Peakedness Comparison for Hyperbolic Cosine and Sine Ratio Random Fields

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- Hyperbolic types of

1. Laplace transforms of infinitely divisible nonnegative random variables
2. Laplace transforms of Lévy processes on $[0, \infty)$
3. Finite-dimensional characteristic functions of elliptically contoured random fields

- Peakedness comparison

⋆ Laplace transforms of five infinitely divisible nonnegative random variables

Suppose that U is a nonnegative random variable,
$\nu, \alpha, \alpha_1, \alpha_2$ are positive constants, and $\alpha_1 < \alpha_2$.

(i). (Biane, Pitman and Yor (2001))

$$\mathrm{E} \exp(-U\omega) = \left(\frac{\alpha\sqrt{\omega}}{\sinh(\alpha\sqrt{\omega})} \right)^{\nu}, \quad \omega \geq 0.$$

(ii). (Biane, Pitman and Yor (2001))

$$\mathrm{E} \exp(-U\omega) = \left(\cosh\left(\alpha\sqrt{\omega}\right)\right)^{-\nu}, \quad \omega \geq 0.$$

(iii). (Biane, Pitman and Yor (2001))

$$\mathrm{E} \exp(-U\omega) = \left(\frac{\tanh(\alpha\sqrt{\omega})}{\alpha\sqrt{\omega}} \right)^{\nu}, \quad \omega \geq 0.$$

(iv). (New, except for $\nu = 1$ in Biane, Pitman and Yor (2001))

$$\mathrm{E} \exp(-U\omega) = \left(\frac{\cosh(\alpha_1\sqrt{\omega})}{\cosh(\alpha_2\sqrt{\omega})} \right)^{\nu}, \quad \omega \geq 0.$$

(v). (New, except for $\nu = 1$ in Biane, Pitman and Yor (2001))

$$\mathrm{E} \exp(-U\omega) = \left(\frac{\alpha_2 \sinh(\alpha_1\sqrt{\omega})}{\alpha_1 \sinh(\alpha_2\sqrt{\omega})} \right)^{\nu}, \quad \omega \geq 0.$$

⋆⋆ Five Lévy processes $\{Z(x), x \in [0,\infty)\}$

Suppose that $\nu, \alpha, \alpha_1, \alpha_2$ are positive constants, and $\alpha_1 < \alpha_2$.

(i). (Biane, Pitman and Yor (2001))

$$\mathrm{E}\exp\left(-Z(x)\,\omega\right) = \left(\frac{\alpha\sqrt{\omega}}{\sinh\left(\alpha\sqrt{\omega}\right)}\right)^{\nu x}, \quad \omega \geq 0,\ x \geq 0.$$

(ii). (Biane, Pitman and Yor (2001))

$$\mathrm{E}\exp\left(-Z(x)\,\omega\right) = \left(\cosh\left(\alpha\sqrt{\omega}\right)\right)^{-\nu x}, \quad \omega \geq 0,\ x \geq 0.$$

(iii). (Biane, Pitman and Yor (2001))

$$\mathrm{E}\exp\left(-Z(x)\,\omega\right) = \left(\frac{\tanh\left(\alpha\sqrt{\omega}\right)}{\alpha\sqrt{\omega}}\right)^{\nu x}, \quad \omega \geq 0,\ x \geq 0.$$

(iv). (New)

$$\mathrm{E}\exp\left(-Z(x)\,\omega\right) = \left(\frac{\cosh\left(\alpha_1\sqrt{\omega}\right)}{\cosh\left(\alpha_2\sqrt{\omega}\right)}\right)^{\nu x}, \quad \omega \geq 0,\ x \geq 0.$$

(v). (New)

$$\mathrm{E}\exp\left(-Z(x)\,\omega\right) = \left(\frac{\alpha_2\sinh(\alpha_1\sqrt{\omega})}{\alpha_1\sinh\left(\alpha_2\sqrt{\omega}\right)}\right)^{\nu x}, \quad \omega \geq 0,\ x \geq 0.$$

⋆⋆⋆ Five elliptically contoured random fields $\{Z(x), x \in \mathbb{D}\}$

$$Z(x) = \sqrt{2U}\,Y(x) + \mu(x), \quad x \in \mathbb{D},$$

where U is a nonnegative random variable with one of the five Laplace transforms,
$\{Y(x), x \in \mathbb{D}\}$ is a Gaussian random field
with mean 0 and covariance function $C(x_1, x_2)$,
U and $\{Y(x), x \in \mathbb{D}\}$ are independent,
and $\mu(x)$ is a (non-random) function on $\mathbb{D}$.

Finite-dimensional characteristic functions

$$\mathrm{E}\left(\imath \sum_{k=1}^{n} \omega_k Z(x_k)\right) = \exp\left(\imath \sum_{k=1}^{n} \omega_k \mu(x_k)\right) \ell_U\left(\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i C(x_i, x_j)\omega_j\right),$$

$$\omega_1, \ldots, \omega_n \in \mathbb{R},$$

where $\ell_U(\omega)$ is the Laplace transform of U.

