Exact Results on Regularity of Gaussian Random Fields

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3. Exact results on regularity of Gaussian random fields

For a Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$, we study:

(i) Local modulus of continuity: law of the iterated logarithm (LIL)

(ii) Chung's law of the iterated logarithm

(iii) Uniform modulus of continuity

(iv) Modulus of non-diffenerability

3.1 Local modulus of continuity

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered, real-valued Gaussian random field. For local oscillation of $X(t)$ near a fixed point $t^0 \in \mathbb{R}^N$, we may study the following question: Are there functions $\varphi_1, \varphi_2 : \mathbb{R}^N \to \mathbb{R}_+$ and constants $c_1, c_2 \in (0, \infty)$ such that

$$\limsup_{r\to 0} \max_{|h|\le r} \frac{|X(t^0+h) - X(t^0)|}{\varphi_1(h)} = \kappa_1, \qquad \text{a.s.,}$$

and

$$\liminf_{r\to 0} \max_{|h|\le r} \frac{|X(t^0+h) - X(t^0)|}{\varphi_2(h)} = \kappa_2, \qquad \text{a.s.?}$$

The answers to these questions are referred to as the LIL and Chung's LIL, respectively. They describe different aspects of X near t^0 and rely on different methods.

Many authors have studied these questions for Gaussian random fields, usually under the extra condition of stationarity, or stationarity of increments. See, e.g., the book by Marcus and Rosen (2006) for Gaussian processes, Li and Shao (2001), Meerschaert, Wang and X. (2013) for Gaussian random fields.

We will use the following setting from Dalang, Mueller and X. (2017), which does not require stationarity of the Gaussian random field nor its increments, and can handle anisotropy. It is more convenient for applications to the solutions of linear SPDEs.

Condition (A1)

Consider a compact interval $T \subset \mathbb{R}^N$. There exists a Gaussian random field $\{v(A,t) : A \in \mathscr{B}(\mathbb{R}_+),\ t \in T\}$ such that

(a) For all $t \in T$, $A \mapsto v(A,t)$ is a real-valued Gaussian noise, $v(\mathbb{R}_+, t) = X(t)$, and $v(A, \cdot)$ and $v(B, \cdot)$ are independent whenever A and B are disjoint.

Condition (A1) (continued)

(b) There are constants $a_0 \geq 0$ and $\gamma_j > 0$, $j = 1, \ldots, N$ such that for all $a_0 \leq a \leq b \leq \infty$ and $s = (s_1, \ldots, s_N)$, $t = (t_1, \ldots, t_N) \in T$,

$$\begin{aligned} &\Big\| v([a,b),s) - X(s) - v([a,b),t) + X(t) \Big\|_{L^2} \\ &\leq C\Big(\sum_{j=1}^{N} a^{\gamma_j} |s_j - t_j| + b^{-1} \Big), \end{aligned} \tag{1}$$

where $\|Y\|_{L^2} = \big[\mathbb{E}(Y^2)\big]^{1/2}$ for a random variable Y and

$$\Big\| v([0,a_0),s) - v([0,a_0),t) \Big\|_{L^2} \leq C \sum_{j=1}^{N} |s_j - t_j|. \tag{2}$$

The parameters γ_j ($j = 1, \ldots, N$) are important for characterizing sample path properties of $X(t)$.

Let

$$H_j = (\gamma_j + 1)^{-1} \quad \text{and} \quad Q = \sum_{j=1}^{N} H_j^{-1}.$$

Define the metric $\rho(s, t)$ on $\mathbb{R}^N$ by

$$\rho(s, t) = \sum_{j=1}^{N} |s_j - t_j|^{H_j}.$$

In order to see that (A1) is satisfied by the solution of an SPED, one needs to construct the random field $v(A, x)$.

As an example, consider the solution of the linear one-dimensional heat equation driven by space-time white noise.

In this case, $\mathbb{R}^N$ is replaced by $\mathbb{R}_+ \times \mathbb{R}$, and $X(t)$ is $u(t, x)$. Dalang, Mueller and X. (2017) defined

$$v(A, t, x) = \iint_{\max(|\tau|^{\frac{1}{4}}, |\xi|^{\frac{1}{2}}) \in A} e^{-i\xi x} \frac{e^{-i\tau t} - e^{-t\xi^2}}{|\xi|^2 - i\tau} W(d\tau, d\xi),$$

and verified that (A1) is satisfied with $\gamma_1 = 3$, $\gamma_2 = 1$. Thus, $H_1 = 1/4$ and $H_2 = 1/2$.

The following lemmas are needed for applying general Gaussian methods. For example, Lemma 2.1 can be applied to derive an upper bound for the uniform modulus of continuity for $\{X(t), t \in T\}$.

Lemma 3.1 [Dalang, Mueller, X. (2017)]

Under (A1), there is a constant $c \in (0, \infty)$ such that $\rho(s,t)$

$$d_X(s,t) \le c\,\rho(s,t), \quad \forall\, s,t \in T, \tag{3}$$

where $d_X(s,t) = \|X(s) - X(t)\|_{L^2}$ is the canonical metric.

