Stochastic heat equation with general nonlinear spatial rough Gaussian noise

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Outline of the talk

1. Problem
2. Difficulty
3. Background
4. Additive noise
5. General case
6. Some key estimates

1. Problem

$$\frac{\partial u(t,x)}{\partial t} = \Delta u(t,x) + \sigma(u(t,x))\dot{W}, \quad t > 0, x \in \mathbb{R}.$$

- $\Delta = \frac{\partial^2}{\partial x^2}$ is the Laplacian and $\sigma : \mathbb{R} \to \mathbb{R}$ is a nice function (Lipschitz).
- initial condition $u(0,x) = u_0(x)$ is continuous and bounded.
- $\dot{W} = \frac{\partial^2 W}{\partial t \partial x}$ is centered Gaussian field with covariance

$$\mathbb{E}(\dot{W}(s,x)\dot{W}(t,y)) = \delta(s-t)\,|x-y|^{2H-2}.$$

  Here $1/4 < H < 1/2$
- The product $\sigma(u)\dot{W}$ is taken in Skorohod sense.

Stochastic integral

For a function $\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$, the Marchaud fractional derivative D_-^β is defined as:

$$\begin{aligned} D_-^\beta \phi(t,x) &= \lim_{\varepsilon \downarrow 0} D_{-,\varepsilon}^\beta \phi(t,x) \\ &= \lim_{\varepsilon \downarrow 0} \frac{\beta}{\Gamma(1-\beta)} \int_\varepsilon^\infty \frac{\phi(t,x) - \phi(t,x+y)}{y^{1+\beta}} dy. \end{aligned}$$

The Riemann-Liouville fractional integral is defined by

$$I_-^\beta \phi(t,x) = \frac{1}{\Gamma(\beta)} \int_x^\infty \phi(t,y)(y-x)^{\beta-1} dy.$$

Set

$$\mathbb{H} = \{\phi : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R} \mid \exists \psi \in L^2(\mathbb{R}_+ \times \mathbb{R}) \; s.t. \; \phi(t,x) = I_-^{\frac{1}{2}-H} \psi(t,x)\}.$$

Proposition

$\mathbb{H}$ is a Hilbert space equipped with the scalar product

$$\begin{aligned}
\langle \phi, \psi \rangle_{\mathbb{H}} &= c_{1,H} \int_{\mathbb{R}_+ \times \mathbb{R}} \mathcal{F}\phi(s,\xi) \overline{\mathcal{F}\psi(s,\xi)} |\xi|^{1-2H} d\xi ds \\
&= c_{2,H} \int_{\mathbb{R}_+ \times \mathbb{R}} D_-^{\frac{1}{2}-H} \phi(t,x) D_-^{\frac{1}{2}-H} \psi(t,x) dx dt \\
&= c_{3,\beta}^2 \int_{\mathbb{R}^2} [\phi(x+y) - \phi(x)][\psi(x+y) - \psi(x)] |y|^{2H-2} dx dy \,,
\end{aligned}$$

where

$$
\begin{aligned}
c_{1,H} &= \frac{1}{2\pi}\Gamma(2H+1)\sin(\pi H)\,; \\
c_{2,H} &= \left[\Gamma\left(H+\frac{1}{2}\right)\right]^2 \left(\int_0^\infty \left[(1+t)^{H-\frac{1}{2}} - t^{H-\frac{1}{2}}\right]^2 dt + \frac{1}{2H}\right)^{-1}\,; \\
c_{3,\beta}^2 &= (\frac{1}{2}-\beta)\beta c_{2,\frac{1}{2}-\beta}^{-1}\,.
\end{aligned}
$$

The space $D(\mathbb{R}_+ \times \mathbb{R})$ is dense in $\mathbb{H}$.

