Some Local Properties of A Nonlinear Stochastic Heat Equation

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Outline

1. Parabolic SPDEs: Background
2. Approximation Theorems
3. Local Properties of the Solution

10.1 A nonlinear stochastic heat equation

Consider a parabolic SPDE of the following form:

$$\frac{\partial}{\partial t} u_t(x) = \tfrac{1}{2}\Delta_{\alpha/2} u_t(x) + \sigma(u_t(x))\, \dot{W}_t(x), \qquad (1)$$

subject to $u_0(x) := U_0$ for all $x \in \mathbb{R}$, for some non-negative constant U_0.

In the above $\Delta_{\alpha/2} = -(-\Delta)^{\alpha/2}$ denotes the fractional Laplacian of index $\alpha/2$, $\sigma : \mathbb{R} \to \mathbb{R}$ is non random and Lipschitz continuous, and $\dot{W}$ denotes space-time white noise.

We assume $1 < \alpha \leq 2$. According to Dalang (1999), this is a sufficient and necessary condition for (1) to have a mild solution that is a random field.

Let $p_t(x)$ denote the fundamental solution to the fractional heat operator $(\partial/\partial t) - \frac{1}{2}\Delta_{\alpha/2}$. Then

$$\widehat{p}_t(\xi) = \exp\left(-t|\xi|^\alpha/2\right) \qquad (t \geq 0, \xi \in \mathbb{R}).$$

The Plancherel theorem implies that: For all $t > 0$,

$$\|p_t\|^2_{L^2(\mathbb{R})} = \frac{1}{2\pi}\|\widehat{p}_t\|^2_{L^2(\mathbb{R})} = \frac{\Gamma(1/\alpha)}{\alpha\pi t^{1/\alpha}}.$$

Moreover,

$$p_t(0) = \sup_{x\in\mathbb{R}} p_t(x) = \frac{2^{1/\alpha}\Gamma(1/\alpha)}{\alpha\pi t^{1/\alpha}} \qquad (t > 0).$$

The linear case

When $\sigma \equiv 1$ and $U_0 = 0$, the mild solution to (1) is given by

$$v_t(x) = \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\, W(\mathrm{d}s\, \mathrm{d}y). \tag{2}$$

Then $\{v_t(x)\}$ is a continuous, centered Gaussian random field, and many of its properties can be established (e.g., when $\alpha = 2$, Swanson (2007),Tudor and X. (2007), Lai and Nualart (2009), etc).

We will relate the local properties of $\{v_t(x)\}$ to those of the solution to (1).

General case: moment estimates

In general (1) is interpreted as

$$u_t(x) = U_0 + \int_{(0,t)\times\mathbb{R}} p_{t-s}(y-x)\sigma(u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y). \tag{3}$$

We will make use of the following two results:

Lemma 10.1 [Dalang, 1999; Foondun and Khoshnevisan, 2009]

For all $k \in [2\,,\infty)$ there exists a constant $A_{k,T}$ such that:

$$\mathbb{E}\left(|u_t(x)|^k\right) \le A_{k,T}; \qquad \text{and}$$

$$\mathbb{E}\left(|u_t(x) - u_{t'}(x')|^k\right) \le A_{k,T}\left(|x-x'|^{(\alpha-1)k/2} + |t-t'|^{(\alpha-1)k/(2\alpha)}\right)$$

uniformly for all $t, t' \in [0\,,T]$ and $x, x' \in \mathbb{R}$.

When $t > 0$ is fixed, Foondun, Khoshnevisan and Mahboubi (2015) have studied some properties of the function $x \mapsto u_t(x)$ by relating it to a fractional Brownian motion through the Gaussian process $\{v_t(x), x \in \mathbb{R}\}$.

Khoshnevisan, Swanson, X. and Zhang have considered some properties of the function $t \mapsto u_t(x)$, when $x \in \mathbb{R}$ is fixed.
By combining the results of the two papers, one can derive some local properties of the sample function $(t, x) \mapsto u_t(x)$.

In the following, we will focus on the behavior of $t \mapsto u_t(x)$, when $x \in \mathbb{R}$ is fixed, and present some results in Khoshnevisan, Swanson, X. and Zhang.