Proof. For any $a > a_0$,

$$d_X(s,t) \le \|X(s) - v([a_0, a[, s) - X(t) + v([a_0, a[, t)\|_{L^2} + \|v([a_0, a[, s) - v([a_0, a[, t)\|_{L^2}.$$

By (1) in (A1)(b), we have

$$\begin{aligned}
&\|v([a_0,a[,s)-v([a_0,a[,t)\|_{L^2}\\
&\le \|X(s)-v([a,\infty[,s)-X(t)+v([a,\infty[,t)\|_{L^2}\\
&\qquad + \|-v([0,a_0[,s)+v([0,a_0[,t)\|_{L^2}.
\end{aligned}$$

Applying (2) in (A1)(b), we see that

$$d_X(s,t) \le C\left[\sum_{j=1}^{N}\left(a_0^{H_j^{-1}-1}+a^{H_j^{-1}-1}\right)|s_j-t_j| + a^{-1} + \sum_{j=1}^{N}|s_j-t_j|\right].$$

By hypothesis, $\max_{j=1,\dots,N}|s_j-t_j|^{H_j} \le \rho(s,t) \le C\,a_0^{-1}$, so we choose $a \ge a_0$ such that $\max\limits_{j=1,\dots,N}|s_j-t_j|^{H_j} = a^{-1}$.

Notice that

$$
\begin{aligned}
&(a_0^{H_j^{-1}-1} + a^{H_j^{-1}-1})|s_j - t_j| \\
&= \left[\left(a_0\,|s_j - t_j|^{H_j}\right)^{\frac{1-H_j}{H_j}} + \left(a\,|s_j - t_j|^{H_j}\right)^{\frac{1-H_j}{H_j}}\right]|s_j - t_j|^{H_j} \\
&\le 2\left(a\,|s_j - t_j|^{H_j}\right)^{\frac{1-H_j}{H_j}}\,|s_j - t_j|^{H_j} \\
&\le 2|s_j - t_j|^{H_j}
\end{aligned}
$$

by the choice of a. This proves (3).

Condition (A1) indicates that $X(t)$ can be approximated by $v([a,b],t)$. The following lemma quantifies the approximation error.

Lemma 3.2 [Dalang, Mueller, X. (2017)]

Assume that (A1) holds. Consider $b > a > 1$ and $r > 0$ small. Set

$$A = \sum_{j=1}^{N} a^{H_j^{-1}-1}\, r^{H_j^{-1}} + b^{-1}.$$

There are constants A_0, K and c such that for $A \le A_0 r$ and

$$u \ge KA \log^{1/2}\left(\frac{r}{A}\right), \tag{4}$$

Lemma 3.2 (Continued)

we have for all $t^0 \in T$,

$$\begin{aligned} &\mathbb{P}\Big\{ \sup_{t \in S(t^0,r)} \big|X(t) - X(t^0) - (v([a,b],t) - v([a,b],t^0))\big| \geq u \Big\} \\ &\leq \exp\Big(- \frac{u^2}{cA^2} \Big), \end{aligned} \tag{5}$$

where $S(t^0, r) = \{t \in T : \rho(t,\ t^0) \leq r\}$.

The proof of Lemma 3.2 makes use of the following important inequality from Lemma 2.1 in Talagrand (1995).

Lemma 3.3

Let D be the d_X-diameter of a subset $S \subset \mathbb{R}^N$. There is a universal constant K such that, for all $u > 0$, we have

$$\mathbb{P}\left\{ \sup_{s,t \in S} \big|X(s) - X(t)\big| \ge K\Big(u + \int_0^D \sqrt{\log N(S, d_X, \varepsilon)}\, d\varepsilon\Big)\right\} \le \exp\Big(-\frac{u^2}{D^2}\Big). \tag{6}$$

Proof of Lemma 3.2. Set

$$\tilde{d}(s,t) = \big\|X(s) - X(t) - (v([a,b[,s) - v([a,b[,t)\big\|_{L^2}.$$

Then

$$\tilde{d}(s,t) \le \big\|X(s) - X(t)\|_{L^2} + \|v([a,b[,s) - v([a,b[,t)\big\|_{L^2}.$$

Since

$$\begin{aligned} X(s) - X(t) = &\big(v([a,b[,s) - v([a,b[,t)\big) \\ &+ \big(v(\mathbb{R}_+ \setminus [a,b[,s) - v(\mathbb{R}_+ \setminus [a,b[,t)\big), \end{aligned}$$

and the two terms on the right-hand side are independent by (A1)(a), we see that

$$\|v([a,b[,s) - v([a,b[,t)\|_{L^2} \le \|X(s) - X(t)\|_{L^2}.$$

By Lemma 3.1, we obtain

$$\tilde{d}(s,t) \le 2\big\|X(s) - X(t)\big\|_{L^2} \le K\rho(s,t) \tag{7}$$

Therefore, for small $\varepsilon > 0$, the number of ε-balls (in metric $\tilde{d}$) needed to cover $S(t^0, r)$ is

$$N(S(t^0, r), \tilde{d}, \varepsilon) \le c\frac{r^Q}{\varepsilon^Q}.$$

For $t \in S(t^0, r)$, $|t_j - t_j^0| \le r^{H_j^{-1}}$, so by (1), we have

$$\tilde{d}(t,\, t^0) \le CA.$$

Therefore the diameter D of $S(t^0, r)$ satisfies $D \le CA$.

Assuming that we have chosen the constant A_0 and that $A \le A_0 r$, then such that

$$\begin{aligned}
&\int_0^D \sqrt{\log N(S(t^0, r), \tilde{d}, \varepsilon)}\, d\varepsilon \\
&\le K \int_0^{CA} \sqrt{\log\left(\frac{r}{\varepsilon}\right)}\, d\varepsilon \\
&\le KA \sqrt{\log \frac{r}{A}}.
\end{aligned}$$

It follows from this and Lemma 3.3 that (5) holds for all $u \ge KA \log^{1/2}\left(\frac{r}{A}\right)$. This proves Lemma 3.2.

