int_{\mathbb{R}^k} |\mathscr{F}(H(s,\cdot))(\xi)|^2 \frac{d\xi}{|\xi|^{k-\beta}} \tag{17}$$

provided that $s \mapsto H(s,\cdot)$ is a deterministic function with values in the space of nonnegative distributions with rapid decrease and

$$\int_0^t ds \int_{\mathbb{R}^k} |\mathscr{F}(H(s,\cdot)(\xi)|^2 \frac{d\xi}{|\xi|^{k-\beta}} < \infty.$$

In the following, we show that the Gaussian random field $\{u(t,x), t \geq 0, x \in \mathbb{R}^k\}$ satisfies a form of strong local nondeterminism.

Let $0 < a < a' < \infty$ and $0 < b < \infty$ be fixed constants.

Theorem 4.4 (Lee and X. (2019))

There exists a constant $C > 0$ such that for all integers $n \geq 1$ and $(t,x), (t^1,x^1), \ldots, (t^n,x^n)$ in $[a,a'] \times [-b,b]^k$ with $|t - t^j| + |x - x^j| \leq a/2$, we have

$$\begin{aligned} &\mathrm{Var}\left(u(t,x) | u(t^1,x^1), \ldots, u(t^n,x^n)\right) \\ &\geq C \int_{\mathbb{S}^{k-1}} \min_{1 \leq j \leq n} \left|(t - t^j) + (x - x^j) \cdot w\right|^{2-\beta} dw, \end{aligned} \tag{18}$$

where dw is the surface measure on the unit sphere $\mathbb{S}^{k-1}$.

When $k = 1$, the surface measure dw in (18) is supported on $\{-1, 1\}$. It follows that $u(t, x)$ satisfies sectorial local nondeterminism:

$$\begin{aligned}&\mathrm{Var}(u(t,x)|u(t^1,x^1),\ldots,u(t^n,x^n))\\&\geq C\Big(\min_{1\leq j\leq n}|(t-t^j)+(x-x^j)|^{2-\beta}+\min_{1\leq j\leq n}|(t-t^j)-(x-x^j)|^{2-\beta}\Big).\end{aligned}$$

Proof of Theorem 4.4. For each $w \in \mathbb{S}^{k-1}$, let

$$r(w) = \min_{1 \le j \le n} |(t^j - t) - (x^j - x) \cdot w|.$$

Since u is a centered Gaussian random field,it suffices to show that there a exist constant $C > 0$ such that for all $(t, x), (t^1, x^1), \ldots, (t^n, x^n)$ in $[a, a'] \times [-b, b]^k$ with $|t - t^j| + |x - x^j| \le a/2$,

$$\mathbb{E}\Big[\Big(u(t,x) - \sum_{j=1}^{n} \alpha_j u(t^j, x^j)\Big)^2\Big] \ge C \int_{\mathbb{S}^{k-1}} r(w)^{2-\beta}\, dw \tag{19}$$

for all possible choice of real numbers $\alpha_1, \ldots, \alpha_n$.

Using (15), (17) and spherical coordinate $\xi = \rho w$, we have

$$\mathbb{E}\Big[\Big(u(t,x) - \sum_{j=1}^n \alpha_j u(t^j, x^j)\Big)^2\Big] = c\int_0^\infty ds \int_{\mathbb{R}^k} \Big| \sin((t-s)|\xi|)\mathbf{1}_{[0,t]}(s)$$

$$- \sum_{j=1}^n \alpha_j e^{-i(x^j - x)\cdot\xi} \sin((t^j - s)|\xi|)\mathbf{1}_{[0,t^j]}(s)\Big|^2 \frac{d\xi}{|\xi|^{2+k-\beta}}$$

$$\geq c\int_0^{a/2} ds \int_0^\infty \frac{d\rho}{\rho^{3-\beta}} \int_{\mathbb{S}^{k-1}} \Big| \sin((t-s)\rho)$$

$$- \sum_{j=1}^n \alpha_j e^{-i\rho(x^j - x)\cdot w} \sin((t^j - s)\rho)\Big|^2 dw$$

$$= c\int_0^{a/2} ds \int_{-\infty}^\infty \frac{d\rho}{|\rho|^{3-\beta}} \int_{\mathbb{S}^{k-1}} \Big| \big(e^{i(t-s)\rho} - e^{-i(t-s)\rho}\big)$$

$$- \sum_{j=1}^n \alpha_j e^{-i\rho(x^j - x)\cdot w} \big(e^{i(t^j-s)\rho} - e^{-i(t^j-s)\rho}\big)\Big|^2 dw$$