The BDG Inequality

For every $t \geq 0$, let $\mathscr{F}_t^0$ denote the σ-algebra generated by $\int_{(0,t)\times\mathbb{R}} \varphi_s(y)\, W(\mathrm{d}s\,\mathrm{d}y)$ as φ ranges over all elements of $L^2(\mathbb{R}_+ \times \mathbb{R})$. We complete every such σ-algebra, and make the filtration $\{\mathscr{F}_t\}_{t\geq 0}$ right continuous.

The following BDG Inequality is Proposition 4.4 in Khoshnevisan (2014).

Lemma 10.2

If $h \in L^2([0,t] \times \mathbb{R})$ for all $t > 0$ and $\Phi \in \mathcal{L}^{\beta,2}$ for some $\beta > 0$. Then, for every real number $k \in [2\,,\infty)$, we have

$$\left\| \int_{(0,t)\times\mathbb{R}} h_s(y)\Phi_s(y)\, W(\mathrm{d}s\,\mathrm{d}y) \right\|_k^2 \leq 4k \int_0^t \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\, [h_s(y)]^2 \|\Phi_s(y)\|_k^2.$$

10. 2 Approximation Theorems

Notation $\mathcal{D}_\varepsilon$: For any $\varepsilon > 0$ and random field $\{X_t(x)\}_{t\geq 0, x\in\mathbb{R}}$, denote

$$(\mathcal{D}_\varepsilon X)_t(x) := X_{t+\varepsilon}(x) - X_t(x) \qquad (t \geq 0,\ x \in \mathbb{R}).$$

Theorem 10.1 [KSXZ]

For every $k \in [2\,, \infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0\,, 1)$, $x \in \mathbb{R}$, and $t \in [0\,, T]$,

$$\mathbb{E}\left(\left|(\mathcal{D}_\varepsilon u)_t(x) - \sigma(u_t(x))\,(\mathcal{D}_\varepsilon v)_t(x)\right|^k\right) \leq A_{k,T}\, \varepsilon^{\mathcal{G}_\alpha k},$$

where

$$\mathcal{G}_\alpha := \frac{2(\alpha-1)}{3\alpha-1}.$$

Outlines of proof

For $x \in \mathbb{R}$ fixed, we write the increment of $t \mapsto u_t(x)$ as

$$u_{t+\varepsilon}(x) - u_t(x) := \mathscr{J}_1 + \mathscr{J}_2, \tag{4}$$

where

$$\begin{aligned} \mathscr{J}_1 &:= \int_{(0,t)\times\mathbb{R}} \left[p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)\right] \sigma(u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y); \\ \mathscr{J}_2 &:= \int_{(t,t+\varepsilon)\times\mathbb{R}} p_{t+\varepsilon-s}(y-x)\sigma(u_s(y))\, W(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \tag{5}$$

Step 1: Estimation of $\mathscr{J}_2$

Define

$$\widetilde{\mathscr{J}_2} := \sigma(u_t(x)) \int_{(t,t+\varepsilon)\times\mathbb{R}} p_{t+\varepsilon-s}(y-x)\, W(\mathrm{d}s\,\mathrm{d}y).$$

Lemma 10.3

For every $k \in [2\,,\infty)$ there exists a finite constant $A_{k,T}$ such that for all $\varepsilon \in (0\,,1)$,

$$\sup_{x\in\mathbb{R}} \sup_{t\in[0,T]} \mathbb{E}\left(\left|\mathscr{J}_2 - \widetilde{\mathscr{J}_2}\right|^k\right) \le A_{k,T}\,\varepsilon^{(\alpha-1)k/\alpha}. \tag{6}$$

To prove (6), we first consider

$$\mathscr{J}_2 - \mathscr{J}_2' = \int_{(t,t+\varepsilon)\times\mathbb{R}} p_{t+\varepsilon-s}(y-x)[\sigma(u_s(y)) - \sigma(u_s(x))]\, W(\mathrm{d}s\,\mathrm{d}y).$$

By the BDG inequality and Lemma 10.1, we have

$$\begin{aligned}
&\|\mathscr{J}_2 - \mathscr{J}_2'\|_{L^k(\Omega)}^2 \\
&\leq 4k \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_{t+\varepsilon-s}(y-x)]^2\, \|\sigma(u_s(y)) - \sigma(u_s(x))\|_k^2 \\
&\leq A \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_{t+\varepsilon-s}(y-x)]^2\, \|u_s(y) - u_s(x)\|_k^2 \\
&\leq A \int_0^{\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y\ [p_s(y)]^2 \left(|y|^{\alpha-1} \wedge 1\right) \\
&\leq A\varepsilon^{2(\alpha-1)/\alpha}.
\end{aligned}$$