Similarly, Lemma 3.3 implies

Lemma 3.5 [Meerschaert, Wang, X. (2013)]

Assume that (A1) holds. Then there exist positive and finite constants u_0 and C such that for all $t^0 \in T$, and $u \geq u_0$

$$\mathbb{P}\bigg(\sup_{s:|s_j| \leq a_j} |X(t^0 + s) - X(t^0)| \geq u \sum_{j=1}^{N} a_j^{H_j} \bigg) \leq e^{-C\,u^2}$$

for all $a = (a_1, \ldots, a_N) \in (0, 1]^N$ such that $[t^0 - a, t^0 + a] \subseteq T$.

Law of the iterated logarithm

Besides (A1), we also need the following condition.

Condition (A2)

$\|X(t)\|_{L^2} \geq c > 0$ for all $t \in T$ and

$$\mathbb{E}\big[(X(s) - X(t))^2\big] \geq K\rho(s,t)^2 \quad \text{for all } \; s, t \in T.$$

Law of the iterated logarithm

Theorem 3.1

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field that satisfies (A1) and (A2). Then for every $t^0 \in \mathbb{R}^N$, there is a constant $\kappa_1 = \kappa_1(t^0) \in (0, \infty)$ such that

$$\limsup_{|h|\downarrow 0} \sup_{s\in[-h,h]} \frac{|X(t^0+s) - X(t^0)|}{\varphi_1(s)} = \kappa_1, \qquad \text{a.s.}, \tag{8}$$

where

$$\varphi_1(s) = \rho(0,s)\left[\log\log\left(1 + \frac{1}{\prod_{j=1}^N |s_j|^{H_j}}\right)\right]^{\frac{1}{2}}, \qquad \forall\, s \in \mathbb{R}^N.$$

Proof of Theorem 3.1. For any $h \in (0,1)^N$, put

$$M(h) = \sup_{s \in [-h,h]} \frac{|X(t^0+s) - X(t^0)|}{\varphi_1(s)}.$$

We claim that there exist constants $c_{3,1}, c_{3,2} \in (0, \infty)$ such that

$$\limsup_{|h| \to 0} M(h) \le c_{3,1} \qquad \text{a.s.} \tag{9}$$

and

$$\limsup_{|h| \to 0} M(h) \ge c_{3,2} \qquad \text{a.s.} \tag{10}$$

Before proving (9) and (10), let us notice that, (9), (10) and the proof of Lemma 7.1.1 in Marcus and Rosen (2006) imply (8) and the constant $\kappa_1 \in [c_{3,2}, c_{3,1}]$.

Proof of (9). Let $\delta > 0$ be a constant whose value will be determined later. For any $\boldsymbol{n} = (n_1, ..., n_N) \in \mathbb{N}^N$, define the event

$$F_{\boldsymbol{n}} = \left\{ \sup_{s: 2^{-n_j} \le |s_j| \le 2^{-n_j+1}} \varphi_1(s)^{-1} |X(t^0 + s) - X(t^0)| \ge \delta \right\}.$$

By Condition (A1), we see that for any $s \in \mathbb{R}^N$ that satisfies $2^{-n_j} \le |s_j| \le 2^{-n_j+1}$ for $j = 1, \ldots, N$, we have

$$\varphi_1(s) \ge \left(\sum_{j=1}^N 2^{-n_j H_j} \right) \sqrt{\log\log\left(1 + \prod_{j=1}^N 2^{(n_j-1)H_j}\right)}.$$

This and Lemma 3.5 imply

$$\mathbb{P}(F_{\boldsymbol{n}}) \le \exp\left(- C\delta^2 \log\log\Big(1 + \prod_{j=1}^{N} 2^{(n_j-1)H_j}\Big)\right)$$
$$\le K \left(\sum_{j=1}^{N} n_j\right)^{-C\delta^2}.$$

By taking δ large enough such that $C\,\delta^2 > N$, we see that

$$\sum_{\boldsymbol{n}\in\mathbb{N}^N} \mathbb{P}(F_{\boldsymbol{n}}) \le K \sum_{\boldsymbol{n}\in\mathbb{N}^N} |\boldsymbol{n}|^{-C\delta^2} < \infty.$$

Thus, by the Borel-Cantelli lemma, a.s. only finitely many of the events $F_{\boldsymbol{n}}$ occur.

This implies

$$\limsup_{|\mathbf{n}|\to\infty} \sup_{s:2^{-n_j}\le|s_j|\le 2^{-n_j+1}} \frac{|X(t^0+s)-X(t^0)|}{\varphi_1(s)} \le \delta \qquad \text{a.s.}$$

This and a monotonicity argument yield (9).

Proof of (10). It is sufficient to provide a sequence $\{h_n\} \subset (0,1)^N$ such that $|h_n| \to 0$ and

$$\limsup_{n\to\infty} \frac{|X(t^0+h_n)-X(t^0)|}{\varphi_1(h_n)} \ge \sqrt{2} \qquad \text{a.s.} \tag{11}$$

This will be done by using the Borel-Cantelli lemma.

For $0 < \mu < 1$ and $n \geq 1$, define $h_n = (h_{n,1}, \ldots, h_{n,N})$ by

$$h_{n,j} = \exp\left(- H_j^{-1} n^{1+\mu} \right) \quad (j = 1, \ldots, N).$$

Then $\rho(0, h_n) = N \exp(-n^{1+\mu})$.
Let $\beta > 0$ be a constant and let $d_n = \exp(n^{1+\mu}) n^{-\beta}$.
For $s \in \mathbb{R}^N$, we write $X(s) = X_n(s) + \widetilde{X}_n(s)$, where

$$X_n(s) = v([d_n, d_{n+1}), s) \text{ and } \widetilde{X}_n(s) = X(s) - v([d_n, d_{n+1}), s).$$

Then $\{X_n(s), s \in \mathbb{R}^N\}$ and $\{\widetilde{X}_n(s), s \in \mathbb{R}^N\}$ are independent. Moreover, the sequence $\{X_n(s), s \in \mathbb{R}^N\}, n = 1, 2, \ldots$ are independent.