(i). A generalized logistic random field

$$\mathrm{E}\left(\imath \sum_{k=1}^{n} \omega_k Z(x_k)\right) = \exp\left(\imath \sum_{k=1}^{n} \omega_k \mu(x_k)\right) \left\{ \frac{\alpha \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i C(x_i, x_j) \omega_j \right)^{\frac{1}{2}}}{\sinh\left(\alpha \left(\sum_{i=1}^{n} \sum_{j=1}^{n} \omega_i' C(x_i, x_j) \omega_j \right)^{\frac{1}{2}} \right)} \right\}^{\nu},$$

$$\omega_1, \ldots, \omega_n \in \mathbb{R}$$

Special case $\nu = 1$

A logistic random field
Balakrishnan, Ma, and Wang (2015).

(ii). A hyperbolic secant random field

$$\mathrm{E}\left(\imath\sum_{k=1}^{n}\omega_k Z(x_k)\right) = \frac{\exp\left(\imath\sum_{k=1}^{n}\omega_k\mu(x_k)\right)}{\left\{\cosh\left(\alpha\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i C(x_i,x_j)\omega_j\right)^{\frac{1}{2}}\right)\right\}^{\nu}},$$

$$\omega_1,\ldots,\omega_n \in \mathbb{R}$$

(iii). A hyperbolic tangent random field

$$\mathrm{E}\left(\imath\sum_{k=1}^{n}\omega_k Z(x_k)\right) = \exp\left(\imath\sum_{k=1}^{n}\omega_k\mu(x_k)\right)\left\{\frac{\tanh\left(\alpha\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i C(x_i,x_j)\omega_j\right)^{\frac{1}{2}}\right)}{\alpha\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\omega_i C(x_i,x_j)\omega_j\right)^{\frac{1}{2}}}\right\}^{\nu},$$

$$\omega_1,\ldots,\omega_n \in \mathbb{R}$$

(iv). A hyperbolic cosine ratio random field

$$\mathrm{E}\left(\imath \sum_{k=1}^{n} \omega_k Z(x_k)\right) = \exp\left(\imath \sum_{k=1}^{n} \omega_k \mu(x_k)\right) \left\{ \frac{\cosh\left(\alpha_1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i C(x_i,x_j)\omega_j\right)^{\frac{1}{2}}\right)}{\cosh\left(\alpha_2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i C(x_i,x_j)\omega_j\right)^{\frac{1}{2}}\right)} \right\}^{\nu},$$

$$\omega_1,\ldots,\omega_n \in \mathbb{R}$$

(v). A hyperbolic sine ratio random field

$$\mathrm{E}\left(\imath \sum_{k=1}^{n} \omega_k Z(x_k)\right) = \exp\left(\imath \sum_{k=1}^{n} \omega_k \mu(x_k)\right) \left\{ \frac{\alpha_2 \sinh\left(\alpha_1 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i C(x_i,x_j)\omega_j\right)^{\frac{1}{2}}\right)}{\alpha_1 \sinh\left(\alpha_2 \left(\sum_{i=1}^{n}\sum_{j=1}^{n} \omega_i C(x_i,x_j)\omega_j\right)^{\frac{1}{2}}\right)} \right\}^{\nu},$$

$$\omega_1,\ldots,\omega_n \in \mathbb{R}$$

Peakedness comparison

For two real-valued random fields $\{Z_k(x), x \in \mathbb{D}\}$ whose finite-dimensional distributions are symmetric about $\mu_k(x)$ $(k = 1, 2)$, we say that $\{Z_1(x), x \in \mathbb{D}\}$ is more peaked about $\mu_1(x)$ than $\{Z_2(x), x \in \mathbb{D}\}$ about $\mu_2(x)$, and denote it by $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$, if

$$\begin{gathered} P((Z_1(x_1) - \mu_1(x_1), \ldots, Z_1(x_n) - \mu_1(x_n))' \in A_n) \\ \geq P((Z_2(x_1) - \mu_2(x_1), \ldots, Z_2(x_n) - \mu_2(x_n))' \in A_n) \end{gathered}$$

holds for every $n \in \mathbb{N}$, any $x_k \in \mathbb{D}$ $(k = 1, \ldots, n)$, and any $A_n \in \mathscr{A}_n$, where $\mathscr{A}_n$ denotes the class of compact, convex, and symmetric (about the origin) sets in $\mathbb{R}^n$, and $\mathbb{N}$ is the set of positive integers. (Wang and Ma (2018))

Tail comparison, heavy-tail (Birnbaum (1948), Olkin and Tong (1988))

A Student's t distribution is more heavy-tailed than a standard normal distribution, but a Student's t random field and a Gaussian random field are not comparable in terms of the peakedness.

Compare two elliptically contoured random fields of hyperbolic type

$$Z_k(x) = \sqrt{2U_k}\,Y(x) + \mu_k(x), \qquad x \in \mathbb{D},$$

where U_k is a positive random variable with Laplace transform of hyperbolic type, $\{Y(x), x \in \mathbb{D}\}$ is a Gaussian random field with mean 0 and covariance function $C(x_1, x_2)$, $\{Y(x), x \in \mathbb{D}\}$ and U_k are independent each other, and $\mu_k(x)$ is a (non-random) function on $\mathbb{D}$, $k = 1, 2$.