Definition

An elementary process g is a process of the following form

$$g(t,x) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \mathbf{1}_{(a_i,b_i]}(t) \mathbf{1}_{(h_j,l_j]}(x),$$

where n and m are finite positive integers, $-\infty < a_1 < b_1 < \cdots < a_n < b_n < \infty$, $h_j < l_j$ and $X_{i,j}$ are $\mathcal{F}_{a_i}$-measurable random variables for $i = 1, \ldots, n$. The stochastic integral of such an elementary process with respect to W is defined as

$$\int_{\mathbb{R}_+} \int_{\mathbb{R}} g(t,x) W(dx,dt) = \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} W(\mathbf{1}_{(a_i,b_i]} \otimes \mathbf{1}_{(h_j,l_j]})$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} X_{i,j} \big[W(b_i,l_j) - W(a_i,l_j) - W(b_i,h_j) + W(a_i,h_j) \big].$$

Definition

Let Λ_H be the space of predictable processes g defined on $\mathbb{R}_+ \times \mathbb{R}$ such that almost surely $g \in \mathbb{H}$ and $\mathbb{E}[\|g\|_{\mathbb{H}}^2] < \infty$. Then, the space of elementary processes defined as above is dense in Λ_H.

For $g \in \Lambda_H$, the stochastic integral $\int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x)W(dx,dt)$ is defined as the $L^2(\Omega)$-limit of stochastic integrals of the elementary processes approximating $g(t,x)$ in Λ_H, and we have the following isometry equality

$$\mathbb{E}\left(\left[\int_{\mathbb{R}_+ \times \mathbb{R}} g(t,x)W(dx,dt)\right]^2\right) = \mathbb{E}\left(\|g\|_{\mathbb{H}}^2\right) + c_{3,H}^2 \int_0^\infty \int_{\mathbb{R}^2} \mathbb{E}|g(t,x+y) - g(t,x)|^2 |y|^{2H-2} dxdydt\,.$$

Definition (Strong solution)

$u(t,x)$ is a *strong (mild random field) solution* if for all $t \in [0,T]$ and $x \in \mathbb{R}$ the process $\{G_{t-s}(x-y)\sigma(u(s,y))\mathbf{1}_{[0,t]}(s)\}$ is integrable with respect to W, where $G_t(x) := \frac{1}{\sqrt{4\pi t}} \exp\left[-\frac{x^2}{4t}\right]$ is heat kernel, and

$$u(t,x) = G_t * u_0(x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(dy,ds)$$

almost surely, where

$$G_t * u_0(x) = \int_{\mathbb{R}^d} G_t(x-y)u_0(y)dy\,.$$

Definition (Weak solution)

We say the spde has a *weak solution* if there exists a probability space with a filtration $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$, a Gaussian noise $\widetilde{W}$ identical to W in law, and an adapted stochastic process $\{u(t,x)\,, t \geq 0\,, x \in \mathbb{R}\}$ on this probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ such that $u(t,x)$ is a strong (mild) solution with respect to $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{P}}, \widetilde{\mathcal{F}}_t)$ and $\widetilde{W}$.

Want to study the existence and uniqueness of the solution (strong or weak)

2. Difficulty

Denote $\xi_t(x) = G_t * u_0(x)$.

Naive application of Picard iteration ($v = u^{n+1}$ and $u = u^n$):

$$v(t,x) = \xi_t(x) + \int_0^t \int_{\mathbb{R}^d} G_{t-s}(x-y)\sigma(s,y,u(s,y))W(dy,ds)$$

Then following isometry equality

$$\begin{aligned}
\mathbb{E}\left(v^2(t,x)\right) &= \xi_t^2(x) \\
&\quad + c_{3,H}^2 \int_0^t \int_{\mathbb{R}^2} \mathbb{E}|G_{t-s}(x-y-z)\sigma(s,y+z,u(s,y+z)) \\
&\quad - G_{t-s}(x-y)\sigma(s,y,u(s,y))|^2 |z|^{2H-2} dydzds \\
&\leq \cdots + \\
& c_{3,H}^2 \int_0^t \int_{\mathbb{R}^2} \mathbb{E} G_{t-s}^2(x-y)|u(s,y+z)-u(s,y)|^2 |z|^{2H-2} dydzds
\end{aligned}$$

One difficulty is that we cannot no longer bound $|\sigma(x_1) - \sigma(x_2) - \sigma(y_1) + \sigma(y_2)|$ by a multiple of $|x_1 - x_2 - y_1 + y_2|$ (which is possible only in the affine case).