Notice that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{|X(t^0+h_n)-X(t^0)|}{\varphi_1(h_n)}\\
&\geq \limsup_{n\to\infty} \frac{|X_n(t^0+h_n)-X_n(t^0)|}{\varphi_1(h_n)} - \limsup_{n\to\infty} \frac{|\widetilde{X}_n(t^0+h_n)-\widetilde{X}_n(t^0)|}{\varphi_1(h_n)}\\
&:= \limsup_{n\to\infty} I_1(n) - \limsup_{n\to\infty} I_2(n).
\end{aligned}$$

We use (A1) to show that $I_1(n)$ is the main term and $I_2(n)$ is negligible.

By (A1) (b), we have

$$
\begin{aligned}
&\mathbb{E}\big(\widetilde{X}_n(t^0+h_n)-\widetilde{X}_n(t^0)\big)^2\\
&\le C\left(\sum_{j=1}^{N} d_n^{H_j^{-1}-1} h_{n,j}+d_{n+1}^{-1}\right)^2\\
&=\rho(0,h_n)^2\cdot CN^{-2}\exp(2n^{1+\mu})\left(\sum_{j=1}^{N} d_n^{H_j^{-1}-1} h_{n,j}+d_{n+1}^{-1}\right)^2\\
&\le C\rho(0,h_n)^2\cdot\left(\sum_{j=1}^{N} n^{-\beta(H_j^{-1}-1)}+\exp(-n^{\mu})(n+1)^{\beta}\right)^2\\
&\le C\rho(0,h_n)^2\cdot n^{-2\beta(\overline{H}^{-1}-1)},
\end{aligned}
$$

for $n \ge n_0$, where $\overline{H} = \max\limits_{1\le j\le N}\{H_j\}$.

Hence, for any $\eta \in (0, 1)$,

$$\begin{aligned}
&\mathbb{P}\Big(|\widetilde{X}_n(t^0 + h_n) - \widetilde{X}_n(t^0)| \geq \eta\varphi_1(h_n)\Big\}\\
&\leq \mathbb{P}\Big(|N(0, 1)| \geq C\eta\sqrt{\log n}\, n^{\beta(\overline{H}^{-1}-1)}\Big\}\\
&\leq n^{-2}
\end{aligned}$$

for all n large enough.
Thus, the Borel-Cantelli lemma and the arbitrariness of η imply $\limsup_{n\to\infty} I_2(n) = 0$. a.s.

On the other hand, by the independence of X_n and $\widetilde{X}_n$, (A2) and (A1)(b), we have

$$
\begin{aligned}
&\mathbb{E}\big(X_n(t^0 + h_n) - X_n(t^0)\big)^2 \\
&= \mathbb{E}\big(X(t^0 + h_n) - X(t^0)\big)^2 - \mathbb{E}\big(\widetilde{X}_n(t^0 + h_n) - \widetilde{X}_n(t^0)\big)^2 \\
&\geq C\rho(0, h_n)^2
\end{aligned}
$$

for all n large enough. It follows that

$$
\begin{aligned}
&\mathbb{P}\Big(|X_n(t^0 + h_n) - X_n(t^0)| \geq \eta\sqrt{2C}\varphi_1(h_n)\Big\} \\
&\geq \mathbb{P}\Big(|N(0, 1)| \geq \eta\sqrt{(1 + \mu)\log n}\Big\} \\
&\geq \frac{1}{\eta\sqrt{2\pi(1 + \mu)\log n}}\, n^{-\eta^2(1+\mu)}
\end{aligned}
$$

for all n large enough.

If the constants η and μ are chosen such that $\eta^2(1+\mu) \le 1$, we have

$$\sum_{n=1}^{\infty} \mathbb{P}\Big(|X_n(t^0 + h_n) - X_n(t^0)| \ge \eta\sqrt{2C}\varphi_1(h_n)\Big) = \infty.$$

Since the events in the above are independent, the Borel-Cantelli lemma implies that

$$\limsup_{n\to\infty} I_1(n) \ge \eta\sqrt{2C} \quad \text{a.s.}$$

This finishes the proof of Theorem 3.1.

3.2 Chung's law of the iterated logarithm

For studying Chung's LIL at $t^0 \in T$, we need the following assumption on the small ball probability of X

Condition (A3)

There is a constant c such that for all $t^0 \in T$, $r > 0$ and $0 < \varepsilon < r$,

$$\mathbb{P}\Big\{ \max_{\rho(s,t^0)\le r} |X(s) - X(t^0)| \le \varepsilon \Big\} \le \exp\Big(- c\big(\frac{r}{\varepsilon}\big)^Q \Big).$$

A similar lower bound for $\mathbb{P}\big\{ \max_{\rho(s,t^0)\le r} |X(s)| \le \varepsilon \big\}$ is given in Lemma 3.7 below, which can be proved by applying the following general result due to Talagrand (1993) [cf. p. 257, Ledoux (1996)].

