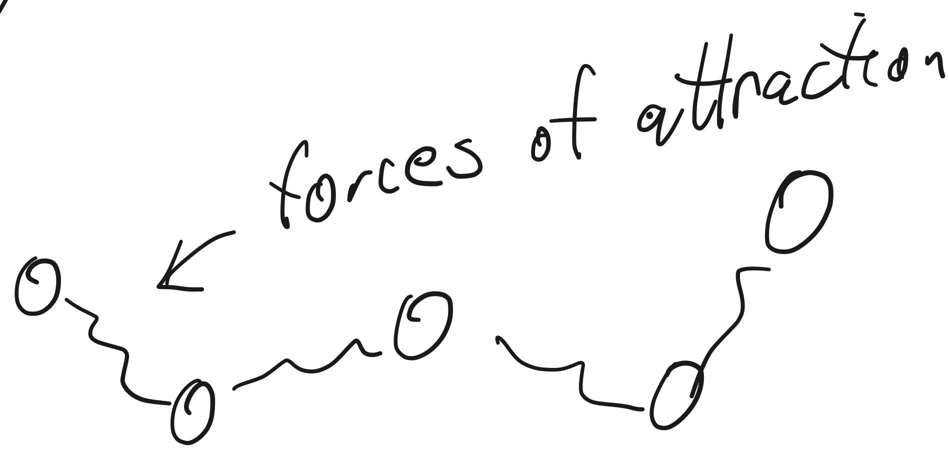
The Radius of a Moving Polymer

with Eyal Neuman

Polymer



SPDE: $\quad u(t,x) \qquad t \in [0,\infty)$

$x \in [0,J]$

$u \in \mathbb{R}^d$

$$\partial_t u = \partial_x^2 u + \dot{W}(t,x)$$

$$\partial_x u(t,0) = \partial_x u(t,J) \qquad \text{Neumann conditions,}$$

$$u(0,x) = u_0 \qquad \dot{W} = (\dot{W}_1, \ldots, \dot{W}_d) \text{ iid}$$

Let $u_0(x) \equiv 0$

Neumann heat kernel

Mild form

$$u(t,x) = \int_0^J G(t,x,y)\, u_0(y)\, dy$$

$$\int_0^t \int_0^J G_{t-s}(x,y)\, W(dy\,ds)$$

$N(t,x)$

Rouse model Δ = discrete Laplacian

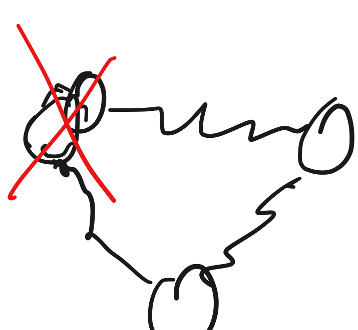
$$dX_i = a\,\Delta X_i\,dt + b\,\underbrace{dB_i}_{\text{iid}}$$

Limit as grid size $\to 0$
gives stochastic heat eq. above

Funaki and polymer scientists

Doi and Edwards, Theory of
Polymer Dynamics

Self-avoidance required

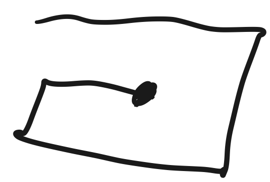


Background, time-independent situation

Self-avoiding walk in $\mathbb{Z}^d$

$(S_k)_{k=0,\dots,n}$

P_n equal prob. on each nearest neighbor self-avoiding path of length n.



trapped

Extension of polymer: $(S_0=0)$

$$E_n\left[|S_n|^2\right] \approx n^{2\nu}$$

rigously known

$$\nu = \begin{cases} 1 & d=1 \quad \checkmark \\ 3/4 & d\doteq 2 \\ .588\dots & d\doteq 3 \\ \frac{1}{2} & d\geq 4 \quad \checkmark \text{ except } d=4 \end{cases}$$

$d=1$

$d>4$ Hara-Slade

$d=4$ partial results, Brydges-Slade

Weakly self-avoiding walk

$\beta > 0$

$$\ell_n(x) = \#\{k \in \{1,\dots,n\} : S_k = x\}$$

$$\frac{dQ_n}{dP_n} = \exp\Big(-\beta \sum_{x\in\mathbb{Z}^d} \ell_n^2(x)\Big)\Big/ Z_n$$

Moving polymer (only in $d=1$)

$u(t,x)$ = position of polymer at time t, at location x along polymer

P_T prob. for SPDE

$$\big(u(t,x)\big)_{(t,x)\in[0,T]\times[0,J]}$$

$$\forall t > 0, \quad x \to u(t,x) = cB_x + \underbrace{\varphi(x)}_{\text{smooth}}$$