3. Background

$\sigma(u) = au + b$: $H > 1/4$.

Balan, R.; Jolis, M. and Quer-Sardanyons, L.

SPDEs with affine multiplicative fractional noise in space with index $\frac{1}{4} < H < \frac{1}{2}$.

Electronic Journal of Probability 20 (2015).

General $\sigma(u)$ but with $\sigma(0) = 0$.

Hu, Yaozhong; Huang, Jingyu; Le, Khoa; Nualart, David; Tindel, Samy

Stochastic heat equation with rough dependence in space.

Ann. Probab. 45 (2017), 4561-4616.

Introduce a norm $\|\cdot\|_{\mathcal{Z}_T^p}$ for a random field $v(t,x)$ as follows:

$$\|v\|_{\mathcal{Z}_T^p} := \sup_{t\in[0,T]} \|v(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,p} v(t),$$

where $p \geq 2$, $\frac{1}{4} < H < \frac{1}{2}$,

$$\|v(t,\cdot)\|_{L^p(\Omega\times\mathbb{R})} = \left[\int_{\mathbb{R}} \mathbb{E}\left[|v(t,x)|^p\right] dx\right]^{\frac{1}{p}},$$

and

$$\mathcal{N}^*_{\frac{1}{2}-H,p} v(t) = \left[\int_{\mathbb{R}} \|v(t,\cdot) - v(t,\cdot+h)\|^2_{L^p(\Omega\times\mathbb{R})} |h|^{2H-2} dh\right]^{\frac{1}{2}}.$$

When $\sigma(0) = 0$ we seek the solution in the space $\mathcal{Z}_T^p$

Theorem (Hu, Huang, Le, Nualart, Tindel, 2017)

When $\sigma(0) = 0$ and some nice conditions, the solution exists uniquely in $\mathcal{Z}_T^p$.

However, when $\sigma(0) \neq 0$, we cannot show the solution is in $\mathcal{Z}_p$. Even when $\sigma(u) = 1$ and $u_0 = 0$ (additive noise) we cannot show that the solution is in $\mathcal{Z}_p$.

We introduce the weighted $\mathcal{Z}_T^p$ space. This weighted space is bigger than $\mathcal{Z}_T^p$

4. Additive noise

Let $u_{\mathrm{aff}}(t,x)$ be the solution to the stochastic heat equation with $\sigma(t,x,u) = 1$ and $u_0(x) = 0$:

$$\frac{\partial u(t,x)}{\partial t} = \frac{1}{2}\Delta u(t,x) + \dot{W}, \quad t > 0, x \in \mathbb{R}.$$

Then, there are two positive constants c_H and C_H, independent of T and L, such that

$$c_H\, \rho(T,L) \le \mathbb{E}\left(\sup_{\substack{0\le t\le T \\ -L\le x\le L}} u_{\mathrm{aff}}(t,x) \right) \le \mathbb{E}\left(\sup_{\substack{0\le t\le T \\ -L\le x\le L}} |u_{\mathrm{aff}}(t,x)| \right) \le C_H\, \rho(T,L)\,,$$

where

$$\rho(T,L)=\begin{cases} T^{\frac{H}{2}}+T^{\frac{H}{2}}\sqrt{\log_2\left[\frac{L}{\sqrt{T}}\right]} & \text{if } L^2>T,\\ T^{\frac{H}{2}} & \text{if } L^2\leq T\,. \end{cases}$$