Lemma 3.6 [Talagrand (1993)]

Let $\{Y(t), t \in S\}$ be an $\mathbb{R}$-valued centered Gaussian process indexed by a bounded set S. If there is a decreasing function $\psi : (0, \delta] \to (0, \infty)$ such that $N(S, d_Y, \varepsilon) \le \psi(\varepsilon)$ for all $\varepsilon \in (0, \delta]$ and there are constants $c_{3,4} \ge c_{3,3} > 1$ such that

$$c_{3,3}\psi(\varepsilon) \le \psi(\varepsilon/2) \le c_{3,4}\psi(\varepsilon) \tag{12}$$

for all $\varepsilon \in (0, \delta]$, then there is a constant K depending only on $c_{3,3}$ and $c_{3,4}$ such that for all $u \in (0, \delta)$,

$$\mathbb{P}\left(\sup_{s,t \in S} |Y(s) - Y(t)| \le u \right) \ge \exp\big(- K\psi(u)\big). \tag{13}$$

Lemma 3.7

Under (A1), there is a constant $c' \in (0, \infty)$ such that for every $t^0 \in T$, $r > 0$ and $0 < \varepsilon < r$,

$$\mathbb{P}\Big\{ \max_{\rho(s,t^0) \le r} |X(s) - X(t^0)| \le \varepsilon \Big\} \ge \exp\Big(- c' \big(\frac{r}{\varepsilon}\big)^Q \Big). \tag{14}$$

Proof. Let $S = \{s \in T : \rho(s, t^0) \le r\}$. It follows from Lemma 2.1 that for all $\varepsilon \in (0, r)$,

$$N(S, d_X, \varepsilon) \le c \prod_{i=1}^{N} \Big(\frac{r}{\varepsilon}\Big)^{\frac{1}{H_i}} = c\, \Big(\frac{r}{\varepsilon}\Big)^Q := \psi(\varepsilon).$$

Clearly $\psi(\varepsilon)$ satisfies the condition (12) in Lemma 3.6. Hence the lower bound in (14) follows from (13).

Theorem 3.2 [Chung's LIL]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian random field that satisfies (A1) and (A3). Then for every $t^0 \in T$, there is a constant $\kappa_2 = \kappa_2(t^0) \in (0, \infty)$ such that

$$\liminf_{r\to 0} \frac{\max_{s:\rho(s,t^0)\le r} |X(t^0+s) - X(t^0)|}{r(\log\log 1/r)^{-1/Q}} = \kappa_2, \qquad \text{a.s.,} \tag{15}$$

where $Q = \sum_{j=1}^N H_j^{-1}$.

Chung's LIL describes the smallest local oscillation of $X(t)$, which is useful for studying hitting probabilities and fractal properties of X.

Proof of Theorem 3.2. Assumption (A1) implies a 0-1 law for the limit in the left hand-side of (15).
We need to prove that $\kappa_2 \in (0, \infty)$. It is sufficient to prove that for some constants $c_{3,5}, c_{3,6} \in (0, \infty)$,

$$\liminf_{r \to 0} \frac{\max_{s:\rho(s,t^0) \le r} |X(t^0 + s) - X(t^0)|}{r(\log \log 1/r)^{-1/Q}} \ge c_{3,5}, \qquad \text{a.s.,} \qquad (16)$$

and

$$\liminf_{r \to 0} \frac{\max_{s:\rho(s,t^0) \le r} |X(t^0 + s) - X(t^0)|}{r(\log \log 1/r)^{-1/Q}} \le c_{3,6}, \qquad \text{a.s.} \qquad (17)$$

In fact, (16) and (17) imply that $\kappa_2 \in [c_{3,5}, c_{3,6}]$.

Proof of (16). For any integer $n \geq 1$, let $r_n = e^{-n}$. Let $\eta > 0$ be a constant and consider the event

$$A_n = \left\{ \max_{\rho(s,t^0) \leq r_n} |X(s) - X(t^0)| \leq \eta r_n (\log\log 1/r_n)^{-1/Q} \right\}.$$

By (A3) we have

$$\mathbb{P}(A_n) \leq \exp\left(- \frac{c}{\eta^Q} \log n \right) = n^{-c/\eta^Q},$$

which is summable if $\eta > 0$ is chosen small enough. Hence, (16) follows from the Borel-Cantelli lemma.

Proof of (17). For every integer $n \geq 1$, we take $r_n = e^{-(n+n^2)}$ and $d_n = e^{n^2}$. Then it follows that

$$r_n \, d_n = e^{-n} \quad \text{and} \quad r_n \, d_{n+1} > e^n.$$

It's sufficient to prove that there exists a finite constant $c_{3,7}$ such that

$$\liminf_{n\to\infty} \frac{\max_{\rho(s,\,t^0)\leq r_n} |X(t^0+s) - X(t^0)|}{r_n(\log\log 1/r_n)^{-1/Q}} \leq c_{3,7} \qquad \text{a.s.} \qquad (18)$$

For proving (18) we will use (A1) to decompose X in a way similar to that in the proof of Theorem 3.1.

Define two Gaussian fields X_n and $\widetilde{X}_n$ by

$$X_n(s) = v([d_n, d_{n+1}), s) \text{ and } \widetilde{X}_n(s) = X(s) - X_n(s).$$

Then the Gaussian fields $\{X_n(s), s \in \mathbb{R}^N\}$ $(n = 1, 2, \cdots)$ are independent and for every $n \geq 1$, X_n and $\widetilde{X}_n$ are independent as well.

