The entry-exit theorem and relaxation oscillations in slow-fast planar systems

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Abstract

The entry-exit theorem for the phenomenon of delay of stability loss for certain types of slow-fast planar systems plays a key role in establishing existence of limit cycles that exhibit relaxation oscillations. The general existing proofs of this theorem depend on Fenichel’s geometric singular perturbation theory and blow-up techniques. In this work, we give a short and elementary proof of the entry-exit theorem based on a direct study of asymptotic formulas of the underlying solutions. We employ this theorem to a broad class of slow-fast planar systems to obtain existence, global uniqueness and asymptotic orbital stability of relaxation oscillations. The results are then applied to a diffusive predator-prey model with Holling type II functional response to establish periodic traveling wave solutions. Furthermore, we extend our work to another class of slow-fast systems that can have multiple orbits exhibiting relaxation oscillations, and subsequently apply the results to a two time-scale Holling-Tanner predator-prey model with Holling type IV functional response. It is generally assumed in the literature that the non-trivial equilibrium points exist uniquely in the interior of the domains bounded by the relaxation oscillations; we do not make this assumption in this paper.

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1 Introduction

Relaxation oscillations are typical phenomena that appear in slow-fast dynamical systems of the form
\[
\frac{du}{dt} = f(u,v), \quad \frac{dv}{dt} = \varepsilon g(u,v)
\] (1)
for small \(\varepsilon > 0\). These systems have state variables that evolve on different time-scales and are characterized by dynamics that have slow and fast episodes. The parameter \(\varepsilon\) determines the ratio between the two time-scales and the variables \(u\) and \(v\) are referred to as the fast and slow variables respectively. To understand the dynamics of slow-fast systems, modern theory of geometric singular perturbation [6] exploits the separation of time-scales by studying “slow and fast flows” that exist in the respective limits as \(\varepsilon \to 0\) of the slow and fast time-scale formulations of these systems (see [13] for preliminaries on slow-fast systems). The slow flow occurs along the set \(\{(u,v) : f(u,v) = 0\}\), which is referred to as the “critical manifold” of such systems.

By “relaxation oscillation”, we mean a periodic solution of (1) whose orbit in the phase plane approaches a certain piecewise smooth curve as \(\varepsilon \to 0\). Such a piecewise smooth curve is referred to as the “singular orbit” [13, 14]. The main focus of this paper is to rigorously study existence, multiplicity and orbital stability of relaxation oscillations of a certain class of (1) (described later) as \(\varepsilon \to 0\).

A classical example where relaxation oscillations are seen is the van der Pol’s equation, written in the system form as
\[
\frac{du}{dt} = v - \frac{1}{3}u^3 - u, \quad \frac{dv}{dt} = -\varepsilon u.
\] (2)
The critical manifold of (2) is the \(S\)-shaped curve defined by \(S := \{(u,v) : v = \frac{1}{3}u^3 - u\}\), which consists of two stable branches \(S^-\) and \(S^+\), an unstable branch \(S_r\) and fold points \(F^\pm = (\pm 1, \mp \frac{2}{3})\), where \(S^- = S \cap \{u < -1\}\), \(S^+ = S \cap \{u > 1\}\), \(S_r = S \cap \{-1 < u < 1\}\). The singular orbit \(J_0\) of system (2) consists of segments of the slow flow that occur along the two branches \(S^\pm\) of the critical manifold, with “jumps” at fold points that occur along the horizontal lines through \(F^\pm\) connecting the two branches. A rigorous proof of existence of relaxation oscillations of (2) is obtained by constructing a narrow closed annular region \(U\) containing the singular orbit \(J_0\) such that \(dist(U, J_0)\) is any preassigned constant and yet for a sufficiently small \(\varepsilon > 0\), the region \(U\) is positively invariant, where the Poincaré-Bendixson theorem can be applied (e.g. [8, 9, 25]). This is a direct and an effective approach to establish existence of relaxation oscillations even for the general system (1). (We note that the existence of the limit cycle of (2) for all \(\varepsilon > 0\) has been established by different techniques [4].)
In the recent literature \([10, 14, 17, 19, 20, 23, 24]\), there are several models of slow-fast systems arising from applications in various fields of study such as ecology, bio-economics and other related subjects that can be written in the system form as

\[
\frac{du}{dt} = u^k f(u, v), \quad \frac{dv}{dt} = \varepsilon g(u, v) \quad (k \geq 1).
\]  

(3)

A new feature of these systems is that the critical manifold now consists of two pieces, \(\{u = 0\}\) and \(\{(u, v) : f(u, v) = 0\}\). The curve \(f(u, v) = 0\) is typically divided into two branches, namely a stable branch and an unstable branch connected by a vertex point (like a facing-down parabola). A special characteristic of system (3) is the existence of turning points. A turning point lies in the intersection of the two pieces of the critical manifold, namely \(\{u = 0\} \cap \{(u, v) : f(u, v) = 0\}\), and splits the \(v\)-axis into stable and unstable segments. More precisely, these points correspond to the set of transcritical bifurcation points of the fast flow. The singular orbit \(\Gamma_0\) of (3), besides containing a part of the stable branch of \(f(u, v) = 0\), also contains a part of the \(v\)-axis (which includes parts of stable as well as unstable segments of \(\{u = 0\}\)) as a piece of the slow flow, and two horizontal line segments connecting the two slow pieces as “fast fibers” \([13]\); see figure 2 in Section 3. An interesting feature observed in the dynamics of such a singular orbit is a delay in concatenation of the slow flow with a fast fiber as the orbit flows along the unstable segment of the \(v\)-axis. This phenomenon is referred to as the delay of stability loss, also known as “Pontryagin delay” \([20, 24]\). Such a phenomenon has also occurred in the study of canard explosion (see \([16, 18, 21]\) and the references therein).

One of the main goals of this paper is to establish existence of relaxation oscillations of (3) for small \(\varepsilon > 0\) also by constructing a positively invariant closed annular region \(U\) in the vicinity of \(\Gamma_0\) such that \(dist(U, \Gamma_0)\) is any preassigned constant. This construction is not as apparent to geometric intuition as it is for system (2) due to the phenomenon of Pontryagin’s delay of loss of stability. To construct \(U\), it turns out that we need two solution orbits of (3) near the \(v\)-axis to form parts of the boundary of \(U\), and these solutions are guaranteed to exist by the entry-exit theorem. As a result, the annular region \(U\) in this case depends on \(\varepsilon\) and does not contain the segment of \(\Gamma_0\) that lies on the \(v\)-axis. We refer the reader to \([11, 22, 24, 27]\) and the references therein for this, the entry-exit theorem and other related topics.

The rigorous study of the entry-exit theorem has appeared in several papers. Among others, the case \(k = 1\) was studied in an earlier paper \([24]\) and in a recent paper \([11]\), the cases \(k = 2\) and \(2 < k \in \mathbb{N}\) were studied in the recent papers \([22]\) and \([27]\) respectively. The proof of \([24]\) depends on a direct study of the solutions of (3) near the \(v\)-axis for sufficiently small \(\varepsilon > 0\), while the proof in \([11]\) is based on the Exchange Lemma \([13]\). The proofs in \([22, 27]\) are based on Fenichel’s geometric singular perturbation theory and blow-up techniques. Comparing the proofs in these
references, one can see that the proof in [24] is much shorter and elementary, though it dealt with the case $k = 1$ and established convergence of the entry-exit function in the $C^0$ sense (which suffices for the purpose of applications). In this note we intend to give a short and elementary proof of the entry-exit theorem (the $C^0$ version) for all $k \geq 1$. Different from the proof of [24], where a comparison argument was used, our proof is based on a direct study of asymptotic formulas of solutions of (3) near the $v$-axis.

We present our proof of the entry-exit theorem in Section 2. Assuming that the nullcline $f(u,v) = 0$ has a unique fold point (which corresponds to the maximum of $f(u,v) = 0$) with an attracting branch on one side of the fold and a repelling branch on the other, we study existence, multiplicity and stability of relaxation oscillations of system (3) in Section 3. The existence of such orbits has been considered for several models in the literature (e.g. [1, 7, 19, 20, 27]) by using the entry-exit theorem and Fenichel’s theory. We employ the aforementioned classical approach, that is, we use the entry-exit theorem and the vector field of (3) to construct a positively invariant closed annular region in the vicinity of the singular orbit, and then apply the Poincaré-Bendixson theorem. We next rigorously derive an asymptotic formula for the nontrivial Floquet multiplier of any relaxation oscillator lying in a neighborhood of the singular orbit, which then yields uniqueness and local asymptotic orbital stability of such limit cycles. We further show non-existence of closed orbits that do not exhibit relaxation oscillations. Combining these results, we obtain a theorem on existence, global uniqueness and asymptotic orbital stability of the relaxation oscillations of system (3). We remark that most of the techniques employed in our proof are different from those in the existing references (on this subject) we are aware of. With the aid of this theorem, one can obtain global asymptotics of dynamics of (3).

We further remark that the assumptions in this theorem are quite general which cover many concrete cases in the literature. Moreover, generally, it is assumed in the literature that the nullclines $f(u,v) = 0$ and $g(u,v) = 0$ intersect at a unique point on the unstable branch of $f(u,v) = 0$. In this article, we do not make any assumptions on the monotonicity of the $v$-nullcline and allow for multiple intersections of the nullclines $f(u,v) = 0$ and $g(u,v) = 0$. Furthermore, we also allow occurrence of non-generic situations that may include a tangential intersection of the two nullclines or a vertical tangency of the $v$-nullcline at the point of intersection. In Section 4, we demonstrate such situations by analyzing a diffusive predator-prey model with Holling type II functional response, where the corresponding homogenous model admits up to three coexistence equilibria and exhibits saddle-node bifurcations of equilibria as a system parameter is varied. With the aid of asymptotic orbital stability of relaxation oscillations, we show that the model has periodic traveling wave solutions with relaxation oscillations profile in a parameter range that allows for multiple intersections of the nullclines.
We extend our results from Section 3 to study multiplicity and stability of relaxation oscillations of system (3), when the nullcline $f(u, v) = 0$ has a maximum as well as a minimum (i.e. the graph of $f(u, v) = 0$ has two folds). Under this assumption, we prove existence of two relaxation oscillators in Section 5. We further apply our results to establish coexistence of stable and unstable relaxation oscillations in a two-time scale Holling-Tanner predator-prey model with Holling type IV functional response.