There are two strictly positive random constants $c_{\varepsilon,H}$ and $C_{\varepsilon,H}H$, independent of T and L, such that almost surely

$$\begin{aligned} c_{\varepsilon,H}\left(T^{\frac{H}{2}}+T^{\frac{H}{2}}\sqrt{\log_2\left[\frac{L}{\sqrt{T}}\right]}\right) &\leq \sup_{(t,x)\in\mathcal{R}_\varepsilon(T,L)} u_{\mathrm{aff}}(t,x)\\ &\leq \sup_{(t,x)\in\mathcal{R}_\varepsilon(T,L)} |u_{\mathrm{aff}}(t,x)| \leq C_{\varepsilon,H}\left(T^{\frac{H}{2}}+T^{\frac{H}{2}}\sqrt{\log_2\left[\frac{L}{\sqrt{T}}\right]}\right), \end{aligned}$$

where $\mathcal{R}_\varepsilon(T,L)=\{(t,x)\in[0,T]\times[-L,L]\ :\ L\geq T^{\frac{1+\varepsilon}{2}}\}$ for any $\varepsilon>0$.

Theorem

Let $u_{\mathrm{aff}}(t,x)$ *be the solution to the equation with* $\sigma(t,x,u) = 1$ *and* $u_0(x) = 0$ *and denote*

$$\Delta_h u_{\mathrm{aff}}(t,x) := u_{\mathrm{aff}}(t,x+h) - u_{\mathrm{aff}}(t,x)\,.$$

Let $\theta \in (0,H)$ *be given and let* $L > \sqrt{t}$*. Then, there are two positive constants* c_H *and* $C_{H,\theta}$ *such that for sufficiently small value of h satisfying* $0 < |h| \le C(\sqrt{t} \wedge 1)$ *for some constant* C*, the following inequalities hold true:*

$$c_H |h|^H \sqrt{\log_2\left[\frac{L}{\sqrt{t}}\right]} \le \mathbb{E}\left(\sup_{-L\le x\le L} \Delta_h u_{\mathrm{aff}}(t,x)\right)$$
$$\le \mathbb{E}\left(\sup_{-L\le x\le L} |\Delta_h u_{\mathrm{aff}}(t,x)|\right) \le C_{H,\theta}\, t^{\frac{H-\theta}{2}} |h|^\theta \sqrt{\log_2\left[\frac{L}{\sqrt{t}}\right]}\,.$$

Theorem

Moreover, there are two (strictly) positive random constants c_H and $C_{H,\theta}$, independent of $L \in \mathbb{R}_+$ and $h \in [-C(\sqrt{t} \wedge 1), C(\sqrt{t} \wedge 1)]$ almost surely

$$c_H |h|^H \sqrt{\log_2 \left[\frac{L}{\sqrt{t}} \right]} \leq \sup_{-L \leq x \leq L} \Delta_h u_{\text{aff}}(t, x)$$

$$\leq \sup_{-L \leq x \leq L} |\Delta_h u_{\text{aff}}(t, x)| \leq C_{H,\theta} t^{\frac{H-\theta}{2}} |h|^\theta \sqrt{\log_2 \left[\frac{L}{\sqrt{t}} \right]}.$$

Let $u_{\mathrm{aff}}(t,x)$ be the solution to the stochastic heat equation with $\sigma(t,x,u)=1$ and $u_0(x)=0$ and denote

$$\Delta_\tau u_{\mathrm{aff}}(t,x) := u_{\mathrm{aff}}(t+\tau,x) - u_{\mathrm{aff}}(t,x)\,.$$

Then, for sufficiently small value of τ such that $0<\tau\le C(t\wedge 1)$ for some constant C, we have

$$\begin{aligned} c_H \tau^{\frac{H}{2}} \sqrt{\log_2\left[\frac{L}{\sqrt{t}}\right]} &\le \mathbb{E}\left(\sup_{-L\le x\le L} \Delta_\tau u_{\mathrm{aff}}(t,x)\right) \\ &\le \mathbb{E}\left(\sup_{-L\le x\le L} |\Delta_\tau u_{\mathrm{aff}}(t,x)|\right) \\ &\le C_{H,\theta}\, t^{\frac{H}{2}-\theta}\tau^{\theta} \sqrt{\log_2\left[\frac{L}{\sqrt{t}}\right]}\,, \end{aligned}$$

where $0<\theta<H/2$ and the constants c_H and $C_{H,\theta}$ are independent of L and τ.