Denote $\gamma(r) = r(\log\log 1/r)^{-1/Q}$. We make the following two claims:
(i). There is a constant $\eta > 0$ such that

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{ \max_{\rho(s,t^0)\le r_n} |X_n(t^0+s) - X_n(t^0)| \le \eta\gamma(r_n) \right\} = \infty. \tag{19}$$

(ii). For every $\eta_1 > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}\left\{ \max_{\rho(s,\,t^0)\le r_n} |\widetilde{X}_n(t^0+s) - X_n(t^0)| > \eta_1\gamma(r_n) \right\} < \infty. \tag{20}$$

Since the events in (19) are independent, we see that (18) follows from (19), (20) and the Borel-Cantelli Lemma.

It remains to verify the claims (i) and (ii) above.

By Lemma 3.7 and Anderson's inequality [see Anderson (1955)], we have

$$
\begin{aligned}
&\mathbb{P}\Big\{ \max_{\rho(s,\,t^0)\le r_n} |X_n(t^0+s) - X_n(t^0)| \le \eta\gamma(r_n) \Big\} \\
&\quad \ge \mathbb{P}\Big\{ \max_{\rho(s,\,t^0)\le r_n} |X(t^0+s) - X(t^0)| \le \eta\gamma(r_n) \Big\} \\
&\quad \ge \exp\Big(-\frac{c'}{\eta^Q} \log(n+n^2) \Big) \\
&\quad = (n+n^2)^{-c'/\eta^Q}.
\end{aligned}
$$

Hence (i) holds for $\eta > (2c')^{1/Q}$.

To prove (ii), we let $S = \{s \in T : \rho(s, t^0) \le r_n\}$ and consider on S the metric

$$\tilde{d}(s,t) = \left\|\widetilde{X}_n(t^0+s) - \widetilde{X}_n(t^0+t)\right\|_{L^2}.$$

By Lemma 3.1 we have $\tilde{d}(s,t) \le c\sum_{i=1}^N |s_i - t_i|^{H_i}$ for all $s, t \in T$ and hence

$$N(S, \tilde{d}, \varepsilon) \le c\left(\frac{r_n}{\varepsilon}\right)^Q.$$

Now we estimate the $\tilde{d}$-diameter $\widetilde{D}$ of S. By (1) in (A1),

$$\tilde{d}(s,t) \le C\left(\sum_{j=1}^N d_n^{H_j^{-1}-1}|s_j - t_j| + d_{n+1}^{-1}\right) \le C\, e^{-n^2-(\overline{H}^{-1}\wedge 2)n}.$$

Thus $\widetilde{D} \le C\, e^{-n^2-(\overline{H}^{-1}\wedge 2)n}.$

Notice that $\widetilde{D} \le r_n e^{-((\overline{H}^{-1} \wedge 2)-1)n}$. The Dudley's integral is

$$\int_0^{\widetilde{D}} \sqrt{\log N(S, \tilde{d}, \varepsilon)}\, d\varepsilon \le \int_0^{\widetilde{D}} \sqrt{\log(\frac{r_n}{\varepsilon})^Q}\, d\varepsilon$$
$$\le C r_n \sqrt{n}\, e^{-((\overline{H}^{-1} \wedge 2)-1)n}.$$

Hence for any $\eta_1 > 0$, it follows from by Lemma 3.3 that for all n large,

$$\mathbb{P}\left\{ \max_{\rho(s,t^0) \le r_n} |\widetilde{X}_n(t^0 + s) - X_n(t^0)| > \eta_1 \gamma(r_n) \right\}$$
$$\le \exp\left(- K \frac{\eta_1^2 \gamma(r_n)^2}{\widetilde{D}^2} \right)$$
$$\le \exp\left(- K \eta_1^2 (\log n)^{-2/Q}\, e^{((\overline{H}^{-1} \wedge 2)-1)n} \right).$$

Therefore (ii) holds. The proof of Theorem 3.2 is finished.

3.3 Uniform modulus of continuity

In order to prove an exact uniform modulus of continuity, we will make use of Condition (A1) and the following:

Condition (A4) [sectorial local nondeterminism]

There exists a constant $c > 0$ such that for all $n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X(u) \,\big|\, X(t^1), \ldots, X(t^n)\big) \geq c \sum_{j=1}^{N} \min_{1 \leq k \leq n} \big|u_j - t_j^k\big|^{2H_j}. \tag{21}$$

Remarks about (A4)

Condition (A4) and the following (A4$'$) are properties of strong local nondeterminism for Gaussian random fields with certain anisotropy.

Condition (A4$'$) [strong local nondeterminism]

There exists a constant $c > 0$ such that $\forall\ n \geq 1$ and $u, t^1, \ldots, t^n \in T$,

$$\mathrm{Var}\big(X(u) \,\big|\, X(t^1), \ldots, X(t^n)\big) \geq c \min_{1 \leq k \leq n} \rho(u,\, t^k)^2. \tag{22}$$

- The concept of local nondeterminism (LND) of a Gaussian process was first introduced by Berman (1973) for studying local times of Gaussian processes.
- Pitt (1978) extended Berman's definition to the setting of random fields.
- Cuzick and DuPreez (1982) introduced strong local ϕ-nonderterminism for Gaussian processes and showed its usefulness in studying local times.
- The "sectorial local nondeterminism" was first discovered by Khoshnevisan and X. (2007) for the Brownian sheet; and extended to fractional Brownian sheets by Wu and X. (2007).
- X. (2009), Luan and X. (2012) proved "strong local nondeterminism" for a large class of Gaussian fields with stationary increments.

Uniform modulus of continuity

Theorem 3.3 [Meerschaert, Wang and X. (2013)]

If a centered Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$ satisfies

$$\mathbb{E}\Big[\big(X(s) - X(t)\big)^2\Big] \le c\rho(s,t)^2 \quad \text{for all } \; s, t \in T \tag{23}$$

and (A4). then

$$\lim_{r \to 0} \sup_{t,s \in T, \rho(s,t) \le r} \frac{|X(s) - X(t)|}{\rho(s,t)\sqrt{\log(1 + \rho(s,t)^{-1})}} = \kappa_3, \tag{24}$$

where $\kappa_3 > 0$ is a constant.