- Let U_k be a positive random variable with Laplace transform

$$\mathrm{E}\exp(-U_k\omega) = \left(\cosh\left(\alpha_k\sqrt{\omega}\right)\right)^{-\nu_k}, \quad \omega \geq 0,$$

respectively, where α_k and ν_k are positive constants, $k = 1, 2$.

(i) If $\alpha_1 \leq \alpha_2$ and $\nu_1 \leq \nu_2$, then $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$.

(ii) Let $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$. If $\alpha_1 = \alpha_2$, then $\nu_1 \leq \nu_2$; and if $\nu_1 = \nu_2$, then $\alpha_1 \leq \alpha_2$.

- Let U_k be a positive random variable with Laplace transform

$$\mathrm{E}\exp(-U_k\omega) = \left(\frac{\alpha_k\sqrt{\omega}}{\sinh\left(\alpha_k\sqrt{\omega}\right)}\right)^{\nu_k}, \quad \omega \geq 0,$$

respectively, where α_k and ν_k are positive constants, $k = 1, 2$.

(i) If $\alpha_1 \leq \alpha_2$ and $\nu_1 \leq \nu_2$, then
$\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$.

(ii) Let $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$. If $\alpha_1 = \alpha_2$, then $\nu_1 \leq \nu_2$; and if $\nu_1 = \nu_2$, then $\alpha_1 \leq \alpha_2$.

• Suppose that U_1, U_2, and U_3 are positive random variables with Laplace transform

$$\mathrm{E}\exp(-U_1\omega) = \left(\cosh\left(\alpha_1\sqrt{\omega}\right)\right)^{-\nu_1}, \quad \mathrm{E}\exp(-U_2\omega) = \left(\frac{\alpha_2\sqrt{\omega}}{\sinh\left(\alpha_2\sqrt{\omega}\right)}\right)^{\nu_2},$$

and

$$\mathrm{E}\exp(-U_3\omega) = \left(\frac{\tanh(\alpha_3\sqrt{\omega})}{\alpha_3\sqrt{\omega}}\right)^{\nu_3}, \quad \omega \geq 0,$$

respectively, where α_k and ν_k are positive constants ($k = 1, 2, 3$).

(i) If $\alpha_1 \leq \frac{\alpha_2}{2}$ and $\nu_1 \leq \nu_2$, then

$$\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}.$$

(ii) Let $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$. If $\sqrt{3}\alpha_1 = \alpha_2$, then $\nu_1 \leq \nu_2$; and if $\nu_1 = \nu_2$, then $\alpha_1 \leq \frac{\alpha_2}{\sqrt{3}}$.

(iii) If $\alpha_1 = \alpha_3$ and $\nu_1 = \nu_3$, then

$$\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_3(x) - \mu_3(x), x \in \mathbb{D}\}.$$

• Let α_k and ν_k $(k = 1, 2)$ are positive constants, and $\alpha_1 < \alpha_2$.

If U_1 and U_2 are positive random variables with Laplace transform

$$\mathrm{E}\exp(-U_1\omega) = \left(\frac{\cosh(\alpha_1\sqrt{\omega})}{\cosh(\alpha_2\sqrt{\omega})}\right)^{\nu_1}, \ \mathrm{E}\exp(-U_2\omega) = \left(\frac{\cosh(\alpha_1\sqrt{\omega})}{\cosh(\alpha_2\sqrt{\omega})}\right)^{\nu_2}, \ \omega \geq 0,$$

respectively, then $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$ if and only if $\nu_1 \leq \nu_2$.

If U_1 and U_2 are positive random variables with Laplace transform

$$\mathrm{E}\exp(-U_1\omega) = \left(\frac{\alpha_2\sinh(\alpha_1\sqrt{\omega})}{\alpha_1\sinh(\alpha_2\sqrt{\omega})}\right)^{\nu_1}, \quad \mathrm{E}\exp(-U_2\omega) = \left(\frac{\alpha_2\sinh(\alpha_1\sqrt{\omega})}{\alpha_1\sinh(\alpha_2\sqrt{\omega})}\right)^{\nu_2},$$

respectively, then $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$ if and only if $\nu_1 \leq \nu_2$.

Assume that U_1 and U_2 are positive random variables with Laplace transform

$$\mathrm{E}\exp(-U_1\omega) = \left(\frac{\cosh(\alpha_1\sqrt{\omega})}{\cosh(\alpha_2\sqrt{\omega})}\right)^{\nu_1}, \quad \mathrm{E}\exp(-U_2\omega) = \left(\frac{\alpha_2\sinh(\alpha_1\sqrt{\omega})}{\alpha_1\sinh(\alpha_2\sqrt{\omega})}\right)^{\nu_2},$$

respectively. If $\nu_1 \leq \nu_2$, then $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$. Conversely, if $\{Z_1(x) - \mu_1(x), x \in \mathbb{D}\} \overset{p}{\succeq} \{Z_2(x) - \mu_2(x), x \in \mathbb{D}\}$, then $\nu_1 \leq \frac{\nu_2}{3}$.