The paper is organized as follows. We prove the entry-exit theorem by studying asymptotic formulas of solutions of system (3) in Section 2. In Section 3, under the assumption that $f(u, v) = 0$ has a maximum, we employ the entry-exit theorem to construct a positively invariant region and prove existence of an orbit that exhibits relaxation oscillations lying in that region. We further prove global uniqueness and asymptotic orbital stability of the cycle. We apply this result to establish existence of periodic traveling wave solutions for a diffusive predator-prey model with Holling type II functional response in Section 4. In Section 5, we extend our results from Section 3 to establish a theorem on coexistence of two relaxation oscillators when $f(u, v) = 0$ has a maximum and a minimum. Finally we apply this theorem to obtain relaxation oscillations in the Holling-Tanner predator-prey model with Holling type IV functional response.

2 The entry-exit theorem

In this section we present a new proof of the entry-exit theorem that has been studied in [11, 22, 24, 27]. We shall use a similar setting and notations as in these references, so we consider the planar system

$$\frac{dx}{dt} = \varepsilon f(x, y, \varepsilon), \quad \frac{dy}{dt} = y^{k+1}g(x, y, \varepsilon) \quad (0 \leq k \in \mathbb{R}),$$

(4)

and assume the following:

(H1) $f$ and $g$ are in $C^1((a_1, a_2) \times [0, b))$ for some $a_1 < 0 < a_2$ and $b > 0$, with

$$f(x, 0, 0) > 0, \quad xg(x, 0, 0) > 0, \quad \forall x \in (a_1, a_2).$$

(H2) There exist constants $y_0 > 0$ and $\varepsilon_0 > 0$, both small, and a unique $C^1$ function $x = \tilde{x}(y, \varepsilon)$ defined on $[0, y_0] \times [0, \varepsilon_0]$ such that $\tilde{x}(0, 0) = 0$ and

$$g(\tilde{x}(y, \varepsilon), y, \varepsilon) = 0 \quad \forall (y, \varepsilon) \in [0, y_0] \times [0, \varepsilon_0].$$

That is, other than $x$-axis, $x = \tilde{x}(y, \varepsilon)$ is the unique slow curve near the $x$-axis.
(H3) Let \( x_0^* < 0 \) and \( x_1^* > 0 \) be such that

\[
\int_{x_0^*}^{x_1^*} \frac{g(x,0,0)}{f(x,0,0)} \, dx = 0.
\]

Note that (H2) is guaranteed to hold by assuming \( g_x(0,0,0) \neq 0 \) and applying the implicit function theorem. We may assume from (H1) that \( f(x,y,\varepsilon) > 0 \) for \((x,y,\varepsilon) \in (a_1,a_2) \times [0,y_0] \times [0,\varepsilon_0]\) and define

\[ h(x,y,\varepsilon) := \frac{g(x,y,\varepsilon)}{f(x,y,\varepsilon)}. \]

It follows that \( h(x,y,\varepsilon) < 0 \) for \( x < \hat{x}(y,\varepsilon) \) and \( h(x,y,\varepsilon) > 0 \) for \( x > \hat{x}(y,\varepsilon) \). Note from (H3) and the implicit function theorem that there exist \( \delta > 0 \) and a \( C^1 \) function \( p_0 : [x_0^* - \delta, x_0^* + \delta] \to (0,\infty) \) such that \( p_0(x_0^*) = x_1^* \) and

\[
\int_{x_0^*}^{p_0(x_0^*)} h(x,0,0) \, dx = 0, \quad p_0'(x_0^*) = -\frac{h(x_0^*,0)}{h(p_0(x_0^*),0,0)} < 0,
\]

so that \( p_0 \) is decreasing on \([x_0^* - \delta, x_0^* + \delta]\). In the literature, this function \( p_0 \) and the function \( p_\varepsilon \) given in Theorem 1 below are referred to as the entry-exit functions. We shall henceforth refer to Theorem 1 as the entry-exit theorem.

**Theorem 1.** Assume (H1)-(H3). Let \( \varepsilon > 0 \) be sufficiently small. For any \( x_0 \in [x_0^* - \delta, x_0^* + \delta] \), let \( y(x) := y(x,x_0,y_0,\varepsilon) \) be the solution of the IVP

\[
\frac{dy}{dx} = \frac{1}{\varepsilon} y^{k+1} h(x,y,\varepsilon), \quad y(x_0) = y_0.
\]

(i) Then there is a continuous function \( x_{1,\varepsilon} := p_\varepsilon(x_0) \) for \( x_0 \in [x_0^* - \delta, x_0^* + \delta] \) such that

\[
|p_\varepsilon(x_0) - p_0(x_0)| \leq M_0 \varepsilon^{1/(k+1)}
\]

for some constant \( M_0 > 0 \) independent of \( \varepsilon \) and \( x_0 \).

(ii) \( y(x) \) is defined on \([x_0,x_{1,\varepsilon}]\) with \( 0 < y(x) < y_0 \) for \( x \in (x_0,x_{1,\varepsilon}) \) and \( y(x_{1,\varepsilon}) = y_0 \), and there exist constants \( m > 0 \) and \( K > 0 \) independent of \( \varepsilon \) and \( x_0 \) such that

\[
y(x) < \left( \frac{1}{mk} \right)^{1/k} \varepsilon^{1/(k+1)} \quad \text{for} \quad x_0 + \varepsilon^{1/(k+1)} \leq x \leq p_0(x_0) - K \varepsilon^{1/(k+1)}.
\]

**Proof.** We only prove the theorem for \( k > 1 \) since the case \( k = 1 \) can be obtained by applying a smooth transformation to the case \( k = 2 \) (see [22] for the transformation).
Figure 1: The black dashed curve and the x-axis represent the equilibria of system (4) for \( \varepsilon = 0 \) with the arrows indicating the corresponding vector field. The dashed red curve represents the solution of (4) for \( \varepsilon > 0 \) that starts at \((x_0, y_0)\) and exits at \((p_\varepsilon(x_0), y_0)\). A similar figure is considered in [22].

To proceed with the proof, we let \( x_{1-} := p_0(x_0^* - \delta) + \delta \), and define

\[
\begin{align*}
    m &:= \min\{|h(x, y, \varepsilon)| : (x, y, \varepsilon) \in ([x_0^* - \delta, -\delta] \cup [\delta, x_1^-]) \times [0, y_0] \times [0, \varepsilon_0]\} > 0, \\
    M &:= \max\{|h(x, y, \varepsilon)| : (x, y, \varepsilon) \in [x_0^* - \delta, x_1^-] \times [0, y_0] \times [0, \varepsilon_0]\}.
\end{align*}
\]

Then for \((x, y, \varepsilon) \in [x_0^* - \delta, x_1^-] \times [0, y_0] \times [0, \varepsilon_0]\),

\[
\begin{align*}
    f(x, y, \varepsilon) &= f(x, 0, 0)[1 + O(y + \varepsilon)], \\
    g(x, y, \varepsilon) &= g(x, 0, 0) + O(y + \varepsilon),
\end{align*}
\]

and so

\[
    h(x, y, \varepsilon) = \frac{g(x, 0, 0) + O(y + \varepsilon)}{f(x, 0, 0)[1 + O(y + \varepsilon)]} = h(x, 0, 0) + O(y + \varepsilon),
\]

where \(|O(y + \varepsilon)| \leq M_1(y + \varepsilon)\) for some constant \(M_1 > 0\).

Now fix \(x_0 \in [x_0^* - \delta, x_0^* + \delta]\) and let \(y(x) = y(x, x_0, y_0, \varepsilon)\). It follows from the sign of \(h\) that \(y(x)\) is defined over \([x_0, \tilde{x}]\), \(dy/dx < 0\) and \(y(x) < y_0\) for \(x \in [x_0, \tilde{x}]\), where at \(\tilde{x}\) the graph of \(y(x)\) intersects with the slow curve \(x = \tilde{x}(y, \varepsilon)\) and \(h(\tilde{x}) = 0\) (see figure 1).

**Step 1.** Let \(x_01 := x_0 + \varepsilon^\alpha\), where \(\alpha = \frac{1}{k+1}\). We write the equation for \(y(x)\) as

\[
    \frac{1}{y^{k+1}} \frac{dy}{dx} = \frac{1}{\varepsilon} h(x, y, \varepsilon),
\]
and integrate (7) over \([x_0, x_{01}]\) to get
\[
\frac{1}{y^k(x_0)} - \frac{1}{y^k_0} = -\frac{k}{\varepsilon} \int_{x_0}^{x_{01}} h(x, y(x), \varepsilon) \, dx \geq \frac{km}{\varepsilon}(x_{01} - x_0) = \frac{mk}{\varepsilon^{1-\alpha}},
\]
and so
\[
y^k(x_{01}) < \frac{1}{mk\varepsilon^{1-\alpha}}.
\]

**Step 2.** Let \(x_1 := p_0(x_0)\). We show that there is a constant \(K > 1\) (independent of \(\varepsilon\)) such that \(y(x)\) is defined on \([x_0, x_{11}]\) with \(x_{11} := x_1 - K\varepsilon^\alpha\) and satisfies the estimates
\[
\frac{1}{3MK\varepsilon^{1-\alpha}} < y^k(x_{11}) < \frac{1}{mk\varepsilon^{1-\alpha}}.
\]