We also have the almost sure version of the above result: if $0 < \tau \leq C(t \wedge 1)$, then we have

$$\begin{aligned} c_H \tau^{\frac{H}{2}} \sqrt{\log_2 \left[\frac{L}{\sqrt{t}} \right]} &\leq \sup_{-L \leq x \leq L} \Delta_\tau u_{\text{aff}}(t,x) \\ &\leq \sup_{-L \leq x \leq L} |\Delta_\tau u_{\text{aff}}(t,x)| \\ &\leq C_{H,\theta} t^{\frac{H}{2}-\theta} \tau^\theta \sqrt{\log_2 \left[\frac{L}{\sqrt{t}} \right]}, \end{aligned}$$

almost surely if $L \to \infty$, where $0 < \theta < H/2$, and random constants c_H and $C_{H,\theta}$ are independent of L and τ.

Theorem (Talagrand majorizing measure theorem)

Let T be a given set and let $\{X_t, t \in T\}$ be a centered Gaussian process indexed by T. Denote $d(t,s) = (\mathbb{E}|X_t - X_s|^2)^{\frac{1}{2}}$, the associated natural metric on T.
Then

$$\mathbb{E}\left[\sup_{t\in T} X_t\right] \approx \gamma_2(T,d) := \inf_{\mathcal{A}} \sup_{t\in T} \sum_{n\geq 0} 2^{n/2}\mathrm{diam}(A_n(t)),$$

where the infimum is taken over all increasing sequence $\mathcal{A} := \{\mathcal{A}_n, n = 1, 2, \cdots\}$ of partitions of T such that $\#\mathcal{A}_n \leq 2^{2^n}$ ($\#A$ denotes the number of elements in the set A), $A_n(t)$ denotes the unique element of $\mathcal{A}_n$ that contains t, and $\mathrm{diam}(A_n(t))$ is the diameter (with respect to the natural distance d) of $A_n(t)$.

Theorem (Sudakov minoration)

Let $\{X_{t_i}, i = 1, \cdots, L\}$ be centered Gaussian family with natural distance d and assume

$$\forall p, q \leq L,\ p \neq q \Rightarrow d(t_p, t_q) \geq \delta.$$

Then, we have

$$\mathbb{E}\Big(\sup_{1 \leq i \leq L} X_{t_i} \Big) \geq \frac{\delta}{C} \sqrt{\log_2(L)},$$

where C is a universal constant.

Theorem (Borell)

Let $\{X_t, t \in T\}$ be a centered separable Gaussian process on some topological index set T with almost surely bounded sample paths. Then $\mathbb{E}\Big(\sup_{t\in T} X_t\Big) < \infty$, and for all $\lambda > 0$

$$\mathbf{P}\left\{\left|\sup_{t\in T} X_t - \mathbb{E}\Big(\sup_{t\in T} X_t\Big)\right| > \lambda\right\} \leq 2\exp\left[-\frac{\lambda^2}{2\sigma_T^2}\right], \qquad (1)$$

where $\sigma_T^2 := \sup_{t\in T} \mathbb{E}(X_t^2)$.

Define the natural metric:

$$d_1((t,x),(s,y)) = \sqrt{\mathbb{E}|u_{\mathrm{aff}}(t,x) - u_{\mathrm{aff}}(s,y)|^2}\,, \tag{2}$$

Then

Lemma

Let $d_1((t,x),(s,y))$ be the natural metric defined by (2). Then, there are positive constants c_H, C_H such that

$$\begin{aligned} c_H(|x-y|^H \wedge (t\wedge s)^{\frac{H}{2}} + |t-s|^{\frac{H}{2}}) &\le d_1((t,x),(s,y)) \\ &\le C_H(|x-y|^H \wedge (t\wedge s)^{\frac{H}{2}} + |t-s|^{\frac{H}{2}}) \end{aligned} \tag{3}$$

for any $(t,x),(s,y) \in \mathbb{R}_+ \times \mathbb{R}$. Or

$$d_1((t,x),(s,y)) \asymp d_{1,H}((t,x),(s,y)) := |x-y|^H \wedge (t\wedge s)^{\frac{H}{2}} + |t-s|^{\frac{H}{2}}. \tag{4}$$

Proof: Need nice bounds of some integrals.

