Local Limit Theorem for Linear Random Fields

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This talk is based on the papers jointly with
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Outline

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Basic Idea

- Let $Z_1, Z_2, ..., Z_n$ be an i.i.d. sequence of standard normal random variables.
- The sum $S_n = \sum_{k=1}^{n} Z_k$ has distribution function

$$f(x) = \frac{1}{\sqrt{2\pi n}} e^{-\frac{1}{2n}x^2}.$$

- Given an interval (a, b), define a sequence of measure by

$$\mu_n(a, b) = \sqrt{2\pi n}\, P(S_n \in (a, b)) = \int_a^b e^{-\frac{1}{2n}x^2}\, dx$$
$$\to b - a.$$

Previous Local Limit Work I

- The LLT has been well-studied for certain cases. Amongst them are the case of lattice random variables and the case of independent, absolutely continuous random variables.
- Some papers to consider include Shepp (1964) and Mineka and Silverman (1970). We also refer the reader to the books by Ibragimov and Linnik (1971), Petrov (1975), and Gnedenko (1962).
- Linear random fields (l.r.f.) have been extensively studied in probability and statistics.
- Mallik and Woodroofe (2011) studied the CLT for l.r.f., Sang and Xiao (2018) established exact moderate and large deviation asymptotics for l. r. f. under moment or regularly varying tail conditions.

Previous Local Limit Work II

- With a conjugate method, Beknazaryan, Sang, and Xiao (2019) studied the Cramér type moderate deviation for l.r.f.
- We refer to Sang and Xiao (2018) for a brief review on the study of asymptotic properties for l.r.f. and to Koul, Mimoto, and Surgailis (2016), Lahiri and Robinson (2016) and the reference therein for recent developments in statistics.
- To the best of our knowledge, the local limit result for l.r.f. has not been established in the literature.

Outline

Preliminary Stuff

- $\{\varepsilon_i : i \in \mathbf{Z}^d\}$ are i.i.d. random variables.
- $\{a_i : i \in \mathbf{Z}^d\}$ is a collection of real numbers.
- The linear random field (l.r.f.)

$$X_j = \sum_{k \in \mathbf{Z}^d} a_k \varepsilon_{j-k} \tag{1}$$

  is defined on $\mathbf{Z}^d$.
- Let Γ_n^d be a sequence of subsets of $\mathbf{Z}^d$. For example, $\Gamma_n^d = [-n, n]^d \cap \mathbf{Z}^d$. For linear random fields,

$$S_n = \sum_{j \in \Gamma_n^d} X_j = \sum_{i \in \mathbf{Z}^d} b_{n,i}\, \varepsilon_i \tag{2}$$

  with coefficients

$$b_{n,i} = \sum_{j \in \Gamma_n^d} a_{j-i}. \tag{3}$$

First case

- The ε_i have mean 0 and finite variance σ_ε^2.
- $\sum_{i \in \mathbf{Z}^d} a_i^2 < \infty$.
- The field has *short memory* if $\sum_{i \in \mathbf{Z}^d} a_i \neq 0$ and $\sum_{i \in \mathbf{Z}^d} |a_i| < \infty$.
- The field has *long memory* if $\sum_{i \in \mathbf{Z}^d} |a_i| = \infty$. In this case, we assume that the sets Γ_n^d are constructed as a disjoint union of J_n discrete rectangles.
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$$B_n^2 = Var(S_n) = \sigma_\varepsilon \sum_{i \in \mathbf{Z}^d} b_{n,i}^2. \tag{4}$$

CLT (Mallik and Woodroofe, 2011)

Let S_n and B_n be defined as in (2) and (4). Assume that $B_n \to \infty$. When the field has long range dependence we additionally require that $J_n = o(B_n^2)$, while otherwise no such restriction is required. Under these conditions, S_n/B_n converges in distribution to the standard normal distribution.