Since \(y(x)\) is decreasing on \([x_0, \bar{x}]\), it follows that \(y^k(x) < y^k(x_{01}) < \frac{1}{mk\varepsilon^{1-\alpha}}\) for \(x \in (x_{01}, \bar{x})\). It is clear that \(y(x)\) is also defined for \(x \in (\bar{x}, x_1]\) as long as \(y < y_0\). We now let
\[
\bar{x} = \sup \{x \in (\bar{x}, x_1]: y^k(s) < \frac{1}{mk\varepsilon^{1-\alpha}} \quad \forall s \in (\bar{x}, x)\}.
\]
Then for \(x \in (\bar{x}, \bar{x}]\), \(y(x) \leq (\frac{1}{mk}\varepsilon^{1-\alpha}) (\text{note that } 1 - \alpha = k\alpha)\) and so from (6) and \(0 < \alpha < 1\),
\[
h(x, y(x), \varepsilon) = h(x, 0, 0) + O(\varepsilon^\alpha).
\]
For any given \(x \in (\bar{x}, \bar{x}]\), we integrate (7) over \([x_0, x]\) to get
\[
\frac{1}{y^k(x)} - \frac{1}{y^k_0} = -\frac{k}{\varepsilon} \int_{x_0}^{x} h(s, y(s), \varepsilon) \, ds - \frac{k}{\varepsilon} \int_{x_0}^{x_{01}} h(s, y(s), \varepsilon) \, ds - \frac{k}{\varepsilon} \int_{x_0}^{x} h(s, y(s), \varepsilon) \, ds
\]
\[
= O\left(\frac{x_{01} - x_0}{\varepsilon}\right) - \frac{k}{\varepsilon} \int_{x_0}^{x} h(s, 0, 0) \, ds + O\left(\frac{1}{\varepsilon^{1-\alpha}}\right)(x - x_{01})
\]
\[
= -\frac{k}{\varepsilon} \int_{x_0}^{x} h(s, 0, 0) \, ds + O\left(\frac{1}{\varepsilon^{1-\alpha}}\right) = \frac{k}{\varepsilon} \int_{x}^{x_{11}} h(s, 0, 0) \, ds + O\left(\frac{1}{\varepsilon^{1-\alpha}}\right)
\]
where we used \(x_1 = p_0(x_0)\) and \(\int_{x_0}^{p_0(x_0)} h(x, 0, 0) \, dx = 0\), hence
\[
\frac{1}{y^k(x)} = \frac{1}{y^k_0} + \frac{k}{\varepsilon} \int_{x}^{x_{11}} h(s, 0, 0) \, ds + O\left(\frac{1}{\varepsilon^{1-\alpha}}\right) = \frac{\varepsilon + y^k_0 \left( \frac{k}{\varepsilon} \int_{x}^{x_{11}} h(s, 0, 0) \, ds + O(\varepsilon^\alpha) \right)}{y^k_0 \varepsilon}
\]
\[
= \frac{k}{\varepsilon} \int_{x}^{x_{11}} h(s, 0, 0) \, ds + O(\varepsilon^\alpha),
\]
and hence
\[
y^k(x) = \frac{\varepsilon}{k \int_{x}^{x_{11}} h(s, 0, 0) \, ds + O(\varepsilon^\alpha)},
\]
We note that by checking the above proof we see that the constants involved in all the above big “$O$” terms can be taken independent of $\varepsilon$ and $x \in (\bar{x}, \tilde{x}]$. Hence there is a constant $N > 0$ independent of $\varepsilon$ and $x \in (\tilde{x}, \bar{x}]$ such that the big “$O$” term in (9) satisfies

$$|O(\varepsilon^\alpha)| \leq N\varepsilon^\alpha.$$ 

Now we may take a constant $N > 1 + mk$ in the above inequality and let $K := 2N/(km)$ and let $x_{11} := x_1 - K\varepsilon^\alpha$. We claim that $\bar{x} \geq x_{11}$. Supposing that this is false, we would have $\bar{x} < x_{11}$, $y_k(\bar{x}) = \frac{1}{mk}\varepsilon^{1-\alpha}$ (by the definition of $\bar{x}$), and

$$k \int_{\tilde{x}}^{x_1} h(s, 0, 0) \, ds > k \int_{x_{11}}^{x_1} h(s, 0, 0) \, ds \geq km(x_1 - x_{11}) \geq 2N\varepsilon^\alpha,$$

and so from (9)

$$y_k(\bar{x}) \leq \frac{\varepsilon}{k \int_{\tilde{x}}^{x_1} h(s, 0, 0) \, ds - N\varepsilon^\alpha} < \frac{\varepsilon}{k \int_{x_{11}}^{x_1} h(s, 0, 0) \, ds - N\varepsilon^\alpha} \leq \frac{\varepsilon}{2N\varepsilon^\alpha - N\varepsilon^\alpha} = \frac{1}{N} \varepsilon^{1-\alpha} < \frac{1}{mk}\varepsilon^{1-\alpha},$$

a contradiction. Therefore, the above claim holds. This claim together with the definition of $\bar{x}$ yields the second inequality in (8).

To obtain the lower bound estimate for $y(x_{11})$ as given in (8), we use the estimate from (5) which gives

$$\int_{x_{11}}^{x_1} h(s, 0, 0) \, ds \leq M(x_1 - x_{11}) = MK\varepsilon^\alpha,$$

and the asymptotic formula (9) with $x = x_{11}$ to obtain

$$y_k(x_{11}) \geq \frac{\varepsilon}{kMK\varepsilon^\alpha + N\varepsilon^\alpha} > \frac{\varepsilon^{1-\alpha}}{3kMK}.$$ 

This completes the proof of Step 2.

**Step 3.** As long as $y(x) \leq y_0$ with $x \in (x_{11}, x_{11} + \delta]$, we have

$$0 < \frac{dx}{dy} = \frac{\varepsilon}{y^{k+1}h(x, y, \varepsilon)} \leq \frac{\varepsilon}{my^{k+1}},$$

and so we integrate on $[y(x_{11}), y(x)]$ and use an estimate in (8) to get

$$0 < x - x_{11} \leq \int_{y(x_{11})}^{y(x)} \frac{\varepsilon}{my^{k+1}} \, dy \leq \frac{\varepsilon}{mk} \left( \frac{1}{y_k(x_{11})} - \frac{1}{y_0} \right) < \frac{3MK}{mk} \varepsilon^\alpha.$$
from which we conclude the existence of $x_{1, \varepsilon}$ such that $y(x_{1, \varepsilon}) = y_0$ and

$$x_1 - K \varepsilon^\alpha = x_{11} < x_{1, \varepsilon} < x_{11} + \frac{3MK}{mk} \varepsilon^\alpha = x_1 - K \varepsilon^\alpha + \frac{3MK}{mk} \varepsilon^\alpha,$$

hence, $|x_{1, \varepsilon} - x_1| \leq \frac{3MK}{mk} \varepsilon^\alpha$. Let $p_\varepsilon(x_0) := x_{1, \varepsilon}$. For any small $\varepsilon > 0$, the continuity of $p_\varepsilon(x_0)$ on $x_0$ follows from $y'(x_{1, \varepsilon}) \neq 0$ and the continuous dependence of solutions on initial data. This completes the proof of Theorem 1.

\[\square\]

3 Existence, uniqueness, and stability of relaxation oscillation cycles

We consider system (3):

$$\frac{du}{dt} = u^k f(u, v), \quad \frac{dv}{dt} = \varepsilon g(u, v) \quad (k \geq 1)$$

under the following assumptions:

\[(D0)\] \( f \) and \( g \) are \( C^1 \) in the rectangle \( \mathcal{R} := [0, a] \times [b_1, b_2] \) where \( a > 0 \) and \( 0 \leq b_1 < b_2 < \infty \).

\[(D1)\] The nontrivial \( u \)-nullcline \( f(u, v) = 0 \) lying in \( \mathcal{R} \) is the graph of a \( C^1 \) function \( v = v_f(u) \) for \( u \in [0, a] \) such that (i) \( v_f(u) \) has the global maximum point \( P(u_1^*, v_1^*) \in (0, a) \times (b_1, b_2) \), (ii) \( v_f(u) \) is strictly decreasing on \( [u_1^*, a] \), (iii) \( b_1 \leq v_R < v_2^* < v_1^* \) where \( v_2^* := \min\{v_f(u) : u \in [0, u_1^*]\} \) and \( v_R := v_f(a) \), and (iv)

\[\frac{\partial f}{\partial v}(u, v_f(u)) < 0 \quad \forall \ 0 < u < a. \quad (10)\]

\[(D2)\] The \( v \)-nullcline \( g(u, v) = 0 \) lying in \( \mathcal{R} \) can be written as the graph of a \( C^1 \) function \( u = u_g(v) \) defined for \( v \in [b_1, b_2] \) and satisfies

\[\frac{\partial g}{\partial u}(u_g(v), v) > 0 \quad \forall \ v \in (b_1, b_2). \quad (11)\]

\[(D3)\] There is \( v_0^* \in (v_R, v_2^*) \) such that the Pontryagin’s delay of loss of stability is expressed by the integral

\[\int_{v_0^*}^{v_1^*} \frac{f(0, v)}{g(0, v)} \, dv = 0. \]

(Note that \( v_0^* \) is uniquely determined from the assumptions (D0)-(D2).)
(D4) The graph of \( u = u_g(v) \) intersects the \( u \)-nullcline \( f(u, v) = 0 \) at finitely many points (at least one), all lying strictly to the left of the maximum point \( P(u^*_1, v^*_1) \).

We note that (D1) (iv) implies that \( f(u, v) > 0 \) for \((u, v) \in \mathcal{R} \) and \( v > v_f(u) \) and \( f(u, v) < 0 \) for \((u, v) \in \mathcal{R} \) and \( v < v_f(u) \). Condition (11) implies that \( g(u, v) < 0 \) for \((u, v) \in \mathcal{R} \) with \( u < u_g(v) \) and \( g(u, v) > 0 \) for \((u, v) \in \mathcal{R} \) with \( u > u_g(v) \). Assumption (D4) implies that the portion of the graph \( u = u_g(v) \) for \( v \in (b_1, v^*_1) \) lies entirely to the left of the stable branch of \( f(u, v) = 0 \) defined by \( v = v_f(u) \) for \( u \in [u^*_1, a] \). Also since the nullcline \( g = 0 \) is a graph of \( u = u_g(v) \) for \( v \in (b_1, b_2) \) and \( g_u \neq 0 \), it follows that this nullcline intersects each horizontal line \( v = c \) with \( c \in (b_1, b_2) \) exactly at one point and the intersection is transversal.

![Figure 2: Representation of the singular orbit \( \Gamma_0 \) (magenta) and a relaxation oscillation orbit \( \Gamma_\epsilon \) (green) for \( \epsilon > 0 \).](image)

To prove the existence of relaxation oscillations for system (3), we will first define the singular orbit of (3) which is obtained by concatenating the singular limits of the solutions of the fast and the slow time-scale formulations of (3). Let \( u = u_r(v) \) be the inverse function of \( v = v_f(u) \) with \( u \in [u^*_1, a] \) and \( \overrightarrow{PR} \) be its graph where \( \mathcal{R} = R(a, v_R) \). Denoting the singular orbit by \( \Gamma_0 \), we define it as

\[
\Gamma_0 := L_1 \cup L_2 \cup L_3 \cup L_4, \tag{12}
\]

where \( L_1 \) is the arc joining the points \((u_r(v^*_0), v^*_0)\) and \( P(u^*_1, v^*_1) \) along \( \overrightarrow{PR} \), \( L_2 \) is the horizontal line segment connecting the vertex \( P \) to the point \((0, v^*_1)\), \( L_3 \) is the
vertical line segment along the $v$-axis joining the points $(0,v_1^*)$ and $(0,v_0^*)$ and $L_4$ is the horizontal line segment connecting $(0,v_0^*)$ to $(u_r(v_0^*),v_0^*)$ as shown in figure 2. We note that by the above remark, each of the horizontal segments $L_2$ and $L_4$ intersects the $v$-nullcline $g(u,v) = 0$ only once.

