Gaussian Random Fields and SPDEs: An Introduction

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Outline

Introduction to (Gaussian) random fields

1. Stationary random fields
2. Random fields with stationary increments
3. More examples of non-stationary random fields
4. Multivariate Gaussian random fields

1. An introduction on random fields

A random field $X = \{X(t), t \in T\}$ is a family of random variables with values in state space S, where T is the parameter set.

We consider $T \subseteq \mathbb{R}^N$ and $S = \mathbb{R}^d$ $(d \geq 1)$. Then X is called an (N, d) random field.

Random fields arise naturally in

- turbulence (A. N. Kolmogorov, 1941)
- oceanography
- spatial statistics, spatio-temporal geostatistics
- image and signal processing
- ...

1.1 Stationary random fields and their spectral representations

A real-valued random field $\{X(t), t \in \mathbb{R}^N\}$ is called second-order stationary if $\mathbb{E}(X(t)) \equiv m$, where m is a constant, and the covariance function depends on $s - t$ only:

$$\mathbb{E}\big[(X(s) - m)(X(t) - m)\big] = C(s - t), \quad \forall s, t \in \mathbb{R}^N.$$

Bochner's Theorem (1932) says that a bounded and continuous function C is positive definite if and only if there is a finite Borel measure μ such that

$$C(t) = \int_{\mathbb{R}^N} e^{i\langle t, x\rangle} \, d\mu(x), \qquad \forall t \in \mathbb{R}^N.$$

If $X = \{X(t), t \in \mathbb{R}^N\}$ is a centered, stationary Gaussian random field with values in $\mathbb{R}$ whose covariance function is the Fourier transform of μ, then there is a complex-valued Gaussian random measure $\widetilde{W}$ on $\mathcal{B}(\mathbb{R}^N)$ such that $\mathbb{E}\big(\widetilde{W}(A)\big) = 0$,

$$\mathbb{E}(\widetilde{W}(A)\overline{\widetilde{W}(B)}) = \mu(A \cap B) \quad \text{and} \quad \widetilde{W}(-A) = \overline{\widetilde{W}(A)}$$

and X has the following Wiener integral representation:

$$X(t) = \int_{\mathbb{R}^N} e^{i\langle t, x\rangle}\, d\widetilde{W}(x).$$

The finite measure μ is called the spectral measure of X.

The Matérn class

An important class of isotropic stationary random fields are those with the Matérn covariance function

$$C(t) = \frac{1}{\Gamma(\nu)2^{\nu-1}}\Big(\sqrt{2\nu}\frac{|t|}{\rho}\Big)^{\nu} K_\nu\Big(\sqrt{2\nu}\frac{|t|}{\rho}\Big),$$

where Γ is the Gamma function, K_ν is the modified Bessel function of the second kind, and ρ and ν are non-negative parameters.

Since the covariance function $C(t)$ depends only on the Euclidean norm $|t|$, the corresponding Gaussian field X is called isotropic.

By the inverse Fourier transform, one can show that the spectral measure of X has the following density function:

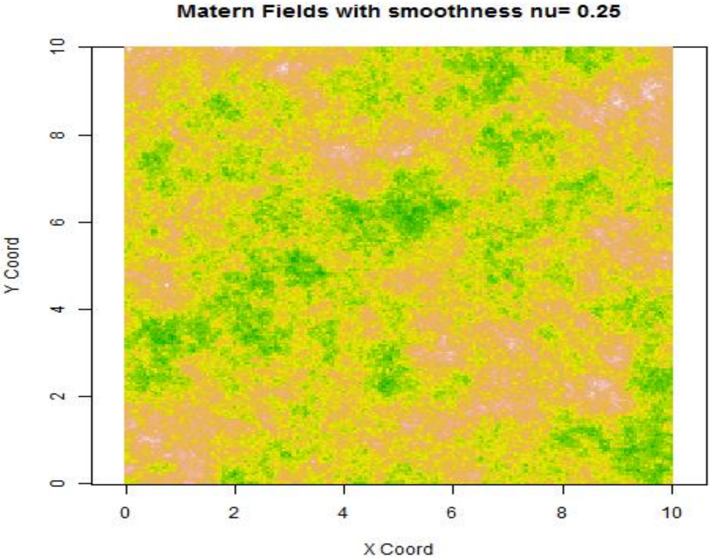
$$f(\lambda) = \frac{1}{(2\pi)^N} \frac{1}{(|\lambda|^2 + \frac{\rho^2}{2\nu})^{\nu + \frac{N}{2}}}, \quad \forall \lambda \in \mathbb{R}^N.$$