Non-lattice distribution and Cramér condition

- Denote the characteristic function of ε_0 by $\varphi_\varepsilon(t) := \mathbb{E}(\exp\{it\varepsilon_0\})$.
- It is well known that ε_0 not having a lattice distribution is equivalent to $|\varphi_\varepsilon(t)| < 1$ for all $t \neq 0$.
- The Cramér condition means that $\limsup_{|t|\to\infty} |\varphi_\varepsilon(t)| < 1$.
- ε_0 has a non-lattice distribution whenever $\varphi_\varepsilon(t)$ satisfies the Cramér condition.

Local limit theorem (Fortune, Peligrad and S., 2021)

Assume that the innovations have non-lattice distribution and $B_n \to \infty$. In the long range dependence case, we additionally assume that the innovations satisfy the Cramér condition and $\frac{J_n^{2/d}\log(B_n)}{\sup_{i\in\mathbb{Z}^d}|b_{n,i}|^{2/d}} \to 0$ as $n \to \infty$. Under these conditions, for all continuous complex-valued functions $h(x)$ with $|h| \in L^1(\mathbb{R})$ and with Fourier transform $\hat{h}$ real and with compact support,

$$\lim_{n\to\infty}\sup_{u\in\mathbb{R}}\left|\sqrt{2\pi}B_n\mathbb{E}h(S_n-u)-[\exp(-u^2/2B_n^2)]\int h(x)\lambda(dx)\right|=0, \quad (5)$$

where λ is the Lebesgue measure.
Note: the condition can be improved to $J_n = o(B_n^2)$.

Remark

- The LLT is new also for $d = 1$.
- For short memory case, we only require $B_n \to \infty$.
- We have CLT & LLT for long memory l.r.f. over a sequence of regions Γ_n^d which are a disjoint union of J_n discrete rectangles with $J_n = o(B_n^2)$.
- In practice it allows us to have disjoint discrete rectangles as spatial sampling regions, and the number of sampling regions may increase as the sample size increases.
- The discrete spatial rectangular sampling regions also include $(\prod_{k=1}^{d}[\underline{n}_k, \overline{n}_k]) \cap \mathbb{Z}^d$ where $\underline{n}_k = \overline{n}_k$ for some k's. We may have a single point region if the equality holds for all k's.

Remark

By Hafouta and Kifer (2016) this result implies that (5) also holds for the class of real continuous functions with compact support. So

$$\lim_{n\to\infty} \sup_{u\in\mathbb{R}} \left| \sqrt{2\pi} B_n P(a+u \le S_n \le b+u) - [\exp(-u^2/2B_n^2)](b-a) \right| = 0,$$

for any $a < b$. In particular, since $B_n \to \infty$ as $n \to \infty$, then for fixed $A > 0$,

$$\lim_{n\to\infty} \sup_{|u|\le A} \left| \sqrt{2\pi} B_n P(a+u \le S_n \le b+u) - (b-a) \right| = 0.$$

If we further take $u = 0$, then,

$$\lim_{n\to\infty} \sqrt{2\pi} B_n P(S_n \in [a,b]) = b - a.$$

One example

- Assume that Γ_n^d are cubic, i.e., $\Gamma_n^d = [-n, n]^d \cap \mathbb{Z}^d$.
- Put $a_i = l(\|i\|)G(i/\|i\|)\|i\|^{-\beta}$ with $\beta \in (d/2, d)$, where $l(x)$ is slowly varying at ∞ and $G : \mathbb{S}_{d-1} \to \mathbb{R}^+$ is continuous on its domain $\big($the unit sphere in d-dimensional space$\big)$.
- For this example we know that $B_n \propto n^{\frac{3d}{2}-\beta}\ l(n)$ (see Surgailis, 1982, Theorem 2).

Domain of stable law case

- The innovations ε_i satisfy $\mathbb{P}(|\varepsilon_1| > x) = x^{-\alpha} L(x)$, where $L(x)$ is a slowly varying function at ∞, $0 < \alpha < 2$,

$$\frac{\mathbb{P}(\varepsilon_1 > x)}{\mathbb{P}(|\varepsilon_1| > x)} \to c^+ \quad \text{and} \quad \frac{\mathbb{P}(\varepsilon_1 < -x)}{\mathbb{P}(|\varepsilon_1| > x)} \to c^- \text{ as } x \to \infty.$$

  Here $0 \leq c^+ \leq 1$ and $c^+ + c^- = 1$.

