Regularity of Gaussian Random Fields: Some General Methods

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Outline

1. Continuity of Gaussian random fields
   - The entropy method
   - Majorizing measure
2. Differentiability of Gaussian random fields

2. Regularity of Gaussian random fields: some general methods

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a random field. For each $\omega \in \Omega$, the function $X(\cdot, \omega) : \mathbb{R}^N \to \mathbb{R}^d$, $t \mapsto X(t, \omega)$, is called a sample function of X.

The following are natural questions:

(i) When are the sample functions of X bounded, or continuous?

(ii) When are the sample functions of X differentiable?

(iii) How to characterize the analytic and geometric properties of $X(\cdot)$ precisely?

2.1 The entropy method

We start with some general methods for Gaussian fields.

Let $X = \{X(t), t \in T\}$ be a centered Gaussian process with values in $\mathbb{R}$, where (T, τ) is a metric space; e.g., $T = [0, 1]^N$, or $T = \mathbb{S}^{N-1}$.

We define a pseudo metric $d_X(\cdot, \cdot) : T \times T \to [0, \infty)$ by

$$d_X(s,t) = \sqrt{\mathbb{E}\left[\left(X(t) - X(s)\right)^2\right]}.$$

(d_X is often called the canonical metric for X.)

Let $D = \sup_{t,s \in T} d_X(s,t)$ be the diameter of T, under d_X. For any $\varepsilon > 0$, let $N(T, d_X, \varepsilon)$ be the minimum number of d_X-balls of radius ε that cover T.

$N(T, d_X, \varepsilon)$ is also called the metric entropy of T

Theorem 2.1 [Dudley, 1967]

Assume $N(T, d_X, \varepsilon) < \infty$ for every $\varepsilon > 0$. If

$$\int_0^D \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon < \infty.$$

Then $\exists$ a modification of X, still denoted by X, such that

$$\mathbb{E}\Big(\sup_{t \in T} X(t)\Big) \le 16\sqrt{2} \int_0^{\frac{D}{2}} \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon. \qquad (1)$$

The proof of Dudley's Theorem is based on a chaining argument. See Talagrand (2005), Marcus and Rosen (2007).

The proof of Dudley's Theorem gives an upper bound for the uniform modulus of continuity of X:

$$\omega_{X,\tau}(\delta) = \sup_{s,t\in T,\tau(s,t)\le\delta} \left|X(s) - X(t)\right|.$$

Theorem 2.2

Under the condition of Dudley's theorem, there is a random variable $\eta \in (0, \infty)$ such that for all $0 < \delta < \eta$,

$$\omega_{X,d_X}(\delta) \le K \int_0^{\delta} \sqrt{\log N(T, d_X, \varepsilon)}\, d\varepsilon,$$

where $\omega_{X,d_X}(\delta)$ is the modulus of continuity of $X(t)$ on $(T,\, d_X)$ and K is a universal constant.

Fernique (1975) proved that (1) is also necessary if X is a Gaussian process which is stationary or has stationary increments.

Theorem 2.2 can be applied easily to a wide class of Gaussian processes.
For example,

- fractional Brownian motion (see below)
- solutions of linear stochastic heat and wave equations
- for a Gaussian random field $\{X(t), t \in T\}$ satisfying

$$d_X(s,t) \asymp \left(\log \frac{1}{|s-t|}\right)^{-\gamma},$$

its sample functions are continuous if $\gamma > 1/2$.

Corollary 2.3

Let $B^H = \{B^H(t), t \in \mathbb{R}^N\}$ be a fractional Brownian motion with index $H \in (0, 1)$. Then B^H has a modification, still denoted by B^H, whose sample functions are almost surely continuous. Moreover,

$$\limsup_{\varepsilon \to 0} \frac{\max_{t \in [0,1]^N, |s| \le \varepsilon} |B^H(t+s) - B^H(t)|}{\varepsilon^H \sqrt{\log 1/\varepsilon}} \le K, \quad a.s.$$

Proof: Recall that $d_{B^H}(s,t) = |s-t|^H$ and $\forall\, \varepsilon > 0$,

$$N\big([0,1]^N, d_{B^H}, \varepsilon\big) \le \ K \left(\frac{1}{\varepsilon^{1/H}}\right)^N.$$

It follows from Theorem 2.2 that $\exists$ a random variable $\eta > 0$ and a constant $K > 0$ such that for all $0 < \delta < \eta$,

$$\omega_{B^H}(\delta) \le K \int_0^\delta \sqrt{\log\left(\frac{1}{\varepsilon^{1/H}}\right)}\, d\varepsilon$$

$$\le K\,\delta \sqrt{\log \frac{1}{\delta}} \quad \text{a.s.}$$

Returning to the Euclidean metric and noticing

$$d_{B^H}(s,t) \le \delta \iff |s-t| \le \delta^{1/H},$$

yields the desired result.

Later on, we will prove that there is a constant $K \in (0, \infty)$ such that

$$\limsup_{\varepsilon \to 0} \frac{\max_{t \in [0,1]^N, |s| \le \varepsilon} |B^H(t+s) - B^H(t)|}{\varepsilon^H \sqrt{\log 1/\varepsilon}} = K, \quad a.s.$$

This is an analogue of Lévy's uniform modulus of continuity for Brownian motion.

2.2 Majorizing measure

In general, (1) is not necessary for sample continuity. Talagrand (1987) proved the following necessary and sufficient for the boundedness and continuity.