Whittle (1954) showed that the Gaussian random field X can be obtained as the solution to the following fractional SPDE

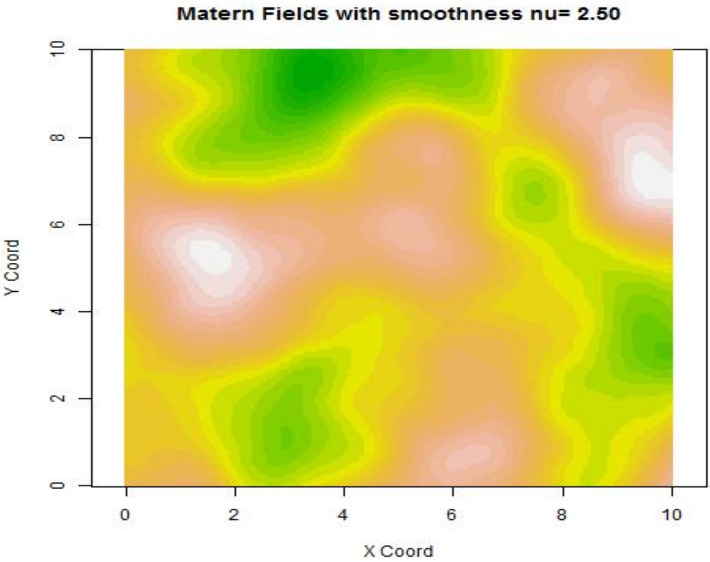
$$\Big(\Delta + \frac{\rho^2}{2\nu}\Big)^{\frac{\nu}{2} + \frac{N}{4}} X(t) = \dot{W}(t),$$

where $\Delta = \frac{\partial^2}{dt_1^2} + \cdots + \frac{\partial^2}{dt_N^2}$ is the N-dimensional Laplacian, and $\dot{W}(t)$ is the white noise.

A smooth Gaussian field: $N = 2, \nu = 0.25$



A smooth Gaussian field: $N = 2, \nu = 2.5$



1.2 Gaussian fields with stationary increments

Let $X = \{X(t),\ t \in \mathbb{R}^N\}$ be a centered Gaussian random field with stationary increments and $X(0) = 0$. Yaglom (1954) showed that, if $R(s,t) = \mathbb{E}\big[X(s)X(t)\big]$ is continuous, then $R(s,t)$ can be written as

$$R(s,t) = \langle s, At\rangle + \int_{\mathbb{R}^N} (e^{i\langle s,\lambda\rangle} - 1)(e^{-i\langle t,\lambda\rangle} - 1)\Delta(d\lambda),$$

where A is a nonnegative definite real $N \times N$ matrix and $\Delta(d\lambda)$ is a Borel measure which satisfies

$$\int_{\mathbb{R}^N} (1 \wedge |\lambda|^2)\, \Delta(d\lambda) < \infty. \tag{1}$$

The measure Δ is called the *spectral measure* of X.

We assume that $A = 0$. Then

$$\mathbb{E}\big[(X(s) - X(t))^2\big] = 2\int_{\mathbb{R}^N} \big(1 - \cos\langle s - t, \lambda\rangle\big)\Delta(d\lambda);$$

and X has the stochastic integral representation:

$$X(t) \stackrel{d}{=} \int_{\mathbb{R}^N} \big(e^{i\langle t,\lambda\rangle} - 1\big)\, \widetilde{W}(d\lambda),$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions, $\widetilde{W}(d\lambda)$ is a centered complex-valued Gaussian random measure with Δ as its control measure.

Gaussian fields with stationary increments can be constructed by choosing spectral measures Δ.

Two examples

Example 1 If Δ has a density function

$$f_H(\lambda) = c(H, N)|\lambda|^{-(2H+N)},$$

where $H \in (0,1)$ and $c(H,N) > 0$, then X is fractional Brownian motion with index H.
It can be verified that (for proper choice of $c(H,N)$),

$$\begin{aligned} \mathbb{E}\big[(X(s) - X(t))^2\big] &= 2c(H,N) \int_{\mathbb{R}^N} \frac{1 - \cos\langle s - t, \lambda\rangle}{|\lambda|^{2H+N}}\, d\lambda \\ &= |s - t|^{2H}. \end{aligned}$$

For the last identity, see, e.g., Schoenberg (1939).