Next we consider $\mathscr{J}_2' - \widetilde{\mathscr{J}_2}$. The same argument gives:

$$\begin{aligned}
&\left\| \mathscr{J}_2' - \widetilde{\mathscr{J}_2} \right\|_{L^k(\Omega)}^2 \\
&\le A \int_t^{t+\varepsilon} \mathrm{d}s \int_{-\infty}^{\infty} \mathrm{d}y \ \left[p_{t+\varepsilon-s}(y)\right]^2 \left\| u_s(x) - u_t(x) \right\|_{L^k(\Omega)}^2 \\
&\le A \int_t^{t+\varepsilon} \left\| p_{t+\varepsilon-s} \right\|_{L^2(\mathbb{R})}^2 |s-t|^{(\alpha-1)/\alpha} \ \mathrm{d}s \\
&\le A \varepsilon^{2(\alpha-1)/\alpha}.
\end{aligned}$$

Step 2: Estimation of $\mathscr{J}_1$

Let $a = 2\alpha/(3\alpha - 1) \in (0\,,1)$, and write

$$\mathscr{J}_1 = \mathscr{J}_{1,a} + \mathscr{J}'_{1,a},$$

where

$$\mathscr{J}_{1,a} := \int_{(0,t-\varepsilon^a)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\,\sigma(u_s(y))\,W(\mathrm{d}s\,\mathrm{d}y),$$

$$\mathscr{J}'_{1,a} := \int_{(t-\varepsilon^a,t)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\,\sigma(u_s(y))\,W(\mathrm{d}s\,\mathrm{d}y).$$

By applying the BDG inequality and Lemma 10.1, one can verify that

$$\sup_{x\in\mathbb{R}}\sup_{t\in[0,T]}\mathbb{E}\Big(\,|\mathscr{J}_{1,a}|^k\Big)\le A\,\varepsilon^{\left(\mathcal{G}_\alpha+\frac{1}{3\alpha-1}\right)k},$$

which is a lot smaller than $A\,\varepsilon^{\mathcal{G}_\alpha k}$.

To estimate $\mathscr{J}'_{1,a}$, we use the same strategy and introduce

$$\mathscr{J}''_{1,a}:=\int_{(t-\varepsilon^a,t)\times\mathbb{R}}[p_{t+\varepsilon-s}(y-x)-p_{t-s}(y-x)]\,\sigma(u_{t-\varepsilon^a}(y))\,W(\mathrm{d}s\,\mathrm{d}y),$$

$$\widetilde{\mathscr{J}}_{1,a}:=\sigma(u_{t-\varepsilon^a}(x))\int_{(t-\varepsilon^a,t)\times\mathbb{R}}[p_{t+\varepsilon-s}(y-x)-p_{t-s}(y-x)]\,W(\mathrm{d}s\,\mathrm{d}y).$$

Note that the moments of $u_{t-\varepsilon^a}(x)-u_t(x)$ is negligible compared with the main term.

By the BDG inequality and Lemma 10.1, we can prove

Lemma 10.4

For every $T > 0$ and $k \in [2, \infty)$ there exists a finite constant $A_{k,T}$ such that uniformly for all $\varepsilon \in (0, 1)$, $x \in \mathbb{R}$, and $t \in [0, T]$,

$$\mathbb{E}\left(\left|\mathscr{J}_1 - \sigma(u_t(x)) \int_{(0,t)\times\mathbb{R}} [p_{t+\varepsilon-s}(y-x) - p_{t-s}(y-x)]\, W(\mathrm{d}s\,\mathrm{d}y)\right|^k\right) \leq A_{k,T}\, \varepsilon^{\mathcal{G}_\alpha k}.$$

Theorem 10.1 follows from Lemmas 10.3 and 10.4.

From Theorem 10.1 and an interpolation argument, we can derive the following result, which is useful for deriving local properties of $t \mapsto u_t(x)$ from those of the Gaussian process $t \mapsto v_t(x)$.