Theorem 2.4 [Talagrand, 1987]

Let $X = \{X(t),\ t \in T\}$ be a centered Gaussian process with values in $\mathbb{R}$. Suppose $D = \sup_{t,s\in T} d_X(s,t) < \infty$. Then

(i) X has a modification which is bounded on T if and only if there exists a probability measure μ on T such that

$$\sup_{t\in T} \int_0^D \left(\log \frac{1}{\mu(B_{d_X}(t,\varepsilon))}\right)^{1/2} d\varepsilon < \infty, \tag{2}$$

where $B_{d_X}(t,\varepsilon) = \{s \in T : d_X(s,t) \le \varepsilon\}$.

Theorem 2.4 (Continued)

Moreover,

$$\mathbb{E}\Big(\sup_{t\in T} X(t)\Big) \le K \inf_{\mu} \sup_{t\in T} \int_0^{\infty} \left(\log \frac{1}{\mu(B_{d_X}(t,\varepsilon))}\right)^{1/2} d\varepsilon.$$

(ii) There exists a modification of X with bounded, uniformly continuous sample functions if and only if there exists a probability measure μ on T such that

$$\lim_{\varepsilon\to 0} \sup_{t\in T} \int_0^{\varepsilon} \left(\log \frac{1}{\mu(B_{d_X}(t,u))}\right)^{1/2} du = 0.$$

Kwapień and Rosiński (2004) provided an upper bound for the uniform modulus of continuity in terms of "weakly majorizing measure".

2.3 Differentiability

(i). Mean-square differentiability: the mean square partial derivative of X at t is defined as

$$\frac{\partial X(t)}{\partial t_j} = \text{l.i.m}_{h \to 0} \frac{X(t + he_j) - X(t)}{h},$$

where e_j is the unit vector in the j-th direction.
For a Gaussian field, sufficient conditions can be given in terms of the differentiability of the covariance function (Adler, 1981).
(ii). Sample path differentiability: the sample function $t \mapsto X(t)$ is differentiable. This is much stronger and more useful than (i).

Sample path differentiability of $X(t)$ can be proved by using criteria for continuity.

Consider a centered Gaussian field with stationary increments whose spectral density function satisfies

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^{N} |\lambda_j|^{\beta_j}\right)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ \ |\lambda| \ge 1, \tag{3}$$

where $(\beta_1, \ldots, \beta_N) \in (0, \infty)^N$ and

$$\gamma > \sum_{j=1}^{N} \frac{1}{\beta_j}.$$

Differentiability

Theorem 2.5 (Xue and X. 2011)

Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a centered Gaussian field with stationary increments and spectral density which satisfies (3). (i) If

$$\beta_j\Big(\gamma - \sum_{i=1}^{N} \frac{1}{\beta_i}\Big) > 2, \tag{4}$$

then the partial derivative $\partial X(t)/\partial t_j$ is continuous almost surely. In particular, if (4) holds for all $1 \le j \le N$, then almost surely $X(t)$ is continuously differentiable.

(ii) If $\max\limits_{1\le j\le N} \beta_j\big(\gamma - \sum_{i=1}^{N} 1/\beta_i\big) \le 2$, then $X(t)$ is not differentiable in any direction.

Differentiability

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Proof of (i): Under (4), we know that the mean square partial derivative $X_j'(t)$ exists. In order to show that $X_j'(t)$ has a continuous version, by Kolmorogov's continuity theorem, it is enough to show that for any compact interval $I \subset \mathbb{R}^N$, there exist constants $c > 0$ and $\eta > 0$ such that

$$\mathbb{E}\big[X_j'(s) - X_j'(t)\big]^2 \le c\,|s-t|^\eta \quad \forall\, s, t \in I. \tag{5}$$

By the spectral representation of X, we have

$$\begin{aligned}\mathbb{E}\big(X_j'(s) - X_j'(t)\big)^2 &= \mathbb{E}\big[(X_j'(s))^2\big] + \mathbb{E}\big[(X_j'(t))^2\big] - 2\mathbb{E}\big[(X_j'(s)X_j'(t))\big] \\ &= 2\int_{\mathbb{R}^N} \lambda_j^2\big(1 - \cos\langle s-t, \lambda\rangle\big) f(\lambda)\,d\lambda.\end{aligned}$$

From this, we can verify that (5) holds under (4).

It follows from (5) that the Gaussian field $X_j' = \{X_j'(t), t \in \mathbb{R}^N\}$ has a continuous version [still denoted by X_j'].

We define a new Gaussian field $\widetilde{X} = \{\widetilde{X}(t), t \in \mathbb{R}^N\}$ by

$$\begin{aligned}\widetilde{X}(t) &= X(t_1, \cdots, t_{j-1}, 0, t_{j+1}, \cdots, t_N) \\ &\qquad + \int_0^{t_j} X_j'(t_1, \cdots, t_{j-1}, s_j, t_{j+1}, \cdots, t_N)\, ds_j.\end{aligned} \tag{6}$$

Then we can verify that $\widetilde{X}$ is a continuous version of X and, for every $t \in \mathbb{R}^N$, $\widetilde{X}_j'(t) = X_j'(t)$ almost surely. This amounts to verify that for every $t \in \mathbb{R}^N$,

$$\mathbb{E}\Big[\big(\widetilde{X}(t) - X(t)\big)^2\Big] = 0,$$

which can be proved by using (6) and the representations for $X(t)$ and $X_j'(t)$. We omit the details.

Thank you!