CBMS Conference: Gaussian Random Fields, Fractals, SPDEs, and Extremes

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Overview of the Lectures

The principle aim of these lectures is to begin with general fundamental notions such as the mathematics of Gaussian random fields and end by presenting a series of recent advances and various powerful methods, as well as various open problems, from mathematics and statistics, surrounding the general topic of "fractals" and "extremes". Some of the problems will be particularly suitable as research projects for graduate students and post-doctoral researchers.

The only prerequisites are a solid course in measure-theoretic probability, and a modest knowledge of Brownian motion.

The following is a more detailed plan for the lectures.

1. *Introduction: overview, applications, and salient features of random fields.* (1 lecture)

   Multivariate random fields (or spatial processes) have recently been the focus of much attention in probability and statistics, due to their extensive applications as spatial or spatio-temporal models in scientific areas where many problems involve data sets with multivariate measurements obtained at spatial locations.

   We present an overview on random fields and provide concrete examples of random fields that are drawn from science and engineering.

   We introduce important statistical characteristics such as self-similarity, operator-self-similarity, anisotropy, long range dependence of random fields.

2. *Construction of random fields.* (1 lecture)

   The mathematical theory of random fields developed by Itô (1954), Yaglom (1957, 1987), Gihman and Skorohod (1974) provides an excellent framework for constructing and studying multivariate random fields. This lecture will introduce systematic methods for constructing univariate and multivariate Gaussian random fields, including characterization of cross-covariance matrices and the spectral method. Interesting examples of multivariate Gaussian random fields that can be constructed by using these methods include multivariate stationary Gaussian random fields with Matérn cross-covariance matrix and operator fractional Brownian motion.

Another natural way to define multivariate random fields is through systems of stochastic partial differential equations.

3. *General methods for Gaussian random fields.* (1 lecture)

   This lecture introduces some fundamental methods for Gaussian random fields. These include: reproducing kernel Hilbert space; the Karhunen-Loève expansion for univariate and multivariate Gaussian random fields; the entropy method and chaining argument; Dudley's entropy bounds for supremum of Gaussian processes; and concentration inequalities.

4. *Regularity of Gaussian random fields and exact modulus of continuity.* (1 lecture)

   Regularity properties such as continuity and differentiability of the sample functions of Gaussian processes are important topics in probability theory and essential for statistical applications.

   Necessary and sufficient conditions for sample path continuity based on the metric entropy or majorizing measure were established by Dudley (1967), Fernique (1975), and Talagrand (1987).

   The purpose of this module is to present methods for establishing exact uniform and local modulus of continuity results for Gaussian random fields. The main technical tool is the property of strong local nondeterminism.

5. *Fractal properties of random fields.* (1 lecture)

   Fractal geometry is important for studying random fields with non-differentiable sample functions. We introduce Hausdorff and packing measure and dimensions and main techniques for their computation.

   We determine the Hausdorff dimensions of various random sets generated by multivariate Gaussian random fields including the range, graph, level sets, and set of multiple points.

   Local times and self-intersections local times are introduced for studying fractal properties of the level sets and the set of multiple points. The properties of strong local nondeterminism are applied for establishing sharp regularity results on the local times.

6. *Potential theory: Hitting probabilities and self-intersections.* (1 lecture)

   The lecture is concerned with hitting probabilities of Gaussian random fields and their applications in studying the existence and Hausdorff dimensions of intersections.

   Let $X = \{X(t), t \in \mathbb{R}^N\}$ be a Gaussian random field with values in $\mathbb{R}^d$. For any compact sets $E \subset \mathbb{R}^N$ and $F \subset \mathbb{R}^d$, we study conditions on E and F for $\mathbb{P}\big\{X(E) \cap F \neq \varnothing\big\} > 0$.

Only in a few special cases, this hitting probability problem has been solved. We will present the necessary and sufficient conditions due to Khoshnevisan and Shi (1999), Khoshnevisan and Xiao (2007, 2015), Dalang, Mueller, and Xiao (2017).

7. *Analysis of stochastic partial differential equations* (2 lectures)

   In the first lecture on this module, linear SPDEs with Gaussian noise (including "white noise" and "colored noise") are introduced and studied. In this simple setting one can learn many techniques that are useful for analyzing more complicated SPDEs. We describe various structural properties of the solutions to linear SPDEs that highlight the effect of noise in the behavior of the solution.

   Non-linear equations are introduced, and defined rigorously. General issues of existence and uniqueness are addressed.

   In the second lecture we study in detail more concrete families of SPDE models and various local properties of these solutions. Typical examples of such local properties are regularity theory (smoothness of the solution), the analysis of the local effect of noise, and several of their consequences.

8. *Extreme value theory of random fields.* (2 lectures)

   In the first lecture on this module, we introduce general bounds for the excursion probabilities that can be obtained by applying the general methods for Gaussian random fields.

   In the second lecture, we apply the double sum method and the Euler characteristic method to establish more precise approximations to the excursion probabilities. Some tools from integral geometry will be introduced.

   We will give examples of applications of excursion probabilities in statistics including control of the false discovery rate in multiple testing and construction of confidence bands for linear regression.