So we can define local time

$$L_t(A) = \left| \{ x \in [0,J] : u(t,x) \in A \} \right|$$

$$\ell_t(y) = \frac{L_t(dy)}{dy}$$

$$\mathcal{E}_T = \exp\left(-\beta \int_0^T \int_{\mathbb{R}} \ell_t^2(y)\, dy\, dt \right)$$

$$\frac{dQ_T}{dP_T} = \frac{\mathcal{E}_T}{Z_T} ; \quad Z_T = E^{P_T}[\mathcal{E}_T]$$

Let $\bar{u}(t) = \frac{1}{J} \int_0^J u(t,x)\, dx$

Radius

$$R(T,J) = \left[\frac{1}{TJ} \int_0^T \int_0^J \left(u(t,x) - \bar{u}(t) \right)^2 dx\, dt \right]^{\frac{1}{2}}$$

<u>Methods:</u> (for $d=1$)

→ Bolthausen : Jensen's inequality

Greven-den Hollander : Ray-Knight theorem

Ideas of proof :

$$\frac{dQ_T}{dP_T} = \frac{\mathcal{E}_T}{Z_T}$$

Let $A_T = \{ R(T,J) \gg \beta^{\frac{1}{3}} J^{\frac{2}{3}}$ or $\ll \beta^{\frac{1}{3}} J^{\frac{2}{3}} \}$

Goal: Upper bound on $Q_T(A_T)$
($Q_T(A_T) \approx 0$)

$$Q_T(A_T) = \frac{E^{P_T}[\mathcal{E}_T \mathbb{1}_{A_T}]}{Z_T}$$

Need, ① upper bound on numerator
② lower bound on denominator

(2) Impose a drift on u, which gives u the shape we think it will have with self-avoidance

$\hat{P}_T$ prob. with drift

$$Z_T = E^{P_T}[\mathcal{E}_T]$$

$$= E^{\hat{P}_T}\left[\mathcal{E}_T \frac{dP_T}{d\hat{P}_T}\right]$$

Take log, Jensen

$$\log Z_T \geq E^{\hat{P}_T}\left[\log \mathcal{E}_T - \log \frac{d\hat{P}_T}{dP_T}\right]$$

$$\log \mathcal{E}_T = -\underbrace{\beta \iint \ell_T^2(y)\,dy}$$

under $\hat{P}_T$, this term should be close to its value under Q_T

(1) similar to Flory argument for 2-d self-avoiding ~~RW~~ BM.

$(B_t)_{0 \leq t \leq T}$ $\quad B \in \mathbb{R}^2$

Let $A_{T,R} = \{ \sup_{0 \leq t \leq T} |B_t| \approx R \}$

D_R = disk in $\mathbb{R}^2$ of radius R centered at 0

$$Q_T(A_{T,R}) = \frac{E^{P_T}[\mathbb{1}_{A_{T,R}} \mathcal{E}_T]}{Z_T}$$

Consider numerator

On $A_{T,R}$ $\quad B_t \in D_R$

$$\mathcal{E}_T = \exp\left(-\beta \int_{\mathbb{R}^2} l_t^2(y)\,dy\right)$$

$$\leq \exp\left(-\beta \pi R^2 \left(\frac{T}{\pi R^2}\right)^2\right)$$

maximized when ℓ is equally spread over D_R

$$= \exp\left(-\beta \frac{T^2}{\pi R^2}\right)$$

To get this, must have

$$\sup_{0\leq t\leq T} |B_t| \approx R$$

$$P\left(\quad 〃 \quad \right) \approx \exp\left(-c\frac{R^2}{T}\right)$$

$$Q_T(A_T) = \frac{1}{Z_T}\exp\left(-\beta\frac{T^2}{\pi R^2} - \frac{cR^2}{T}\right)$$

Optimize over R

In such problems, we usually set the two exponents equal.

$$\beta\,\frac{T^2}{R^2} \approx \frac{R^2}{T}$$

$$R = \beta^{\frac{1}{4}} T^{\frac{5}{4}}$$ exponent predicted when $d=2$

For SPDE

$d=1$ Our result, $R \approx \beta^{\frac{1}{3}} J^{\frac{5}{3}}$

$d \geq 1$ we predict

$$R \approx \beta^{\frac{1}{2+d}} J^{\frac{5}{2+d}}$$

$$\left(= \beta^{\frac{1}{4}} J^{\frac{5}{4}} \quad \text{if } d=2 \right)$$