Let $\lambda = \min\{1, a/[2(a' + 2\sqrt{k}b)]\}$ and consider the bump function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by

$$\varphi(y) = \begin{cases} \exp\left(1 - \frac{1}{1-|\lambda^{-1}y|^2}\right), & |y| < \lambda, \\ 0, & |y| \geq \lambda. \end{cases}$$

Let $\varphi_r(y) = r^{-1}\varphi(y/r)$. For each $w \in \mathbb{S}^{k-1}$ such that $r(w) > 0$, consider the integral

$$I(w) = \int_0^{a/2} ds \int_{-\infty}^{\infty} \Big[\left(e^{i(t-s)\rho} - e^{-i(t-s)\rho}\right) - \sum_{j=1}^{n} \alpha_j e^{-i\rho(x^j - x)\cdot w}\left(e^{i(t^j-s)\rho} - e^{-i(t^j-s)\rho}\right)\Big] e^{-i(t-s)\rho}\widehat{\varphi}_{r(w)}(\rho)d\rho.$$

By the inverse Fourier transform, we have

$$I(w) = 2\pi \int_0^{a/2} \Big[\varphi_{r(w)}(0) - \varphi_{r(w)}\big(2(t-s)\big) - \sum_{j=1}^{n} \alpha_j \Big\{ \varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t)\big) - \varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t) + 2(t^j - s)\big) \Big\} \Big] \, ds.$$

Note that $r(w) \le |t^j - t| + |x^j - x| \le a' + 2\sqrt{k}b$. For any $s \in [0, a/2]$, we have $2(t-s)/r(w) \ge a/[(a' + 2\sqrt{k}b)]$ and $|(x^j - x)\cdot w - (t^j - t)|/r(w) \ge 1$, thus

$$\varphi_{r(w)}\big(2(t-s)\big) = 0 \;\text{ and }\; \varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t)\big) = 0 \;\text{ for } j = 1, \ldots, n.$$

Also,

$$[(x^j - x)\cdot w - (t^j - t) + 2(t^j - s)]/r(w) \ge (-\delta + a)/[(a' + 2\sqrt{k}b)] \ge \lambda,$$

we have

$$\varphi_{r(w)}\big((x^j - x)\cdot w - (t^j - t) + 2(t^j - s)\big) = 0.$$

It follows that

$$I(w) = a\pi\, r(w)^{-1}.$$

On the other hand, by the Cauchy–Schwarz inequality and scaling, we obtain

$$
\begin{aligned}
(a\pi)^2 r(w)^{-2} = |I(w)|^2 &\le A(w) \times \int_0^{a/2} ds \int_{-\infty}^{\infty} |\widehat{\varphi}(r(w)\rho)|^2 |\rho|^{3-\beta}\, d\rho \\
&= (a/2) A(w) r(w)^{\beta-4} \int_{-\infty}^{\infty} |\widehat{\varphi}(\rho)|^2 |\rho|^{3-\beta}\, d\rho \\
&= C A(w) r(w)^{\beta-4}
\end{aligned}
$$

for some finite constant C. Hence we have

$$A(w) \ge C' r(w)^{2-\beta} \tag{20}$$

and this remains true if $r(w) = 0$. Integrating both sides of (20) over $\mathbb{S}^{k-1}$ yields (19).

As an application of Theorem 4.4., Lee and X. (2019) proved the following uniform modulus of continuity.

Theorem 4.5 [Lee and X. (2019)]

There is a positive finite constant K such that

$$\lim_{\varepsilon\to 0+} \sup_{\substack{(t,x),(t',x')\in I,\\ |(t,x)-(t',x')|\le\varepsilon}} \frac{|u(t,x)-u(t',x')|}{\gamma\big[(t,x),(t',x')\big]} = K, \quad \text{a.s.,} \tag{21}$$

where

$$\gamma\big[(t,x),(t',x')\big] = \big(|t-t'|+|x-x|\big)^{2-\beta}\sqrt{\log\big[|t-t'|+|x-x|\big]^{-1}}.$$

Remark Theorems 4.4 and 4.5 have been extended to SWE with fractional-colored noise by Lee (2021).

Thank you