Proof of the upper bound for expectation

We use Talagrand's majorizing measure theorem.

We choose the admissible sequences $(\mathcal{A}_n)$ as uniform partition of $\mathbb{T} \times \mathbb{L} = [0, T] \times \mathbb{L}$ such that $\mathrm{card}(\mathcal{A}_n) \leq 2^{2^n}$.

$$\begin{cases} [0, T] = \bigcup_{j=0}^{2^{2^{n-1}}-1} \left[j \cdot 2^{-2^{n-1}} T, (j+1) \cdot 2^{-2^{n-1}} T \right) , \\ [-L, L] = \bigcup_{k=-2^{2^{n-2}}}^{2^{2^{n-2}}-1} \left[k \cdot 2^{-2^{n-2}} L, (k+1) \cdot 2^{-2^{n-2}} L \right) . \end{cases}$$

Theorem 8 states

$$\begin{aligned} &\mathbb{E} \left(\sup_{(t,x) \in \mathbb{T} \times \mathbb{L}} u_{\mathrm{aff}}(t, x) \right) \leq C \gamma_2(T, d) \\ &\leq C \sup_{(t,x) \in \mathbb{T} \times \mathbb{L}} \sum_{n \geq 0} 2^{n/2} \mathrm{diam}(A_n(t, x)) . \end{aligned} \tag{5}$$

Here $A_n(t,x)$ is the element of uniform partition $\mathcal{A}_n$ that contains (t,x), i.e.

$$\begin{aligned} A_n(t,x) &= \left[j \cdot 2^{-2^{n-1}} T, (j+1) \cdot 2^{-2^{n-1}} T\right) \\ &\quad \times \left[k \cdot 2^{-2^{n-2}} L, (k+1) \cdot 2^{-2^{n-2}} L\right) \end{aligned}$$

such that $j \cdot 2^{-2^{n-1}} T \leq t < (j+1) \cdot 2^{-2^{n-1}} T$ and $k \cdot 2^{-2^{n-2}} L \leq x < (k+1) \cdot 2^{-2^{n-2}} L$. The diameter of $A_n(t,x)$ with respect to $d_{1,H}((t,x),(s,y))$ defined in (4) can be estimated as

$$\operatorname{diam}(A_n(t,x)) \leq C_H \left[T^{\frac{H}{2}} \wedge (2^{-H2^{n-2}} L^H)\right] + C_H 2^{-H2^{n-2}} T^{\frac{H}{2}}.$$

Let N_0 be the smallest integer such that $2^{-2^{n-2}}L \leq \sqrt{T}$, i.e. $\log_2(\log_2(L/\sqrt{T})) + 2 \leq N_0 < \log_2(\log_2(L/\sqrt{T})) + 3$. By (5) we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} u(t,x)\right) \\
\leq & C_H \sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} \left(\sum_{n=0}^{N_0} 2^{n/2}\mathrm{diam}(A_n(t,x)) + \sum_{n=N_0+1}^{\infty} 2^{n/2}\mathrm{diam}(A_n(t,x))\right) \\
\leq & C_H \sup_{(t,x)\in\mathbb{T}\times\mathbb{L}} T^{\frac{H}{2}} \left(\sum_{n=0}^{N_0} 2^{n/2} + \sum_{n=N_0+1}^{\infty} 2^{n/2}\left[\frac{2^{2^{N_0-2}}}{2^{2^{n-2}}}\right]^H\right) + C_H T^{\frac{H}{2}} \\
\leq & C_H T^{\frac{H}{2}} \left(\sqrt{\log_2\left[\frac{L}{\sqrt{T}}\right]} + 1\right) + C_H T^{\frac{H}{2}}, \qquad (6)
\end{aligned}
$$

where $L > \sqrt{T}$. This concludes proof of the upper bound when $L > \sqrt{T}$.