- In the case $\alpha = 2$, and $\mathbb{E}(\varepsilon_1^2) = \infty$, $\mathbb{P}(|\varepsilon_1| > x) = x^{-2} L(x)$, where $L(x)$ is a slowly varying function at ∞. For this case define

$$h(x) = \mathbb{E}\varepsilon_1^2 I(|\varepsilon_1| \leq x), \quad \text{for} \quad x \geq 0.$$

  $h(x)$ is a slowly varying function at ∞.

Domain of stable law case

- Define
$$H(t) = \begin{cases} L(1/|t|) & \text{if } 0 < \alpha < 2 \\ h(1/|t|) & \text{for } \alpha = 2. \end{cases} \tag{6}$$
- We assume Conditions A : $\mathbb{E}\varepsilon_1 = 0$ if $1 < \alpha \leq 2$ and ε_1 has symmetric distribution if $\alpha = 1$.
- the l.r.f. converges almost surely if and only if $\sum_{i \in \mathbb{Z}^d} |a_i|^\alpha H(a_i) < \infty$. For the one-dimensional case $d = 1$, see Balan, Jakubowski and Louhichi (2016) if $0 < \alpha < 2$ and Peligrad and Sang (2012) if $\alpha = 2$.
- Define
$$B_n = \inf\left\{x \geq 1 : \sum_i (|b_{ni}|/x)^\alpha H(b_{ni}/x) \leq 1\right\}. \tag{7}$$

It is easy to see that
$$\sum_i (|b_{ni}|/B_n)^\alpha H(b_{ni}/B_n) \to 1 \tag{8}$$

Limit theorem (Peligrad, S., Xiao and Yang, 2021+)

Let S_n be the partial sum of (1) and B_n be defined as in (7). Assume Condition A and $B_n \to \infty$ as $n \to \infty$. In the case that $1 < \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$, we additionally require that the sets Γ_n^d are constructed as a disjoint union of J_n discrete rectangles, where $J_n = o(B_n^q)$, $1/p + 1/q = 1$, for some $p > \alpha$ if $1 < \alpha < 2$, and $p = 2$ if $\alpha = 2$. Otherwise no such restriction is required. Under these conditions, S_n/B_n converges in distribution to a stable distribution.

Remark

- In McElroy and Politis (2003), the authors obtained the limit theorem for the partial sums of l.r.f. over one rectangle under the conditions: $1 < \alpha < 2$, the coefficients $\{a_i\}$ are summable and $\min_i n_i \to \infty$, where n_i is the size of the rectangle in the i-th dimension.
- The result here is new even in one-dimensional case. Davis and Resnick (1985) studied the limit theorem for the partial sums $S_n = \sum_{j=1}^{n} X_j$ of one-sided linear process under the condition the coefficients are summable.

Local limit theorem (Peligrad, S., Xiao and Yang, 2021+)

Let S_n be the partial sum of (1) and B_n be defined as in (7). Assume $B_n \to \infty$ as $n \to \infty$. The innovations satisfy Condition A and have a non-lattice distribution. If $1 \leq \alpha \leq 2$ and $\sum_{i \in \mathbb{Z}^d} |a_i| = \infty$, we further assume that the innovations satisfy the Cramér condition, and the sum is over J_n disjoint rectangles with $J_n = o(B_n^q)$, $1/p + 1/q = 1$, for some $p > \alpha$ if $1 < \alpha < 2$, and $p = 2$ if $\alpha = 2$. Otherwise these additional assumptions are not required. Then, for any function g on $\mathbb{R}$ which is continuous and has compact support,

$$\lim_{n \to \infty} \sup_{u \in \mathbb{R}} \left| B_n \mathbb{E} g(S_n + u) - f_L\left(\frac{u}{B_n}\right) \int g(t) \lambda(dt) \right| = 0, \tag{9}$$

where λ is the Lebesgue measure and f_L is the density of limit law L of S_n / B_n.