- FBm X has stationary increments: for any $b \in \mathbb{R}^N$,

$$\Big\{X(t+b) - X(b),\ t \in \mathbb{R}^N\Big\} \stackrel{d}{=} \Big\{X(t),\ t \in \mathbb{R}^N\Big\},$$

where $\stackrel{d}{=}$ means equality in finite dimensional distributions.

- FBm X is H-self-similar: for every constant $c > 0$,

$$\Big\{X(ct),\ t \in \mathbb{R}^N\Big\} \stackrel{d}{=} \Big\{c^H X(t),\ t \in \mathbb{R}^N\Big\}.$$

Example 2 A large class of Gaussian fields can be obtained by letting spectral density functions satisfy (1) and

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{\beta_j}\right)^{\gamma}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ |\lambda| \geq 1, \tag{2}$$

where $(\beta_1, \ldots, \beta_N) \in (0, \infty)^N$ and $\gamma > \sum_{j=1}^N \frac{1}{\beta_j}$.

More conveniently, we re-write (2) as

$$f(\lambda) \asymp \frac{1}{\left(\sum_{j=1}^N |\lambda_j|^{H_j}\right)^{Q+2}}, \qquad \forall\, \lambda \in \mathbb{R}^N,\ |\lambda| \geq 1, \tag{3}$$

where $H_j = \frac{\beta_j}{2}\left(\gamma - \sum_{i=1}^N \frac{1}{\beta_i}\right)$ and $Q = \sum_{j=1}^N H_j^{-1}$.

1.3 More examples of non-stationary Gaussian random fields

1.3.1 The Brownian sheet and fractional Brownian sheets

The Brownian sheet $W = \{W(t),\, t \in \mathbb{R}^N_+\}$ is a centered (N, d)-Gaussian field whose covariance function is

$$\mathbb{E}\big[W_i(s)W_j(t)\big] = \delta_{ij} \prod_{k=1}^{N} s_k \wedge t_k.$$

- When $N = 1$, W is Brownian motion in $\mathbb{R}^d$.
- W is $N/2$-self-similar, but it does not have stationary increments.
- It gives rise to the Gaussian white noise $\dot{W}$, which can be used as a stochastic integrator.

Fractional Brownian sheet $W^{\vec{H}} = \left\{W^{\vec{H}}(t), t \in \mathbb{R}^N\right\}$ is a mean zero Gaussian field in $\mathbb{R}$ with covariance function

$$\mathbb{E}\left[W^{\vec{H}}(s)W^{\vec{H}}(t)\right] = \prod_{j=1}^{N} \frac{1}{2}\left(|s_j|^{2H_j} + |t_j|^{2H_j} - |s_j - t_j|^{2H_j}\right),$$

where $\vec{H} = (H_1, \ldots, H_N) \in (0, 1)^N$.
For all constants $c > 0$,

$$\left\{W^{\vec{H}}(c^E t),\ t \in \mathbb{R}^N\right\} \stackrel{d}{=} \left\{c\, W^{\vec{H}}(t),\ t \in \mathbb{R}^N\right\},$$

where $E = (a_{ij})$ is the $N \times N$ diagonal matrix with $a_{ii} = 1/(NH_i)$ for all $1 \le i \le N$ and $a_{ij} = 0$ if $i \neq j$. This is referred to as an"operator-scaling" property.

1.3.2 Linear stochastic heat equation

Consider the linear stochastic heat equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \frac{1}{2}\Delta u(t,x) + \sigma\,\dot{W}, \quad t \geq 0,\ x \in \mathbb{R}^k, \\ u(0,x) &\equiv 0, \end{aligned} \tag{4}$$

where Δ is the Laplacian operator in the spatial variables, σ is a constant or a deterministic function, and $\dot{W}$ is a Gaussian noise that is white in time and has a spatially homogeneous covariance [Dalang (1999)] given by the Riesz kernel with exponent $\beta \in (0, k \wedge 2)$, i.e.

$$\mathbb{E}(\dot{W}(t,x)\dot{W}(s,y)) = \delta(t-s)|x-y|^{-\beta}.$$

If $k = 1 = \beta$, then $\dot{W}$ is the space-time Gaussian white noise considered by Walsh (1986).