When $L \le \sqrt{T}$. The same uniform partition discussed above is still applicable. We have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}}|u(t,x)|\right)\\
\le& C_H\left(\sum_{n=0}^{\infty}2^{n/2}\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}}\operatorname{diam}(A_n(t,x))\right)\\
\le& C_H T^{\frac{H}{2}}\sum_{n=0}^{\infty}2^{n/2}\cdot 2^{-H2^{n-1}}+C_H T^{\frac{H}{2}}\le C_H T^{\frac{H}{2}}\,, \qquad (7)
\end{aligned}
$$

because

$$
\begin{aligned}
\sup_{(t,x)\in\mathbb{T}\times\mathbb{L}}\operatorname{diam}(A_n(t,x)) &\le C_H\left(\left[2^{-2^{n-2}}L\right]^{H}+\left[2^{-2^{n-1}}T\right]^{\frac{H}{2}}\right)\\
&\le C_H 2^{-H2^{n-2}}T^{\frac{H}{2}}\,.
\end{aligned}
$$

This completes the upper bounds in this case.

5. General Case

We introduce the weighted $\mathcal{Z}_T^p$ space.

Let $\lambda(x) \geq 0$ be a Lebesgues integrable positive function with $\int_{\mathbb{R}} \lambda(x)dx = 1$. Introduce a norm $\|\cdot\|_{\mathcal{Z}_{\lambda,T}^p}$ for a random field $v(t,x)$ as follows:

$$\|v\|_{\mathcal{Z}_{\lambda,T}^p} := \sup_{t\in[0,T]} \|v(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})} + \sup_{t\in[0,T]} \mathcal{N}^*_{\frac{1}{2}-H,p} v(t),$$

where $p \geq 2$, $\frac{1}{4} < H < \frac{1}{2}$,

$$\|v(t,\cdot)\|_{L_\lambda^p(\Omega\times\mathbb{R})} = \left[\int_{\mathbb{R}} \mathbb{E}\left(|v(t,x)|^p\right) \lambda(x)dx\right]^{\frac{1}{p}},$$

and

$$\mathcal{N}^*_{\frac{1}{2}-H,p} v(t) = \left[\int_{\mathbb{R}} \|v(t,\cdot) - v(t,\cdot+h)\|^2_{L_\lambda^p(\Omega\times\mathbb{R})} |h|^{2H-2} dh\right]^{\frac{1}{2}}.$$

We make the following assumptions

(H1) $\sigma(u)$ is at most of linear growth in u uniformly in t and x. This means

$$|\sigma(u)| \leq C(|u| + 1)\,,$$

and it is uniformly Lipschitzian in u, i.e. $\forall\, u, v \in \mathbb{R}$

$$|\sigma(u) - \sigma(v)| \leq C|u - v|\,,$$

for some constant $C > 0$.

Theorem

Let $\lambda(x) = c_H(1+|x|^2)^{H-1}$ *satisfy* $\int_{\mathbb{R}} \lambda(x)dx = 1$*. Assume* $\sigma(u)$ *satisfies hypothesis **(H1)** and that the initial data* u_0 *is in* $L^p_\lambda(\mathbb{R})$ *and*

$$\mathcal{N}^*_{\frac{1}{2}-H,p} u_0 = \left[\int_{\mathbb{R}} \|u_0(\cdot) - u_0(\cdot + h)\|^2_{L^p_\lambda(\Omega\times\mathbb{R})} |h|^{2H-2} dh \right]^{\frac{1}{2}}$$

is finite for some $p > \frac{3}{H}$*. Then, there exists a weak solution to the stochastic heat equation with sample paths in* $\mathcal{C}([0,T]\times\mathbb{R})$ *almost surely. In addition, for any* $\gamma < H - \frac{3}{p}$*, the process* $u(\cdot,\cdot)$ *is almost surely Hölder continuous on any compact sets in* $[0,T]\times\mathbb{R}$ *of Hölder exponent* $\gamma/2$ *with respect to the time variable* t *and of Hölder exponent* γ *with respect to the spatial variable* x*.*