Outline

Simulation

- We use the one-dimensional linear process with $a_i = \frac{\Gamma(i+1-\alpha)}{\Gamma(1-\alpha)\Gamma(i+1)}$, where $\frac{1}{2} < \alpha < 1$. In particular, we use the fractionally integrated (FARIMA$(0, 1-\alpha, 0)$) processes $X_j = (1-B)^{\alpha-1}\varepsilon_j = \sum_{i=0}^{\infty} a_i \varepsilon_{j-i}$.
- We assume that the innovations are i.i.d. Student's t random variables with 5 degrees of freedom.
- The variance of the partial sum $S_n = \sum_{j=1}^{n} X_j$ is

$$B_n^2 \sim c_\alpha n^{3-2\alpha} \mathbb{E}\varepsilon^2 / [(1-\alpha)(3-2\alpha)\Gamma^2(1-\alpha)]$$

  where

$$c_\alpha = \int_0^\infty x^{-\alpha}(1+x)^{-\alpha} dx.$$

  The variance formula for the partial sum of FARIMA$(0, 1-\alpha, 0)$ is well known. See, for example, Wang, Lin and Gulati (2001).

Simulation (cont)

- Employing the MATLAB code of Fay et al. (2009), N replicates of linear processes were generated, each of length n.
- Specifically, we generated cases with $N = 5,000$ and $N = 10,000$ cross-referenced with $n = 2^{10}$, $n = 2^{12}$, and $n = 2^{14}$, and this was done for each of the values $\alpha = 0.95$, $\alpha = 0.70$, and $\alpha = 0.55$.
- Once the data were obtained, the local limit measure of various intervals was estimated by using relative frequency to estimate $P(S_n \in (a, b))$ and using the estimate of B_n described above.

Table 1 I

Table: Local limit measure of the intervals (-100,0), (-50,50), and (0,100) using N one-dimensional linear processes, each of length n, employing various long memory cases using the $FARIMA(0,\ 1-\alpha\ ,0)$ model with t_5 innovations.

	$n = 2^{10}$			$n = 2^{12}$		
N	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$
5×10^3	66	105	117	92	99	98
	90	99	115	101	95	108
	67	99	97	90	96	108
1×10^4	67	97	105	91	98	101
	89	98	95	99	103	105
	65	103	101	87	104	108

Table 1 II

Table: Local limit measure of the intervals (-100,0), (-50,50), and (0,100) using N one-dimensional linear processes, each of length n, employing various long memory cases using the $FARIMA(0,\ 1-\alpha\ , 0)$ model with t_5 innovations.

N	$n = 2^{14}$		
	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$
5×10^3	95	91	122
	100	88	110
	98	106	110
1×10^4	96	97	104
	101	97	98
	98	98	92

Table 2 I

Table: Local limit measure of the intervals (-50,0), (-25,25), and (0,50) using N one-dimensional linear processes, each of length n, employing various long memory cases using the $FARIMA(0,\ 1-\alpha\ ,0)$ model with t_5 innovations.

	$n = 2^{10}$			$n = 2^{12}$		
N	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$
5×10^3	46	51	67	51	52	62
	50	47	54	49	49	43
	46	48	48	49	43	46
1×10^4	46	48	50	49	52	43
	48	50	51	50	52	44
	43	51	54	50	51	62

Table 2 II

Table: Local limit measure of the intervals (-50,0), (-25,25), and (0,50) using N one-dimensional linear processes, each of length n, employing various long memory cases using the $FARIMA(0,\ 1-\alpha\ ,0)$ model with t_5 innovations.

	$n = 2^{14}$		
N	$\alpha = 0.95$	$\alpha = 0.70$	$\alpha = 0.55$
5×10^3	49	45	61
	48	40	61
	51	43	49
1×10^4	50	51	55
	50	45	49
	51	47	43

Outline

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