Due to the monotonicity in r, the limit in the left-hand side of (24) exists a.s. We only need to prove that the limit is a positive and finite constant. This is done in three parts:

(a). $$\lim_{r\to 0} \sup_{s,t\in T, \rho(s,t)\le r} \frac{|X(s)-X(t)|}{\rho(s,t)\sqrt{\log(1+\rho(s,t)^{-1})}} \le c_{3,8} < \infty, \quad \text{a.s.}$$

(b). $$\lim_{r\to 0} \sup_{s,t\in T, \rho(s,t)\le r} \frac{|X(s)-X(t)|}{\rho(s,t)\sqrt{\log(1+\rho(s,t)^{-1})}} \ge c_{3,9} > 0, \quad \text{a.s.}$$

(c). Eq. (24) follows from **(a)**, **(b)** and a zero-one law.

The proof of **(a)** relies on the following estimate of the tail probability which follows from Lemma 3.3:
For $\varepsilon > 0$ and $x \geq c\varepsilon\sqrt{\log(1+\varepsilon^{-1})}$,

$$\mathbb{P}\Big\{ \sup_{\substack{s,t \in T,\\ \rho(s,t) \leq \varepsilon}} |X(t) - X(s)| \geq x \Big\} \leq \exp\Big(-K\frac{x^2}{\varepsilon^2}\Big).$$

and a standard Borel-Cantelli argument.

Or one can apply Theorem 2.2 (after Dudley's theorem).

Proof of (b). For any $n \geq 2$, we choose a sequence of points $\{t_{n,k}, 1 \leq k \leq L_n\}$ in T such that

$$\rho(t_{n,k}, t_{n,k-1}) = 2^{-n},$$

and for some direction $i \in \{1, \ldots, N\}$,

$$|t^i_{n,k} - t^i_{n,k-1}| \geq 2^{-n/H_i}, \quad \forall\, 2 \leq k \leq L_n.$$

We take $L_n = \min\{2^{n/H_i}\}$.
We will prove the following stronger statement:

$$\liminf_{n\to\infty} \frac{\max\limits_{2\leq k\leq L_n} |X(t_{n,k}) - X(t_{n,k-1})|}{2^{-n}\sqrt{n}} \geq c_{3,9} > 0, \quad \text{a.s.} \qquad (25)$$

Let $\eta > 0$ be a constant whose value will be chosen later. Consider the events

$$A_n = \Big\{ \max_{2\le k\le L_n} \big|X(t_{n,k}) - X(t_{n,k-1})\big| \le \eta 2^{-n}\sqrt{n} \Big\}$$

and write

$$\begin{aligned}\mathbb{P}\big(A_n\big) = \mathbb{P}\Big\{ \max_{2\le k\le L_n-1} \big|X(t_{n,k}) - X(t_{n,k-1})\big| \le \eta 2^{-n}\sqrt{n} \Big\} \\ \times \mathbb{P}\Big\{ \big|X(t_{n,L_n}) - X(t_{n,L_n-1})\big| \le \eta 2^{-n}\sqrt{n} \big| \widetilde{A}_{L_n-1} \Big\},\end{aligned} \tag{26}$$

where

$$\widetilde{A}_{L_n-1} = \Big\{ \max_{2\le k\le L_n-1} \big|X(t_{n,k}) - X(t_{n,k-1})\big| \le \eta 2^{-n}\sqrt{n} \Big\}.$$

The conditional distribution of the Gaussian random variable $X(t_{n,L_n}) - X(t_{n,L_n-1})$ under $\widetilde{A}_{L_n-1}$ is still Gaussian and, by (A4), its conditional variance satisfies

$$\mathrm{Var}\Big(X(t_{n,L_n}) - X(t_{n,L_n-1})\Big|\widetilde{A}_{L_n-1}\Big) \ge c\, 2^{-2n}.$$

This and Anderson's inequality (1955) imply

$$\begin{aligned}
&\mathbb{P}\Big\{\Big|X(t_{n,L_n}) - X(t_{n,L_n-1})\Big| \le \eta 2^{-n}\sqrt{n}\Big|\tilde{A}_{L_n-1}\Big\}\\
&\le \mathbb{P}\Big\{\Big|N(0,1)\Big| \le c\,\eta\sqrt{n}\Big\} && \big(\text{use Mill's ratio}\big)\\
&\le 1 - \frac{1}{c\eta\sqrt{n}}\exp\Big(-\frac{c^2\eta^2 n}{2}\Big) && \big(\text{use } 1-x \le e^{-x} \text{ for } x>0\big)\\
&\le \exp\Big(-\frac{1}{c\eta\sqrt{n}}\exp\Big(-\frac{c^2\eta^2 n}{2}\Big)\Big).
\end{aligned}$$

Iterating this procedure in (26) for L_n times, we obtain

$$\mathbb{P}(A_n) \le \exp\left(- \frac{1}{c\eta\sqrt{n}} L_n \exp\left(- \frac{c^2\eta^2 n}{2} \right) \right).$$

By taking $\eta > 0$ small enough, we have

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty.$$

Hence the Borel-Cantelli lemma implies (25).