Strong soluton

(H2) Assume that $\sigma(t,x,u) \in \mathcal{C}^{0,1,1}([0,T]\times\mathbb{R}\times\mathbb{R})$ satisfies the following conditions: $|\sigma'_u(t,x,u)|$ and $|\sigma''_{xu}(t,x,u)|$ are uniformly bounded:

$$\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}} |\sigma'_u(t,x,u)| \le C\,; \tag{8}$$

$$\sup_{t\in[0,T],x\in\mathbb{R},u\in\mathbb{R}} |\sigma''_{xu}(t,x,u)| \le C\,. \tag{9}$$

Moreover, assume

$$\sup_{t\in[0,T],x\in\mathbb{R}} \lambda^{-\frac{1}{p}}(x)\left|\sigma'_u(t,x,u_1)-\sigma'_u(t,x,u_2)\right| \le C|u_2-u_1|\,, \tag{10}$$

where $\lambda(x) = c_H(1+|x|^2)^{H-1}$.

Theorem

*Let σ satisfy the above hypothesis **(H2)** and that for some $p > \frac{6}{4H-1}$, $\|u_0\|_{L^p_\lambda(\mathbb{R})}$ and $\mathcal{N}^*_{\frac{1}{2}-H,p} u_0$ are finite. Then the equation has a unique strong solution. Moreover, for any $\gamma < H - \frac{3}{p}$, the process $u(\cdot,\cdot)$ is almost surely Hölder continuous on any compact sets in $[0,T] \times \mathbb{R}$ of Hölder exponent $\gamma/2$ with respect to the time variable t and of Hölder exponent γ with respect to the spatial variable x.*

6. Some key estimates

Lemma

For any $\lambda \in \mathbb{R}$, $\lambda(x) = \frac{1}{(1+|x|^2)^\lambda}$ *and* $T > 0$, *we have*

$$\sup_{0\le t\le T} \sup_{x\in\mathbb{R}} \frac{1}{\lambda(x)} \int_{\mathbb{R}} G_t(x-y)\lambda(y)dy < \infty.$$

Denote

$$D_t(x,h) := G_t(x+h) - G_t(x)\,, \quad D(x,h) = \sqrt{\pi} D_{1/4}(x,h)$$

$$\Box_t(x,y,h) := G_t(x+y+h) - G_t(x+y) - G_t(x+h) + G_t(x)\,.$$

$$\Box(x,y,h) = \sqrt{\pi}\Box_{1/4}(x,y,h)\,.$$

Then

Lemma

For any $\alpha, \beta \in (0,1)$, *we have*

$$\int_{\mathbb{R}^2} |D_t(x,h)|^2 |h|^{-1-2\beta} dh dx = \frac{C_\beta}{t^{\frac{1}{2}+\beta}}$$

and

$$\int_{\mathbb{R}^3} |\Box_t(x,y,h)|^2 |h|^{-1-2\alpha} |y|^{-1-2\beta} dy dh dx = \frac{C_{\alpha,\beta}}{t^{\frac{1}{2}+\alpha+\beta}}.$$

Lemma

$$\int_{\mathbb{R}^2} |D_t(x,h)|^2 |h|^{2H-2} \lambda(z-x) dx dh \le C_{T,H} t^{H-1} \lambda(z),$$

$$\int_{\mathbb{R}^3} |\Box_t(x,y,h)|^2 |h|^{2H-2} |y|^{2H-2} \lambda(z-x) dx dy dh \le C_{T,H} t^{2H-\frac{3}{2}} \lambda(z).$$

THANKS