The main theorem of this section is the following:

**Theorem 2.** (I) Assume (D0)-(D4). Then for sufficiently small $\varepsilon > 0$, system (3) has a unique relaxation cycle $\Gamma_\varepsilon$ in $\mathcal{R}$ that approaches the singular orbit $\Gamma_0$ defined by (12) as $\varepsilon \to 0$, and furthermore $\Gamma_\varepsilon$ is locally asymptotically orbitally stable.

(II) Assume further that

$(D5)$ (i) $v_f(u)$ is strictly increasing in $(0,u_1^*)$. (ii) Let $u = u_r(v)$ and $u = u_l(v)$ be the inverse functions of $v = v_f(u)$ with $u \in [u_1^*,a]$ and $u \in [0,u_1^*]$ respectively with $f$ satisfying

$$\frac{\partial f}{\partial u}(u_r(v),v) < 0 \quad \forall v_R < v < v_1^*, \quad \frac{\partial f}{\partial u}(u_l(v),v) > 0 \quad \forall v_Q := v_f(0) < v < v_1^*. \quad (13)$$

Then $\Gamma_\varepsilon$ is the unique limit cycle of (3) in $\mathcal{R}$.

**Remark 1.** Under the assumption $(D5)$, we point out the qualitative types of the equilibrium points of (3) which will be used in the proof of Theorem 2 (II). Let $E^*(u^*,v^*) \in \mathcal{R}$ be an equilibrium of (3). The Jacobian of (3) at $E^*(u^*,v^*)$ is

$$J|_{E^*} = \begin{pmatrix} (u^*)^k f_u(u^*,v^*) & (u^*)^k f_v(u^*,v^*) \\ \varepsilon g_u(u^*,v^*) & \varepsilon g_v(u^*,v^*) \end{pmatrix},$$

and so

$$\text{tr} J|_{E^*} = (u^*)^k f_u(u^*,v^*) + \varepsilon g_v(u^*,v^*),$$

$$\text{det} J|_{E^*} = \varepsilon (u^*)^k [f_u(u^*,v^*)g_v(u^*,v^*) - f_v(u^*,v^*)g_u(u^*,v^*)].$$

The second inequality in (13) yields that for sufficiently small $\varepsilon > 0$, $\text{tr} J|_{E^*} > 0$. We have three cases to discuss based on the sign of $\text{det} J|_{E^*}$. Note that the vectors $[f_u(u^*,v^*), f_v(u^*,v^*)]$ and $[g_u(u^*,v^*), g_v(u^*,v^*)]$ are the normal vectors of the curves $f = 0$ and $g = 0$ at $E^*$ respectively; equivalently, by the implicit function theorem,

$$\frac{du_l(v^*)}{dv} = -\frac{f_v(u^*,v^*)}{f_u(u^*,v^*)}, \quad \frac{du_r(v^*)}{dv} = \frac{g_v(u^*,v^*)}{g_u(u^*,v^*)}.$$

**Case 1.** $\frac{du_r(v^*)}{dv} = \frac{du_l(v^*)}{dv}$. In this case, the two nullclines $f = 0$ and $g = 0$ are tangent to each other at $E^*$, and $f_u(u^*,v^*)g_v(u^*,v^*) - f_v(u^*,v^*)g_u(u^*,v^*) = 0$, and
Figure 3: Different possibilities depicting the intersection between the curves $f(u_l(v), v) = 0$, denoted in red, and $g(u_g(v), v) = 0$, denoted in blue. (a)-(d) represent Case 1, (e) represents Case 2, (f) represents Case 3(a), (g)-(j) represent Case 3(b) and (k) represents Case 3(c).

so $\det J|_{E^*} = 0$. (Note that Fig. 3 (a)-(d) show four possible figures for the graph of $u = u_g(v)$ near $E^*$: the graph lies just on either side of $f = 0$ or on both sides of $f = 0$.) For sufficiently small $\varepsilon > 0$, the Jacobian $J|_{E^*}$ has a positive eigenvalue and a zero eigenvalue.

Case 2. $\frac{du_g(v^*)}{dv} > \frac{du_l(v^*)}{dv}$. In this case, as $v$ increases, the nullcline $g = 0$ crosses the curve $f = 0$ at $E^*$ from its left side to its right side (see (e) in figure 3). It follows that $g_v(u^*, v^*) < 0$ and $f_u(u^*, v^*)g_v(u^*, v^*) - f_v(u^*, v^*)g_u(u^*, v^*) < 0$ so that
\[ \text{Case 3. } \frac{d u_g(v^*)}{dv} < \frac{d u_l(v^*)}{dv}. \text{ In this case, as } v \text{ increases, the nullcline } g = 0 \text{ crosses the curve } f = 0 \text{ at } E^* \text{ from its right side to its left side (see (f)-(k) in figure 3). We have three cases to discuss}
\]

(a) \( \frac{d u_g(v^*)}{dv} > 0. \) Then we have \( g_v(u^*, v^*) < 0 \) and
\[
f_u(u^*, v^*) g_v(u^*, v^*) - f_v(u^*, v^*) g_u(u^*, v^*) > 0
\]
so that \( \det J|_{E^*} > 0. \)

(b) \( \frac{d u_g(v^*)}{dv} = 0. \) In this case we have \( g_v(u^*, v^*) = 0 \) and
\[
\det J|_{E^*} = -\varepsilon (u^*)^k f_v(u^*, v^*) g_u(u^*, v^*) > 0.
\]
(Note that \( E^* \) is the critical point of the graph of \( u = u_g(v) \), so based on whether it corresponds to a local minimum or local maximum or neither, figure 3 (g)-(j) show four possible representations of the graph of \( u = u_g(v) \) near \( E^* \).)

(c) \( \frac{d u_g(v^*)}{dv} < 0. \) We have \( g_v(u^*, v^*) > 0 \) and the signs of entries of \( J|_{E^*} \) directly gives \( \det J|_{E^*} > 0. \)

Hence for all the sub-cases in Case 3, we have for sufficiently small \( \varepsilon > 0, E^* \) is an unstable node or an unstable spiral.

We shall prove Theorem 2 in the following subsections.

### 3.1 Existence of relaxation oscillations

In this section, we will construct a closed annular region \( U \) in the vicinity of the singular orbit and show that it is positively invariant under the flow of (3). A similar approach was taken in [24] to study relaxation oscillations of a population model that arose in genetics. We remark that Theorem 1 will be employed in the construction of \( U \), where we put system (3) into the framework of system (4) by letting \( x = v - v_Q \) and \( y = u. \)

**Lemma 1.** Assume (D0)-(D4). Then for sufficiently small \( \varepsilon > 0, \) system (3) has a limit cycle \( \Gamma_\varepsilon \) that approaches the singular orbit \( \Gamma_0 \) as \( \varepsilon \to 0. \)

**Proof.** To prove this, for every sufficiently small \( \varepsilon > 0 \) we construct a closed annular region \( U := U_\varepsilon \) in a vicinity of the singular orbit \( \Gamma_0 \) such that the maximum width of \( U \) is any preassigned constant and yet the region \( U \) is positively invariant for the
flows of (3). An application of the Poincaré Bendixson theorem will then guarantee the existence of the limit cycle $\Gamma_\varepsilon$ lying entirely in $U$. To this end, we fix $h > 0$ small and consider the vertical line $u = u_0$, where $0 < u_0 < h$ is a fixed real number such that $g(u, v) < 0$ for $0 \leq u \leq u_0$ and $b_1 < v < b_2$. Consider the interval $[v_0^-, v_1^+]$, where $v_1^\pm = v_1^\pm + h$. The region $U$ is constructed as follows (see figure 4). Consider the solution of (3) that starts at $A_0(u_0, v_0^-)$ and continue till it exits the line segment $u = u_0$ at $A_1(u_0, p_\varepsilon(v_1^-))$. We know that such a solution exists by Theorem 1 and satisfies $|p_\varepsilon(v_1^-) - p_0(v_1^-)| = O(\varepsilon^{1/k})$ as $\varepsilon \to 0$, with

$$\int_{p_0(v_1^-)}^{v_1^-} \frac{f(0, v)}{g(0, v)} \, dv = 0.$$ 

Denoting the solution by $\overrightarrow{A_0A_1}$, let the solution orbit $\overrightarrow{A_0A_1}$ form a segment of the inner boundary of $U$. Similarly, consider the solution orbit $\overrightarrow{B_0B_1}$ of system (3) through the point $B_0$ to form a segment of the outer boundary of $U$, where $B_0$ and $B_1$ have coordinates $(u_0, v_1^+)$ and $(u_0, p_\varepsilon(v_1^+))$ respectively. Let $\bar{h} = \max\{|p_0(v_1^-) - v_0^-)|, |p_0(v_1^+) - v_0^+)|\}$, where $v_0^+ = p_0(v_1^+)$. Note that $\bar{h} \to 0$ as $h \to 0$. Hence we assume that $v_R < v_0^- - 2\bar{h} < v_0^+ + 2\bar{h} < v_Q$. We next define the other parts of the inner and outer boundaries of the region $U$.

---

**Figure 4:** A positively invariant annular region $U$ with its boundary marked in red. The arrows designate the direction segments of the boundaries crossed. We note that the construction of $U$ depends only on the nuclines $f = 0$ and $g = 0$ in a small neighborhood of the singular orbit $\Gamma_0$. 

---
where the solution trajectory \( \hat{u} \) on the nullcline \( f \) of system (3), similarly the vector field along the boundary segment \( \hat{v} \) connects the vertex \( A_5 \) that connects \( A_1 \) and the horizontal line segment \( \hat{B}_6 \) that connects \( A_4 \) with the nullcline \( f(u, v) = 0 \) at \( A_5 \); the arc \( \hat{A}_5 \) that lies on the arc \( \hat{P} \hat{R} \), where \( A_6 := P \); the horizontal line segment \( \hat{A}_6 \) that connects \( A_4 \) with the nullcline \( g(u, v) = 0 \) at \( A_7 \); and finally the line segment \( \hat{A}_7 \) that connects \( A_7 \) with the root line \( u = u_0 \) at \( A_7 \). Together with these segments and the solution trajectory \( \hat{A}_7 \), the inner boundary of \( U \) forms a closed curve.