Theorem 10.2 [KSXZ]

For all $T > 0$, $M > 0$, and $q \in (0\,, \mathcal{G}_\alpha)$,

$$\mathbb{E}\left(\sup_{t\in[0,T]} \sup_{\varepsilon\in[0,\eta]} \sup_{x\in[-M,M]} \left|(\mathcal{D}_\varepsilon u)_t(x) - \sigma(u_t(x))(\mathcal{D}_\varepsilon v)_t(x)\right|^k \right) = o\left(\eta^{kq}\right),$$

as $\eta \to 0^+$.

By a Borel-Cantelli argument, we obtain the following a.s. uniform approximation bound:

Corollary 10.3

For all $T > 0$, $M > 0$, and $q \in (0\,, \mathcal{G}_\alpha)$,

$$\sup_{t\in[0,T]}\sup_{\varepsilon\in[0,\eta]}\sup_{x\in[-M,M]}\left|(\mathcal{D}_\varepsilon u)_t(x) - \sigma(u_t(x))(\mathcal{D}_\varepsilon v)_t(x)\right| = o\left(\eta^q\right),$$

as $\eta \to 0^+$, almost surely.

Notice that $\mathcal{G}_\alpha > (\alpha - 1)(2\alpha)$, we derive from Corollary 10.3 and local properties of the Gaussian process $t \mapsto v_t(x)$ the following results.

10.3 Local Properties of the Solution

1. Law of the iterated logarithm

Theorem 10.4 [KSXZ]

Let $x \in \mathbb{R}$ be fixed. The following hold almost surely:

1. If $t > 0$, then

$$\limsup_{\varepsilon \to 0^+} \frac{u_{t+\varepsilon}(x) - u_t(x)}{\varepsilon^{(\alpha-1)/(2\alpha)}\sqrt{2\log|\log \varepsilon|}} = \sigma(u_t(x))\sqrt{\frac{2^{1/\alpha}\Gamma(1/\alpha)}{(\alpha-1)\pi}}.$$

2. If $t = 0$, then

$$\limsup_{\varepsilon \to 0^+} \frac{u_\varepsilon(x) - U_0}{\varepsilon^{(\alpha-1)/(2\alpha)}\sqrt{2\log|\log \varepsilon|}} = \sigma(U_0)\sqrt{\frac{\Gamma(1/\alpha)}{(\alpha-1)\pi}}.$$

2. Weighted variation

For any $t > 0$ fixed and some integer $n > 1$, consider a partition $\{t_{j:n}\}_{j=0}^{k_n}$ of $[0\,,t]$ by letting

$$t_{j:n} := jt\varepsilon_n \quad (0 \le j < k_n := \lfloor \varepsilon_n^{-1} \rfloor),\ t_{k_n:n} := t,$$

with "mesh size" ε_n.
For a fixed $x \in \mathbb{R}$, we consider the following function

$$V_t^{(n,\varphi)}(x) := \sum_{j=0}^{k_n-1} \varphi\left(u_{t_{j:n}}(x)\right) \cdot \left|u_{t_{j+1:n}}(x) - u_{t_{j:n}}(x)\right|^{2\alpha/(\alpha-1)}.$$

Here, $\varphi : \mathbb{R} \to \mathbb{R}$ is a non random and Lipschitz continuous function. When $\varphi \equiv 1$, $V_t^{(n,\varphi)}(x)$ is the "β-variation" of the function $s \mapsto u_s(x)$, in $[0\,,t]$, where $\beta := 2\alpha/(\alpha-1)$.

Theorem 10.5 [KSXZ]

Choose and fix $x \in \mathbb{R}$, $t > 0$, and a non random and Lipschitz continuous function $\varphi : \mathbb{R} \to \mathbb{R}$. Then,

$$\lim_{n\to\infty} V_t^{(n,\varphi)}(x) = \mathfrak{V}(\alpha) \int_0^t \varphi(u_s(x))\,|\sigma(u_s(x))|^{2\alpha/(\alpha-1)}\,\mathrm{d}s$$

in $L^2(\Omega)$ as $n \to \infty$, where $\mathfrak{V}(\alpha)$ is an explicit constant depending on α.

Moreover, if

$$\sum_{n=1}^{\infty} \varepsilon_n^{(\alpha-1)/(3\alpha-1)} < \infty,$$

then the preceding can be strengthened to almost-sure convergence.

Thank You!