A question

For any $\lambda > 0$, define the set of "fast points"

$$F(\lambda) = \left\{ t \in [0,1]^N : \limsup_{r \to 0} \frac{|X(t+h) - X(t)|}{\rho(0,h)\sqrt{\log \frac{1}{\rho(0,h)}}} \geq \lambda \right\}.$$

Questions:

- What is the Hausdorff dimension of $F(\lambda)$?
- For a given set $E \subset [0,1]^N$, when is

$$\mathbb{P}\{F(\lambda) \cap E \neq \emptyset\} > 0?$$

3.4. Modulus of non-differentiability

Result for FBM (Wang and X. 2019)

For any compact rectangle $T \subseteq \mathbb{R}^N$,

$$\liminf_{\varepsilon \to 0+} \inf_{t \in T} \frac{\sup_{s \in B(t,\varepsilon)} |B^H(s) - B^H(t)|}{\varepsilon^H |\log 1/\varepsilon|^{-H/N}} = \kappa_4, \quad \text{a.s.},$$

where $\kappa_4 \in (0, \infty)$ is a constant related to the small ball probability of B^H.

This result was extended to a large class of (approximately isotropic) Gaussian random fields with stationary increments by Wang, Su and X. (2020). The following theorem is more general and can be applied to SPDEs.

Theorem 3.4 Wang and X. (2021+)

If a centered Gaussian field $X = \{X(t), t \in \mathbb{R}^N\}$ satisfies Condition (A1), (A4′) and a regularity condition on the second order derivative of the covariance function $K(s,t) = \mathbb{E}\big[X(s)X(t)\big]$ on $T \times T \backslash \{(s,s), s \in T\}$. Then

$$\liminf_{r\to 0} \inf_{t\in T} \frac{\max_{\rho(s,t)\le r} |X(s) - X(t)|}{r(\log r^{-1})^{-1/Q}} = \kappa_5, \tag{27}$$

where $\kappa_5 \in (0, \infty)$ is a constant. In particular, the sample function of X is almost surely nowhere differentiable in any direction.

The proof of Theorem 3.4 has three parts:

(a). $$\liminf_{r\to 0} \inf_{t\in T} \frac{\max_{\rho(s,t)\le r} |X(s)-X(t)|}{r(\log r^{-1})^{-1/Q}} \ge c_{3,10} > 0, \quad \text{a.s.}$$

(b). $$\liminf_{r\to 0} \inf_{t\in T} \frac{\max_{\rho(s,t)\le r} |X(s)-X(t)|}{r(\log r^{-1})^{-1/Q}} \le c_{3,11} < \infty, \quad \text{a.s.}$$

(c). A zero-one law for the modulus of non-differentiability [This can be proved under Condition (A1).]

The proof of (a) relies on the following small ball probability estimate and the Borel-Cantelli lemma. Without loss of generality, we assume $T = [0, 1]^N$.

Lemma 3.8 [X. 2009]

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field that satisfies (A1) and (A4′) on $T = [0, 1]^N$. Then there exist constants C and C' such that for every $t \in T$ and $0 < \varepsilon \le r$,

$$\exp\left(-C'\left(\frac{r}{\varepsilon}\right)^Q\right) \le \mathbb{P}\left\{\max_{s \in T: \rho(s,t) \le r} |X(s) - X(t)| \le \varepsilon\right\} \le \exp\left(-C\left(\frac{r}{\varepsilon}\right)^Q\right),$$

where $Q = \sum_{j=1}^N \frac{1}{H_j}$.

Proof of (a). Let $\theta > 1$ be a constant. For any $n \geq 2$, let $\varepsilon_n = \theta^{-n}$. Divide T into L_n rectangles of sides ε_n^{1/H_j} ($j = 1, \ldots, N$). Denote these rectangles by $I_{\boldsymbol{i}}$, where $\boldsymbol{i} = (i_1, \ldots, i_N)$ and $i_j \in \{1, \ldots, \varepsilon_n^{-1/H_j}\}$. Denote the lower-left vertex by $t_{\boldsymbol{i}}$.

Let $\gamma(\varepsilon_n) = \varepsilon_n(\log(1/\varepsilon_n))^{-1/Q}$. By Lemma 3.8, we have

$$\begin{aligned}
&\mathbb{P}\Big(\min_{\boldsymbol{i}} \max_{s \in I_{\boldsymbol{i}}} |X(s) - X(t_{\boldsymbol{i}})| \leq \eta\gamma(\varepsilon_n)\Big) \\
&\leq \sum_{\boldsymbol{i}} \mathbb{P}\Big(\max_{s \in I_{\boldsymbol{i}}} |X(s) - X(t_{\boldsymbol{i}})| \leq \eta\gamma(\varepsilon_n)\Big) \\
&\leq L_n \exp\Big(-C\eta^{-Q} \log(1/\varepsilon_n)\Big) = \theta^{n(Q - C\eta^{-Q})},
\end{aligned}$$

which is summable if $\eta > 0$ is chosen small enough. This and the Borel-Cantelli lemma yield (a).

Proof of (b). Using the notation in the last page, it is sufficient to prove that

$$\liminf_{n\to\infty} \min_{i} \frac{\max_{s,t\in I_i} |X(s) - X(t)|}{\varepsilon_n (\log \varepsilon_n^{-1})^{-1/Q}} \le c_{3,11} < \infty, \quad \text{a.s.}$$

This is more difficult to prove. Besides small ball probability estimates, we make use of the following tools:

- due to non-stationarity, a general framework on limsup random fractals that extends that of Khoshnevisan, Peres, and X. (2000) is needed. This was done by Hu, Li, and X. (2021) for studying random covering sets.
- a correlation inequality of Shao (2003).

Since the proof quite lengthy, we omit the details here.

Thank you!