The outer boundary of \( U \) is defined similarly and consists of the solution orbit \( \hat{B}_0 \hat{B}_1 \), the vertical line segment \( \hat{B}_1 \hat{B}_2 \) defined by \( \{ u = u_0 : v_0^* - 3h/2 \leq v \leq p_\varepsilon(v_1^*) \} \), the line segment \( \hat{B}_2 \hat{B}_3 \), where \( B_3 \) lies on intersection of the nullcline \( g(u, v) = 0 \) and the line \( v = v_0^* - 2h \) (see figure 5), the horizontal line segment \( \hat{B}_3 \hat{B}_4 \) with \( B_4 \) lying on the nullcline \( f(u, v) = 0 \), the vertical line segment \( \hat{B}_4 \hat{B}_5 \) of height \( h \), the arc \( \hat{B}_5 \hat{B}_6 \) which is a vertical translate of the right branch of the curve \( f(u, v) = 0 \) by \( h \) such that \( v_0^* - 2h + h \leq v \leq v_1^* \) and \( B_6 \) lies on the line \( v = v_1^* \), the line segment \( \hat{B}_6 \hat{B}_7 \), where \( B_7(v_0(v_1^*), v_1^*) \) lies on the nullcline \( g(u, v) = 0 \), and the horizontal line segment \( \hat{B}_7 \hat{B}_0 \), where \( B_0 \) lies on the vertical line \( u = u_0 \).

We next show that except for the segments \( \hat{A}_0 \hat{A}_1 \) and \( \hat{B}_0 \hat{B}_1 \), the orbits cross the boundary of \( U \) inward if \( \varepsilon > 0 \) is chosen small enough as shown in figures 4-5. It is clear that since \( \hat{A}_0 \hat{A}_1 \) and \( \hat{B}_0 \hat{B}_1 \) are solution orbits of system (3), any orbit that lies in between these segments cannot escape through these boundaries. The directional arrows along the horizontal segments \( \hat{A}_2 \hat{A}_3 \), \( \hat{A}_6 \hat{A}_7 \), \( \hat{B}_7 \hat{B}_0 \), \( \hat{B}_3 \hat{B}_4 \) and the vertical line segment \( \hat{A}_1 \hat{A}_2 \), \( \hat{A}_4 \hat{A}_5 \), \( \hat{B}_1 \hat{B}_2 \), \( \hat{B}_4 \hat{B}_5 \) are directly obtained by studying the vector field of system (3). Similarly, the vector field along the boundary segment \( \hat{A}_5 \hat{A}_6 \) that lies on the \( u \)-nullcline can be easily obtained. It remains to check the vector field along the segments \( \hat{A}_3 \hat{A}_4 \), \( \hat{A}_7 \hat{A}_0 \), \( \hat{B}_3 \hat{B}_6 \), \( \hat{B}_3 \hat{B}_6 \) and \( \hat{B}_6 \hat{B}_7 \). Since on these segments \( u^k|f(u, v)| \geq m \) for some constant \( m > 0 \), it follows that the slopes of the \( \varepsilon \) vector field on these segments satisfy

\[
\left| \frac{dv}{du} \right| = \varepsilon \left| \frac{g(u, v)}{u^k f(u, v)} \right| \leq M \varepsilon
\]

for some constant \( M > 0 \). This inequality together with the signs of \( \frac{dv}{du} \) on these segments and the fact that these segments have slopes whose absolute values have
a positive lower bound independent of \(\varepsilon > 0\), yield that for sufficiently small \(\varepsilon > 0\), the vector field on these line segments points inward of \(U\). Thus, we showed that for sufficiently small \(\varepsilon\) small, \(U\) forms a trapping region for the flows of (3). Since by construction \(U\) does not contain any equilibrium points, applying the Poincaré Bendixson theorem yields that there exists a limit cycle \(\Gamma_\varepsilon\) that lies in \(U\). Furthermore, since \(h\) can be arbitrarily small (and so is \(\bar{h}\)) in the construction of \(U\), it follows that \(\Gamma_\varepsilon\) approaches the singular orbit \(\Gamma_0\) as \(\varepsilon \to 0\).

3.2 Stability and uniqueness of relaxation oscillations

Assume (D0)-(D4). In this section we derive the asymptotic formula for the nontrivial Floquet multiplier of the relaxation oscillation orbit \(\Gamma_\varepsilon\) and show that \(\Gamma_\varepsilon\) is asymptotically orbitally stable, thus giving us the uniqueness of such limit cycles for system (3). A similar approach was taken in [9] to prove uniqueness of relaxation oscillations for slow-fast planar systems with S-shaped critical manifolds (such as the van der Pol’s system given by (2)).

Let \(\delta > 0\) be sufficiently small. We construct a \(\delta\)-neighborhood \(\Omega_\delta\) around the singular orbit \(\Gamma_0\) by considering vertical translates of the segments \(L_1, L_2\) and \(L_4\) of the singular orbit \(\Gamma_0\), and a horizontal translate of the segment \(L_3\). More precisely,
we define the outer boundary of $\Omega_\delta$ by the union of the arc $L_1$ shifted up by $\delta$, the horizontal line segment parallel to $L_2$ shifted up by $\delta$, the vertical line segment $L_3$ stretched out along the $v$-axis and the horizontal line segment parallel to $L_4$ shifted down by $\delta$ in such a way that the curves meet to form a closed boundary. Similarly, the inner boundary of $\Omega_\delta$ is defined by appropriate shifts of the segments of $\Gamma_0$; see figure 6. Note that the right and the top arcs of the inner boundary of $\Omega_\delta$ are on the singular orbit $\Gamma_0$. We also note that by taking $h > 0$ and $\varepsilon > 0$ small, the set $U$ constructed in Lemma 1 is contained in $\Omega_\delta$.

Lemma 2. Assume (D0)-(D4). Let $\delta > 0$ be sufficiently small and $\Omega_\delta$ be defined as above. Then there is $\varepsilon_\delta > 0$ small such that for any $0 < \varepsilon < \varepsilon_\delta$, if $\Gamma_\varepsilon$ is a closed orbit of (3) parameterized by $(u(t), v(t))$ with the least period $T > 0$ and contained in the interior of $\Omega_\delta$, then $\Gamma_\varepsilon$ is locally asymptotically orbitally stable with

$$
\lambda := \int_0^T \left[ \frac{\partial (u f)}{\partial u} + \varepsilon \frac{\partial g}{\partial v} \right] (u(t), v(t)) \, dt = \frac{1}{\varepsilon} \int_{v_0^*}^{v_1^*} u_r(v) f_u(u_r(v), v) \frac{g(u_r(v), v)}{g(u_r(v), v)} \, dv \left[ 1 + \Delta(\varepsilon) \right] < 0,
$$

where $|\Delta(\varepsilon)| \leq M\delta$ if $k = 1$ and $|\Delta(\varepsilon)| \leq M(\delta^{k-1} + \delta)$ if $k > 1$ for some $M > 0$ independent of $\varepsilon > 0$ and $\delta > 0$.

Proof. We first fix $\delta_0 > 0$ small and let $0 < \delta \leq \delta_0$. Let $\Gamma_\varepsilon$ start on the vertical line segment $C_0 := \{(u, v) : u = \delta_0, v_1^* \leq v \leq v_1^* + \delta\}$; let $t_1 > 0$ be the first time that $\Gamma_\varepsilon$

Figure 6: An annular neighborhood $\Omega_\delta$ of $\Gamma_\varepsilon$ with its boundaries marked in magenta.
intersects with the vertical line segment \( C_1 := \{(u, v) : u = \delta_0, v_0^* - \delta/2 \leq v \leq v_0^* + \delta \} \); let \( t_2 > t_1 \) be the first time that \( \Gamma_\varepsilon \) reaches the vertical line segment \( C_2 := \{(u, v) : u = u_r(v_0^*) - \delta_0, v_0^* - \delta \leq v \leq v_0^* + \delta \} \); let \( t_3 > t_2 \) be the first time that \( \Gamma_\varepsilon \) reaches the horizontal line segment \( C_3 := \{(u, v) : v = v_1^*, u_1^* \leq u \leq u_r(v_1^* - \delta) \} \); let \( t_4 > t_3 \) be the first time that \( \Gamma_\varepsilon \) reaches the vertical line segment \( C_4 := \{(u, v) : u = u_1^{**}, v_1^* \leq v \leq v_1^* + \delta \} \) where \( u_1^{**} = \frac{1}{2}(u_1^* + u_g(v_1^*)) \) and \( u_g(v_1^*) \) is the \( u \)-coordinate of the intersection point of the nullcline \( g(u, v) = 0 \) with the line \( v = v_1^* \).

The construction of \( \Omega_\delta \) implies that as \( \delta \) decreases, \( \Omega_\delta \) becomes smaller, so do the rectangles contained in \( \Omega_\delta \) and bounded by the line segments \( C_1 \) and \( C_2 \) and respectively by \( C_4 \) and \( C_0 \) (since the \( u \)-coordinates of the vertical segments \( C_0, C_1, C_2 \) and \( C_4 \) are independent of \( \delta \)). We note that the functions \( u^k f, g \) and their partial derivatives are bounded on \( \Omega_{\delta_0} \) and, for some constant \( m_0 > 0 \), \( u^k |f(u, v)| \geq m_0 \) in the rectangles contained in \( \Omega_{\delta_0} \) as described above and \( |g(u, v)| \geq m_0 \) in \( \Omega_{\delta_0} \) outside the rectangles.

We compute \( \lambda \) through five steps described below, with any given \( 0 < \delta \leq \delta_0 \).