It follows from Walsh (1986) and Dalang (1999) that the mild solution of (4) is the mean zero Gaussian random field $u = \{u(t,x), t \geq 0, x \in \mathbb{R}\}$ defined by

$$u(t,x) = \int_0^t \int_{\mathbb{R}} \widetilde{G}_{t-r}(x-y)\, \sigma W(drdy), \quad t \geq 0, x \in \mathbb{R},$$

where $\widetilde{G}_t(x)$ is the Green kernel given by

$$\widetilde{G}_t(x) = (2\pi t)^{-1/2} \exp\Big(- \frac{|x|^2}{2t}\Big), \qquad \forall\, t > 0, x \in \mathbb{R}^k.$$

1.3.3 Linear stochastic wave equation

The linear stochastic wave equation

$$\begin{cases} \frac{\partial^2}{\partial t^2} u(t,x) = \Delta u(t,x) + \dot{W}(t,x), \quad t \geq 0, \ x \in \mathbb{R}^k, \\ u(0,x) = \frac{\partial}{\partial t} u(0,x) = 0, \end{cases} \tag{5}$$

where $\dot{W}$ is a Gaussian noise as in the previous example with exponent $\beta \in (0, k \wedge 2)$.

The existence of real-valued process solution to (5) was studied by Walsh (1986) for the space-time white noise and by Dalang (1999) in the more general setting.

We recall briefly some known results.

Let G be the fundamental solution of the wave equation. Then

$$G(t,x) = \frac{1}{2}\mathbf{1}_{\{|x|<t\}} \quad \text{if } \ k = 1;$$

$$G(t,x) = c_k \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(k-2)/2} (t^2 - |x|^2)_+^{-1/2}, \quad \text{if } k \geq 2 \text{ is even;}$$

$$G(t,x) = c_k \Big(\frac{1}{t}\frac{\partial}{\partial t}\Big)^{(k-3)/2} \frac{\sigma_t^k(dx)}{t}, \quad \text{if } k \geq 3 \text{ is odd,}$$

where σ_t^k is the uniform surface measure on the sphere $\{x \in \mathbb{R}^k : |x| = t\}$.
Note that for $k \geq 3$, G is not a function but a distribution.

For any dimension $k \geq 1$, the Fourier transform of G in variable x is given by

$$\mathscr{F}(G(t,\cdot))(\xi) = \frac{\sin(t|\xi|)}{|\xi|}, \quad t \geq 0,\ \xi \in \mathbb{R}^k. \tag{6}$$

Dalang (1999) proved that the real-valued process solution of equation (5) is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^k} G(t-s, x-y)\, W(ds\, dy), \tag{7}$$

The range of β has been chosen so that the stochastic integral exists.

The solution $u = \{u(t,x), t \geq 0, x \in \mathbb{R}^k\}$ is a centered Gaussian random field.

Recall from Theorem 2 of Dalang (1999) that

$$\mathbb{E}\Big[\Big(\int_0^t\int_{\mathbb{R}^k} H(s,y)W(ds\,dy)\Big)^2\Big] = c\int_0^t ds\int_{\mathbb{R}^k}\frac{d\xi}{|\xi|^{k-\beta}}|\mathscr{F}(H(s,\cdot))(\xi)|^2 \tag{8}$$

provided that $s\mapsto H(s,\cdot)$ is a deterministic function with values in the space of nonnegative distributions with rapid decrease and

$$\int_0^t ds\int_{\mathbb{R}^k}\frac{d\xi}{|\xi|^{k-\beta}}|\mathscr{F}(H(s,\cdot)(\xi)|^2 < \infty.$$

Eq. (8) is a basic tool for studying the Gaussian random field $u=\{u(t,x), t\ge 0, x\in\mathbb{R}^k\}$.

1.3.4 Non-linear stochastic heat & wave eqs

Many authors have studied the following nonlinear SPDE:

$$\begin{cases} \mathcal{L}u = b(u) + \sigma(u)\dot{W}, & t \geq 0,\ x \in \mathbb{R}^k, \\ u(0,x) = \dfrac{\partial}{\partial t}u(0,x) = 0, \end{cases} \tag{9}$$

where $\mathcal{L}$ is a partial differential operator, σ and b are non-random functions that satisfy some regularity conditions [e.g., σ and b are Lipschitz continuous.]