**Step 1.** On the arc \( \Gamma_\varepsilon \lbrack 0, t_1 \rbrack \), we have \( u(t) \leq \delta \) and

\[
ku^{k-1} f(u, v) + u^k f_u(u, v) = \begin{cases} f(u, v) + O(\delta) & \text{if } k = 1, \\ O(\delta^{k-1}) & \text{if } k > 1. \end{cases}
\]

Since \( v' = \varepsilon g(u, v) \geq m_0 \varepsilon \), we may regard \( u = u(t) \) is a function of \( v \) to get, for \( k = 1 \),

\[
\int_0^{t_1} \frac{\partial (u^k f)}{\partial u} dt = \frac{1}{\varepsilon} \int_{v(0)}^{v(t_1)} \frac{f(u(v), v) + O(\delta)}{g(u(v), v)} dv = \frac{1}{\varepsilon} O(\delta) + \int_{v_0^*}^{v_1^*} \frac{f(0, v)}{g(0, v)} dv \]

where \( \varepsilon = \frac{1}{\varepsilon} O(\delta) \);

if \( k > 1 \), we have

\[
\int_0^{t_1} \frac{\partial (u^k f)}{\partial u} dt = \frac{1}{\varepsilon} \int_{v(0)}^{v(t_1)} \frac{O(\delta^{k-1})}{g(u(v), v)} dv = \frac{1}{\varepsilon} O(\delta^{k-1}).
\]

**Step 2.** On the arc \( \Gamma_\varepsilon \lbrack t_1, t_2 \rbrack \), we have \( u'(t) > m_0 \) and so we can regard \( v \) as function of \( u \) to get

\[
\int_{t_1}^{t_2} \frac{\partial (u^k f)}{\partial u} dt = \int_{u(t_1)}^{u(t_2)} \frac{1}{u^k f(u, v(u))} du \leq M'
\]

where \( M' \) is a constant.
for some constant $M' > 0$.

**Step 3.** On the arc $\Gamma_{\varepsilon}|_{[t_2,t_4]}$, we have $v'(t) = \varepsilon g(u,v) > \varepsilon m_0$ so we can regard $u$ as a function of $v$, and write $u = u(v)$. Note that for $t_2 < t < t_3$, we have $u(v) - u_r(v) = O(\delta)$, and so using $f(u_r(v),v) = 0$ we have

\[
\begin{align*}
    f(u(v),v) &= f(u(v),v) - f(u_r(v),v) = f_u(u(v),v)(u(v) - u_r(v)) = O(\delta), \\
    f_u(u(v),v) &= f_u(u_r(v),v) + f_u(u(v),v) - f_u(u_r(v),v) \\
    &= f_u(u_r(v),v) + f_u(u(v),v)(u(v) - u_r(v)) \\
    &= f_u(u_r(v),v) + O(\delta),
\end{align*}
\]

and $g(u(v),v) = g(u_r(v),v) + O(\delta)$. For $t_3 < t < t_4$, we have $v(t_4) - v(t_3) \leq \delta$. Hence,

\[
\int_{t_2}^{t_4} \frac{\partial (u^k f)}{\partial u} \, dt = \frac{1}{\varepsilon} \int_{v(t_2)}^{v(t_4)} \frac{k u^{k-1} f(u(v),v) + u^k f_u(u(v),v)}{g(u(v),v)} \, dv \\
+ \frac{1}{\varepsilon} \int_{v(t_3)}^{v(t_4)} \frac{k u^{k-1} f(u(v),v) + u^k f_u(u(v),v)}{g(u(v),v)} \, dv \\
= \frac{1}{\varepsilon} \left[ O(\delta) + \int_{v_5}^{v_4} \frac{u^k f_u(u_r(v),v)}{g(u_r(v),v)} \, dv \right].
\]

**Step 4.** The proof in this step is very similar to that of Step 2. On the arc $\Gamma_{\varepsilon}|_{[t_4,T]}$, we have $u'(t) < -m_0 < 0$ and so we can regard $v$ as function of $u$ to get

\[
\int_{t_4}^{T} \frac{\partial (u^k f)}{\partial u} \, dt = - \int_{u(T)}^{u(t_4)} \frac{k u^{k-1} f(u(v),v) + u^k f_u(u(v),v)}{u^k f(u, u)} \, du \\
= -k \ln \frac{u(t_4)}{\delta_0} + \int_{\delta_0}^{u(t_4)} \frac{f_u(u,v)}{-f(u,v)} \, du \leq M''
\]

for some constant $M'' > 0$, where we used that $u(t_4) = u^*_s = \frac{1}{2}(u^*_s + u_g(v^*_s))$.

**Step 5.** Using the similar arguments of above steps we get

\[
\int_0^T \frac{\partial g}{\partial v} \, dt \leq \frac{M''}{\varepsilon}.
\]

Combining the estimates in the above steps, we have, for $k = 1$,

\[
\lambda = \frac{1}{\varepsilon} \left( O(\delta) + \int_{v_5}^{v_4} \frac{u^k(v) f_u(u_r(v), v)}{g(u_r(v), v)} \, dv + \varepsilon \tilde{M} \right), \quad (14)
\]

and, for $k > 1$,

\[
\lambda = \frac{1}{\varepsilon} \left( O(\delta^{k-1}) + O(\delta) + \int_{v_5}^{v_4} \frac{u^k(v) f_u(u_r(v), v)}{g(u_r(v), v)} \, dv + \varepsilon \tilde{M} \right), \quad (15)
\]
where $\tilde{M} := M' + M'' + M'''$. Checking the above proof one can see that the constants in all big \(O(\delta)\) and \(O(\delta^{k-1})\) terms can be taken independent of \(\delta > 0\) and \(\varepsilon > 0\) (depending on \(\delta_0\) though). This observation together with (14) and (15) gives the desired asymptotic formula of \(\lambda\) as stated in the lemma, from which we conclude that \(\Gamma_\varepsilon\) is locally asymptotically orbitally stable.

\[\square\]

3.3 Nonexistence of non-relaxation oscillation cycles

We will finally prove that under assumptions (D0)-(D5), the only periodic orbits of system (3) are relaxation oscillations. This, when combined with Lemma 2, yields uniqueness of periodic solutions of (3).

**Lemma 3.** Assume (D0)-(D5). Then for sufficiently small \(\varepsilon > 0\), any closed orbit of system (3) lies inside the annular region \(U\) constructed in the proof of Lemma 1.

**Proof.** Suppose that for a sufficiently small \(\varepsilon > 0\), (3) has a closed orbit \(\gamma\) lying outside the annular region \(U\) in the interior of the region bounded by the inside boundary of \(U\). It follows from the basic theory of planar systems (e.g., the index theory) that \(\gamma\) must contain at least one equilibrium point \(E^*\) other than a saddle point in its

![Figure 7: Representation of the vector field in the region V.](image-url)
interior. Based on Remark 1 we conclude that \( \frac{du_t(v^*)}{dv} \leq \frac{du_l(v^*)}{dv} \) at \( E^* \), which together with the slope of the \( u \)-nullcline at \( E^* \) satisfying

\[
\frac{du_t(v^*)}{dv} = -\frac{f_u(u^*, v^*)}{f_u(u^*, v^*)} > 0,
\]

yields that there is a small \( \delta_0 > 0 \) such that

\[
\frac{du(v)}{dv} \leq 2 \frac{du_t(v)}{dv}, \quad u_g(v) < 2u_l(v) \quad \forall v^* < v < v^* + \delta_0.
\]

Fix a very small number \( 0 < \delta < \min\{\delta_0, -\frac{f_u(u^*, v^*)}{f_u(u^*, v^*)}\} \). Then the estimates in (17) and (16) enable us to construct the set \( V \) bounded by the five arcs as shown in figure 7: (i) the horizontal segment \( \overline{E^*E_1} \), where \( E_1 \) lies in the interior of the arc \( \overline{PR} \); (ii) the segment \( \overline{E^*E_2} : v = v^* + \delta(u - u^*) \) with \( u^* < u \leq u^* + \delta \); (iii) the line segment \( \overline{E_2E_3} \) which has a very small slope \( \delta_1 > 0 \), where \( E_3 \) lies on the arc \( u = u_0(v) - \delta \); (iv) the vertical segment \( \overline{E_3E_4} \) where \( E_4 \) lies on the arc \( \overline{PR} \); (v) the arc \( \overline{E_1E_4} \) lying on the arc \( \overline{PR} \). We show that for sufficiently small \( \varepsilon > 0 \), except for the arc \( \overline{E_1E_4} \), the vector field of (3) points inward of \( V \).

We only need to show that for sufficiently small \( \varepsilon > 0 \), the vector field on the line segment \( \overline{E^*E_2} \) points to the interior of \( V \) as the directional arrows along the other segments can be directly obtained by studying the vector field of (3) as was done in the proof of Lemma 1. For \( (u, v) \in \overline{E^*E_2} \),

\[
f(u, v) = [f_u(u^*, v^*) + o(1)](u - u^*) + [f_v(u^*, v^*) + o(1)](v - v^*)
= \left\{ [f_u(u^*, v^*) + o(1)] + [f_v(u^*, v^*) + o(1)]\delta \right\}(u - u^*),
\]

and

\[
g(u, v) = \left\{ [g_u(u^*, v^*) + o(1)] + [g_v(u^*, v^*) + o(1)]\delta \right\}(u - u^*),
\]

and so, for sufficiently small \( \varepsilon > 0 \),

\[
\frac{dv}{du} = \varepsilon \frac{g(u, v)}{u^k f(u, v)} = \varepsilon \frac{[g_u(u^*, v^*) + o(1)] + [g_v(u^*, v^*) + o(1)]\delta}{[f_u(u^*, v^*) + o(1)] + [f_v(u^*, v^*) + o(1)]\delta} < \delta,
\]

which implies the above statement.

Since \( \gamma \) encloses \( E^* \) in its interior and lies in the interior of the region bounded by the inside boundary of \( U \), \( \gamma \) must intersect the segment \( \overline{E^*E_1} \). Then the vector field of (3) along the boundary of \( V \) yields that \( \gamma \) will exit the region \( V \) through the arc \( \overline{E_1E_4} \), then enter the positively invariant annular region \( U \), and thereafter remain in \( U \). Consequently, \( \gamma \) cannot intersect the segment \( \overline{E^*E_1} \) again, contradicting that
it is closed. It is easy to show from the direction of the vector field of (3) that every orbit starting from the region outside the outer boundary of \( U \) will eventually enter \( U \) and thereafter remain in \( U \). Hence (3) cannot have a closed orbit lying in the region outside the outer boundary of \( U \). This completes the proof of the lemma.

\[ \square \]

**Proof of Theorem 2.** Theorem 2 (I) follows from Lemmas 1 and 2; Theorem 2 (II) follows from Theorem 2 (I) and Lemma 3.

### 4 Applications

#### 4.1 Relaxation oscillations in mathematical models

Relaxation oscillations are observed in numerous models arising in different fields of study, including but not limited to ecology, bio-economics, neurosciences etc. (see [10, 19, 20, 17, 24, 27]). Several of these models can be written as (after time-rescaling)

\[ u' = u^k f_1(u, v)(\phi(u) - v), \quad v' = \varepsilon g_1(u, v)(u - \psi(v)), \quad (18) \]

where we assume

(i) \( f_1(u, v) > 0 \) \( g_1(u, v) > 0 \), \( \forall \ 0 \leq u < A, \ 0 < v < B \);

(ii) The function \( v = \phi(u) \) has a maximum at a point \( u = u^*_1 \in (0, A) \) with \( v^*_1 = \phi(u^*_1) \), \( \phi(0) = 0, \phi(A) = 0 \), \( \phi \) is increasing in \( [0, u^*_1] \) and decreasing in \( [u^*_1, A] \).

(iii) The function \( u = \psi(v) \) is increasing in \( (0, B) \) with \( B > u^*_1 \).

We note that the integral corresponding to the phenomenon of delayed loss of stability for system (18) is

\[ \int_{v^*_0}^{v^*_1} \frac{f_1(0, v)(\phi(0) - v)}{g_2(0, v)\psi(v)} dv = 0, \]

and the Jacobian of (18) at an equilibrium \( E^* = (u^*, v^*) \) is

\[ J = \begin{pmatrix} u^1 f_1(u^*, v^*) \phi'(u^*) & -(u^*_1)^k f_1(u^*, v^*) \\ \varepsilon g_1(u^*, v^*) & -\varepsilon g_1(u^*, v^*) \psi'(v^*) \end{pmatrix}. \]

Below, we give a few examples of mathematical models which can be written in the form of (18).
Example 1. Consider the predator-prey system with Beddington-DeAngelis functional response [3, 5]:

\[ u' = u \left( 1 - u - \frac{mv}{1 + au + bv} \right), \quad v' = \varepsilon v \left( \frac{mu}{1 + au + bv} - \beta \right), \]

where \( m > 0 \), \( a > 0 \) are associated with the capture rate and the handling time respectively, \( b \geq 0 \) is related with the magnitude of interference among predators, \( \beta > 0 \) is associated with the mortality rate of the predator, and \( \varepsilon > 0 \) is the ratio of the growth rate of the predator to its prey. After time-rescaling, this system can be written as (with \( m > \max\{b, \beta\} \))

\[ u' = u \left( m - b(1 - u) \right) \left[ \frac{(1 - u)(1 + au)}{m - b(1 - u)} - v \right], \]

\[ v' = \varepsilon v \left( m - \beta - \gamma v \right) \left[ u - \frac{\beta(1 + bv)}{m - \beta} \right], \]

which is in the form of (18).

Example 2. Consider the generalized Holling-Tanner predator-prey system

\[ u' = u \left( 1 - u - \frac{mv}{1 + au + bv} \right), \quad v' = \varepsilon v \left( \beta - \frac{v}{u + d} \right), \]

where \( m, a > 0 \), \( b \geq 0 \), \( \varepsilon > 0 \) have the same interpretation as before, \( \beta > 0 \) is related to the birth-to-consumption ratio of the predator and \( d \geq 0 \). Here the carrying capacity of the predators depend on the prey population size. When \( d = 0 \), the system reduces to the Holling-Tanner model [26]. After a time rescaling, the system can be rewritten as (with \( m > b \))

\[ u' = u(u + d) \left[ m - b(1 - u) \right] \left[ \frac{(1 - u)(1 + au)}{m - b(1 - u)} - v \right], \quad v' = \varepsilon v \left[ \beta(u + d) - v \right], \]

which is in the form of (18).

Sometimes, it is convenient to write \( g(u, v) \) in (3) as

\[ g(u, v) = g_1(u, v)(\psi(u) - v), \]

where \( v = \psi(u) \) is an increasing function on \((0, A)\), as shown in the following example:

Example 3. In [17], the authors developed a mathematical model to study the coupled social and ecological dynamics of herders in a southern Mongolian rangeland. The herders choose foraging sites for their animals in the dry season based on the
abundance of grass biomass as well as on the higher payoff (profitability of moving to an alternative rangeland versus staying staying in the focal rangeland by comparing the incurred costs). Denoting the abundance of grass biomass in the key resource area (focal rangeland) by \( u \), let \( v \) be the fraction of herders who stay in the focal rangeland so that \( 1 - v \) fraction of herders move to an alternative rangeland. The dynamics of grass biomass and the fraction of herders that stay in the same site were modeled by the following system of equations:

\[
\begin{align*}
\frac{du}{dt} &= u \left( b - cu - \frac{aNv}{1 + hu} \right), \\
\frac{dv}{dt} &= \varepsilon \left( \frac{1}{1 + \exp \left[ -\beta \left( \frac{\lambda au}{1 + hu} - u_m \right) \right]} - v \right),
\end{align*}
\]  

(19)

where \( b \) is the intrinsic growth rate of the grass biomass, \( b/c \) is its carrying capacity, \( N \) is the total number of cattle owned by the herders, \( a \) is the feeding rate and \( h \) is the handling time of the animals, \( \varepsilon \) is the fraction of herders that chooses between the two foraging site options, \( \lambda > 0 \) is a constant, \( \beta \) indicates the sensitivity of the herder to the payoff difference between staying and moving, and \( u_m \) is the utility of using an alternative rangeland. The system (19) can be written as

\[
\begin{align*}
\frac{du}{dt} &= \frac{aNu}{1 + hu} \left( (b - cu)(1 + hu) - v \right), \\
\frac{dv}{dt} &= \varepsilon \left( \frac{1}{1 + \exp \left[ -\beta \left( \frac{\lambda au}{1 + hu} - u_m \right) \right]} - v \right).
\end{align*}
\]  

(21)

4.2 Existence of periodic traveling waves in predator-prey systems

In this section, we discuss about the existence of periodic traveling waves to predator-prey systems of the form

\[
\begin{align*}
&u_t = D_u u_{xx} + u^k f(u, v), &v_t = D_v v_{xx} + \varepsilon g(u, v), &k \geq 1,
\end{align*}
\]  

(20)

where \( t \) is time, \( x \in \mathbb{R} \) is spatial location, \( u, v \) are related to population densities of the prey and the predator respectively, \( D_u, D_v \) are positive diffusive coefficients of the two populations and the reaction terms \( f, g \) satisfy assumptions (D0)-(D4) in a rectangle \( \mathcal{R} = [0, a] \times [b_1, b_2] \) for some \( a, b_1, b_2 > 0 \). The parameter \( \varepsilon \) is associated to the ratio of the growth rate of the predator to its prey and is assumed to be small, which is typically observed in the wild. Here we consider \( 0 < \varepsilon \ll 1 \)

Replacing \( x \) in (20) by the moving coordinate \( \zeta = x - ct \), we obtain

\[
\begin{align*}
0 = D_u u_{\zeta \zeta} + cu_{\zeta} + u^k f(u, v), & \quad 0 = D_v v_{\zeta \zeta} + cv_{\zeta} + \varepsilon g(u, v).
\end{align*}
\]  

(21)

Note that system (21) is invariant under the transformation \( (\zeta, c) \rightarrow (-\zeta, -c) \), hence it suffices to consider \( c > 0 \). Letting \( z = \zeta/c, \mu = D_u/c^2, \theta = D_v/D_u \), (21) transforms
to

\[ 0 = \mu u_{zz} + u_z + u^k f(u, v), \quad 0 = \mu \theta v_{zz} + v_z + \varepsilon g(u, v). \] (22)

Assuming that the diffusive constant \( D_u \) is smaller relative to the wave speed \( c \), we obtain that \( 0 < \mu << 1 \). We will also assume that \( \theta > 0 \) is a fixed real number independent of \( \mu \). We are interested in studying periodic traveling waves (wave trains) of system (22). As an illustration, we will study periodic solutions of the following predator-prey system:

\[ \mu u_{zz} + u_z + u \left( 1 - u - \frac{v}{\alpha + u} \right) = 0, \]
\[ \mu \theta v_{zz} + v_z + \varepsilon v \left( \frac{u}{\alpha + u} - \beta - \gamma v \right) = 0, \] (23)

where \( \alpha \) represents the dimensionless semi-saturation constant measured against the prey’s carrying capacity, \( \beta \) measures the ratio of the death rate of the predator to its growth rate, and \( \gamma > 0 \) is associated with the strength of intraspecific competition within the predators. We make the following assumptions on the parameters: \( 0 < \varepsilon << 1, 0 < \alpha < 1 \) and \( 0 < \beta < 1 \). The spatially homogeneous model is commonly referred to as Bazykin’s model and has been analyzed in [2, 15, 23].

Rewriting system (23) as a first-order ODE system, we obtain

\[ \frac{du}{dz} = w_1, \quad \mu \frac{dw_1}{dz} = -w_1 - u \left( 1 - u - \frac{v}{\alpha + u} \right), \]
\[ \frac{dv}{dz} = w_2, \quad \mu \frac{dw_2}{dz} = -\frac{1}{\theta} w_2 - \varepsilon v \left( \frac{u}{\alpha + u} - \beta - \gamma v \right), \] (24)

which is a slow timescale formulation of a slow-fast system with \( \mu \) being treated as the singular parameter. On rescaling the independent variable as \( z = \mu s \), we obtain the equivalent fast timescale formulation of (23) given by

\[ \frac{du}{ds} = \mu w_1, \quad \frac{dw_1}{ds} = -w_1 - u \left( 1 - u - \frac{v}{\alpha + u} \right), \]
\[ \frac{dv}{ds} = \mu w_2, \quad \frac{dw_2}{ds} = -\frac{1}{\theta} w_2 - \varepsilon v \left( \frac{u}{\alpha + u} - \beta - \gamma v \right). \] (25)

The set of equilibria of (25) is the two-dimensional manifold \( \mathcal{M}_0 \), termed as the critical manifold, given by

\[ \mathcal{M}_0 = \left\{ (u, w_1, v, w_2) : w_1 = -u \left( 1 - u - \frac{v}{\alpha + u} \right), w_2 = -\varepsilon v \left( \frac{u}{\alpha + u} - \beta - \gamma v \right) \right\}. \]

The reduced system of (24) restricted to the critical manifold \( \mathcal{M}_0 \) reads as

\[ \frac{du}{dz} = -u \left( 1 - u - \frac{v}{\alpha + u} \right), \quad \frac{dv}{dz} = -\varepsilon v \left( \frac{u}{\alpha + u} - \beta - \gamma v \right). \] (26)
The linearization of (25) for $\mu = 0$ at each point of $\mathcal{M}_0$ has two zero eigenvalues and two negative eigenvalues $-1, -1/\theta$. Therefore $\mathcal{M}_0$ forms a normally hyperbolic attractive manifold of system (25). Hence by Fenichel’s theorem, there exists a locally invariant manifold $\mathcal{M}_\mu$ which is diffeomorphic to $\mathcal{M}_0$ such that $\mathcal{M}_\mu = \mathcal{M}_0 + O(\mu)$ as $\mu \to 0$. Hence the restriction of (25) to $\mathcal{M}_\mu$ up to its lowest order satisfies
\[ \frac{du}{dz} = -u \left(1 - u - \frac{v}{\alpha + u}\right) + O(\mu), \]
\[ \frac{dv}{dz} = -\varepsilon v \left(\frac{u}{\alpha + u} - \beta - \gamma v\right) + O(\mu). \] (27)

Note that system (26) is in the form of system (3) after time reversal, where
\[ f(u, v) = \left(1 - u - \frac{v}{\alpha + u}\right), \quad g(u, v) = v \left(\frac{u}{\alpha + u} - \beta - \gamma v\right) \] and $k = 1$.