For example, $\mathcal{L}u = \frac{\partial u}{\partial t} - \frac{1}{2}\Delta u$ and $\mathcal{L}u = \frac{\partial^2 u}{\partial t^2} - \Delta u$ give the stochastic heat and wave equation, respectively.

The solutions, when they exist, are in general non-Gaussian random fields. We refer to Dalang (1999), Khoshnevisan (2014) for more information.

1.4 Multivariate Gaussian random fields

Consider a multivariate random field $\boldsymbol{X} = \{\boldsymbol{X}(t),\ t \in \mathbb{R}^N\}$ taking values in $\mathbb{R}^d$ defined by

$$\boldsymbol{X}(t) = (X_1(t), \cdots, X_d(t)), \qquad t \in \mathbb{R}^N. \tag{10}$$

Their key features are:

- the components $X_1, \ldots, X_d$ are dependent.
- $X_1, \ldots, X_d$ may have different smoothness properties.

For any $i, j = 1, \ldots, d$, define

$$C_{ij}(s,t) := \mathbb{E}[X_i(s)X_j(t)]. \tag{11}$$

They are called the cross-covariance functions of $\boldsymbol{X}$.

(i) The multivariate Matérn random fields

Gneiting, Kleiber and Schlather (2010) introduced a class of multivariate stationary Matérn models $\{\boldsymbol{X}(t),\ t \in \mathbb{R}^N\}$ in (10) with marginal and cross-covariance functions of the form

$$C_{ij}(s,t) = M(s-t|\nu_{ij}, a_{ij}),$$

where

$$M(h|\nu, a) := \frac{2^{1-\nu}}{\Gamma(\nu)}(a|h|)^{\nu} K_{\nu}(a|h|).$$

and provided conditions for such matrix-valued functions to form legitimate cross-covariance functions.
See also Apanansovich, Genton and Sun (2012), Kleiber and Nychka (2013).

The bivariate Matérn fields

Let $\boldsymbol{X}(t) = (X_1(t), X_2(t))'$ be an $\mathbb{R}^2$-valued Gaussian field whose covariance matrix is determined by

$$C(h) = \begin{pmatrix} c_{11}(h) & c_{12}(h) \\ c_{21}(h) & c_{22}(h) \end{pmatrix}, \tag{12}$$

where $c_{ij}(h) := \mathbb{E}[X_i(s+h)X_j(s)]$ are specified by

$$\begin{aligned} c_{11}(h) &= \sigma_1^2 M(h|\nu_1, a_1), \\ c_{22}(h) &= \sigma_2^2 M(h|\nu_2, a_2), \\ c_{12}(h) &= c_{21}(h) = \rho\sigma_1\sigma_2 M(h|\nu_{12}, a_{12}) \end{aligned} \tag{13}$$

with $a_1, a_2, a_{12}, \sigma_1, \sigma_2 > 0$ and $\rho \in (-1, 1)$.

Gneiting, et al. (2010) gave NSC for (12) to be valid. In particular, if $\rho \neq 0$, one must have

$$\frac{\nu_1 + \nu_2}{2} \le \nu_{12}.$$

The parameters ν_1 and ν_2 control the smoothness of the sample function $t \mapsto \boldsymbol{X}(t)$.

For example, if $\nu_1 > 1$, then a.s. the sample function $t \mapsto X_1(t)$ is continuously differentiable. This can be proved using the spectral density.

Zhou and X. (2017, 2018) studied extreme values and estimation problems for a class of bivariate random fields that includes the bivariate Matérn fields.

(ii). Multivariate random fields with stationary increments

An $\mathbb{R}^d$-valued Gaussian random field $\boldsymbol{X} = \{\boldsymbol{X}(t), t \in \mathbb{R}^N\}$ is said to have stationary increments if $\forall\, t_0 \in \mathbb{R}^N$,

$$\{\boldsymbol{X}(t + t_0) - \boldsymbol{X}(t_0), t \in \mathbb{R}^N\} \overset{d}{=} \{\boldsymbol{X}(t) - \boldsymbol{X}(0), t \in \mathbb{R}^N\}.$$

A general framework for multivariate random fields with stationary increments was provided by Yaglom (1957).

As an example, we consider a spacial class of *operator fractional Brownian motions*.

Operator fractional Brownian motions

Let D be a linear operator on $\mathbb{R}^d$ (or a $d \times d$ real matrix). The operator norm of D is defined by

$$\|D\| = \max_{|x|=1} |Dx|.$$

Denote the eigenvalues of D by

$$\lambda_k = \alpha_k + i\beta_k, \quad (k = 1, \ldots, d).$$

We assume that

$$0 < \alpha_1 \le \alpha_2 \le \cdots \le \alpha_d < 1. \tag{14}$$

For any $c > 0$, we define the linear operator c^D by

$$c^D = \sum_{k=0}^{\infty} \frac{(\ln c)^k}{k!} D^k.$$

(a). Moving average representation

One can define ofBm $\boldsymbol{X} = \{\boldsymbol{X}(t), t \in \mathbb{R}\}$ in $\mathbb{R}^d$ by using the stochastic integration method:

$$\boldsymbol{X}(t) = \int_{\mathbb{R}} \left[(t-r)_+^{D-\frac{1}{2}I} - (-r)_+^{D-\frac{1}{2}I}\right] W(dr), \tag{15}$$

where W is d-dimensional Brownian motion, is an operator fractional Brownian motion with exponent D.

It has the following properties:

- stationary increments.
- (operator self-similarity) For every constant $c > 0$,

$$\{\boldsymbol{X}(c\,t), t \in \mathbb{R}\} \overset{d}{=} \{c^D \boldsymbol{X}(t), t \in \mathbb{R}\}.$$

(b). Harmonizable representation

The Gaussian random field $\boldsymbol{Y} = \{\boldsymbol{Y}(t), t \in \mathbb{R}^N\}$ in $\mathbb{R}^d$ defined by

$$\boldsymbol{Y}(t) = \int_{\mathbb{R}^N} \frac{e^{i\langle t,r\rangle} - 1}{|r|^{D+\frac{N}{2}I}} \widetilde{\mathcal{W}}(dr), \tag{16}$$

where $\widetilde{\mathcal{W}}$ is a complex-valued Gaussian random measure on $\mathbb{R}^d$ with Lebesgue control measure and i.i.d. components, is also an operator fractional Brownian motion with exponent D.

In order to verify that the stochastic integrals in (15) and (16) are well defined, it is sufficient to verify respectively that

$$\int_{\mathbb{R}} \left\| (t-r)_+^{D-\frac{1}{2}I} - (-r)_+^{D-\frac{1}{2}I} \right\|^2 dr < \infty,$$

and

$$\int_{\mathbb{R}^N} (1-\cos\langle t, r\rangle) \left\| |r|^{-D-\frac{N}{2}I} \right\|^2 dr < \infty.$$

This is where condition (14) is needed.

(iii). Operator-scaling and operator-self-similar random fields

Li and X. (2011) constructed a large class of more general, namely, operator-scaling and operator-self-similar random fields with stationary increments.

Several authors have studied properties of these random fields. See, for example,

- Ercan Sönmez (2017, 2018, 2020).
- Kremer and Scheffler (2019) for further development and recent results.
- Shen, Stilian, and Hsing (2020).

(iv). Systems of stochastic partial differential equations

There has been a lot of recent research on this topic, which we do not discuss here.

in the subsequent sections, we will consider the systems of stochastic heat and wave equations.

(v). Matrix-valued Gaussian random fields

Let $\xi = \{\xi(t) : t \in \mathbb{R}^N_+\}$ be a centered Gaussian random field and let $\{\xi_{i,j} : i,j \in \mathbb{N}\}$ be a family of independent copies of ξ.

Consider the symmetric $d \times d$ matrix-valued process $\boldsymbol{X} = \{X_{i,j}(t); t \in \mathbb{R}^N_+, 1 \le i,j \le d\}$ defined by

$$X_{i,j}(t) = \begin{cases} \xi_{i,j}(t), & i < j; \\ \sqrt{2}\xi_{i,i}(t), & i = j; \\ \xi_{j,i}(t), & i > j. \end{cases} \tag{17}$$

One may study statistical and sample path properties of the eigenvalues of $\boldsymbol{X}$.

Thank you!