It is easy to check that the functions $f$ and $g$ satisfy assumptions (D1)-(D2). Note that the non-trivial $u$-nullcline, $f(u, v) = 0$, of system (26) is a parabola with vertex $(u_1^*, v_1^*)$, where
\[ u_1^* = \frac{1 - \alpha}{2}, \quad v_1^* = \frac{(1 + \alpha)^2}{4}. \]

The integral condition corresponding to the delay of stability loss
\[ \int_{v_0^*}^{v_1^*} \frac{f(0, v)}{vg(0, v)} dv = - \int_{v_0^*}^{v_1^*} \frac{1 - v/\alpha}{v(\beta + \gamma v)} dv = 0 \]
yields the relationship between $v_0^*$ and $v_1^*$, namely
\[ \frac{v_1^*}{v_0^*} = \left(\frac{\beta + \gamma v_1^*}{\beta + \gamma v_0^*}\right)^{1 + \frac{2}{\alpha \gamma}}. \]

The coexistence equilibrium point $E^*(u^*, v^*)$ lies in the intersection of the nontrivial nullclines $f(u, v) = 0$ and $g(u, v) = 0$ and satisfies the cubic equation $q(u^*) = 0$, where
\[ q(u) := \gamma u^3 + (2\alpha \gamma - \gamma)u^2 + (1 - \beta - 2\alpha \gamma + \alpha^2 \gamma)u - (\alpha \beta + \alpha^2 \gamma). \] (28)

Since $\alpha, \beta, \gamma > 0$, it is clear that (28) can have at most three roots with at least one being positive. Hence system (26) can admit up to three positive equilibria. To analyze this, we study the roots of $q'(u)$. Note that
\[ q'(u) = 3\gamma u^2 + 2(2\alpha - 1)\gamma u + (1 - \beta - 2\alpha \gamma + \alpha^2 \gamma). \]
Setting \( q'(u) = 0 \), we have

\[
u^\pm = \frac{1}{6\gamma}[-(2\alpha - 1)\gamma \pm \sqrt{(2\alpha - 1)^2\gamma^2 - 3\gamma(1 - \beta - 2\alpha \gamma + \alpha^2 \gamma)}].
\]

It follows that \( q'(u) = 0 \) does not have real roots if and only if

\[
(2\alpha - 1)^2\gamma^2 - 3\gamma(1 - \beta - 2\alpha \gamma + \alpha^2 \gamma) < 0,
\]
yielding

\[
0 < \gamma < \frac{3(1 - \beta)}{(1 + \alpha)^2},
\]  
(29)

under which we have \( q'(u) > 0 \) for all \( u \). Since \( q(0) < 0 \) and \( q(\infty) = \infty \), therefore (28) will have exactly one positive root if (29) holds. Also, note that

\[
q'(0) = 1 - \beta - 2\alpha \gamma + \alpha^2 \gamma > 0 \text{ if } 0 < \gamma < \frac{1 - \beta}{\alpha(2 - \alpha)}.
\]

Hence \( q'(u) = 0 \) has two positive roots, denoted by \( u^\pm \), if and only if

\[
\frac{3(1 - \beta)}{(1 + \alpha)^2} < \gamma < \frac{1 - \beta}{\alpha(2 - \alpha)};
\]  
(30)

which requires \( 0 < \alpha < 1 - \frac{1}{\sqrt{2}} \). Note that if the roots \( u^\pm \) exist, \( q(u) \) attains its local maximum at \( u^- \) and a local minimum at \( u^- \). Hence under (30), the cubic equation \( q(u) = 0 \) will admit two positive roots if \( q(u^-) = 0 \) or \( q(u^+) = 0 \), and three positive roots if \( q(u^-) > 0 \) and \( q(u^+) < 0 \).

Since the \( v \)-nullcline \( g(u,v) = 0 \) increases with \( v \), to guarantee that \( E^* \) lies on the left branch (but not on the right branch) of the nullcline \( f(u,v) = 0 \), it is necessary and sufficient to require that the graph of the nullcline \( g(u,v) = 0 \) at \((u^*_1, v^*_1) \) lies above the vertex of the nullcline \( f(u,v) = 0 \), yielding that the parameters must satisfy

\[
\frac{1}{\gamma}\left(\frac{1 - \alpha}{1 + \alpha} - \beta\right) > \frac{(1 + \alpha)^2}{4},
\]

which is equivalent to the conditions

\[
0 < \beta < \frac{1 - \alpha}{1 + \alpha}, \quad 0 < \gamma < \frac{4}{(1 + \alpha)^2}\left(\frac{1 - \alpha}{1 + \alpha} - \beta\right).
\]  
(31)

Note that (29) and (31) imply that the coexistence equilibrium \( E^* \) exists uniquely on the left branch of the nullcline \( f(u,v) = 0 \) if

\[
0 < \gamma < \min\left\{\frac{3(1 - \beta)}{(1 + \alpha)^2}, \frac{4}{(1 + \alpha)^2}\left(\frac{1 - \alpha}{1 + \alpha} - \beta\right)\right\}, \quad 0 < \beta < \frac{1 - \alpha}{1 + \alpha}, \quad 0 < \alpha < 1.
\]
On the other hand, if

\[
\frac{3(1 - \beta)}{(1 + \alpha)^2} < \gamma < \min \left\{ \frac{4}{(1 + \alpha)^2} \left( \frac{1 - \alpha}{1 + \alpha} - \beta \right), \frac{1 - \beta}{\alpha(2 - \alpha)} \right\},
\]

\[
0 < \beta < \frac{1 - \alpha}{1 + \alpha}, \quad 0 < \alpha < 1 - \frac{1}{\sqrt{2}}
\]

with \( q(u^-) > 0 \) and \( q(u^+) < 0 \), then system (26) will admit three equilibria, all of which lie on the left branch of the nullcline \( f(u, v) = 0 \). For instance, such a case can be realized with the parameter values \( \alpha = 0.1, \beta = 0.05 \) and \( \gamma = 2.5 \) as shown in figure 8, where the three coexistence equilibria are \((0.1, 0.18), (0.3, 0.28)\) and \((0.4, 0.3)\).

Figure 8: Existence of three equilibria, represented by open circles, lying on the left branch of \( f(u, v) = 0 \) for system (26). The parameter values chosen are \( \alpha = 0.1, \beta = 0.05, \gamma = 2.5, \varepsilon = 0.01 \).

The Jacobian of the vector field at \( E^* \) corresponding to the time reversal of (26) reads as

\[
J \Big|_{E^*} = \begin{pmatrix}
\frac{u^*(1 - \alpha - 2\alpha^*)}{\alpha + \alpha u^*} & -\frac{\alpha}{\alpha + \alpha u^*} \\
\frac{\varepsilon \alpha u^*}{\alpha + \alpha u^*} & -\varepsilon \gamma u^*
\end{pmatrix}.
\] (32)

Since \( u^* < u_j^* = (1 - \alpha)/2 \), it follows that for sufficiently small \( \varepsilon > 0 \), \( trJ \Big|_{E^*} > 0 \).

Case 1: The nullclines \( f(u, v) = 0 \) and \( g(u, v) = 0 \) intersect exactly once. In this case if the intersection of the two nullclines is non-tangential then \( det J |_{E^*} > 0 \), and
hence $E^*$ is an unstable node/spiral.

Case 2: The nullclines $f(u, v) = 0$ and $g(u, v) = 0$ intersect three times. Let $E_1^*$, $E_2^*$ and $E_3^*$ be the three intersection points, with $E_2^*$ lying in between $E_1^*$ and $E_3^*$. If the two nullclines intersect non-tangentially, then $E_1^*$ and $E_3^*$ are unstable nodes or unstable spirals, whereas $E_2^*$ is a saddle.

Case 3: The nullclines $f(u, v) = 0$ and $g(u, v) = 0$ intersect twice. In this case, either $E_1^*$ and $E_3^*$ merge through a saddle-node bifurcation of equilibrium points, and is a saddle-node, whereas $E_3^*$ is an unstable node or an unstable spiral, or $E_2^*$ and $E_3^*$ merge through a saddle-node bifurcation of equilibrium points, and is a saddle-node, whereas $E_1^*$ is an unstable node or an unstable spiral.

In all the three cases, it follows by Theorem 2 that for sufficiently small $\varepsilon$, system (26) after time reversal admits a unique stable limit cycle, $\tilde{\Gamma}^0(\varepsilon)$, in the form of a relaxation oscillation orbit.

![Diagram](image-url)

**Figure 9:** An invariant annular region $\tilde{\Omega}^\mu$ of system (27), where $z < 0$. The outer boundary of $\tilde{\Omega}^\mu$ is formed by $P_1^0P_2^\mu$ and $P_2^\muP_1^0$ and the inner boundary is formed by $Q_1^0Q_2^\mu$ and $Q_2^\muQ_1^0$.

We next show that for $\mu > 0$ sufficiently small, system (27) also admits a stable limit cycle $\tilde{\Gamma}^\mu(\varepsilon)$ such that $\tilde{\Gamma}^\mu(\varepsilon) \to \tilde{\Gamma}^0(\varepsilon)$ as $\mu \to 0$ which then guarantees the existence of a periodic traveling wave solution of system (24).

To see this, we fix $\varepsilon > 0$ and construct a closed neighborhood $\tilde{\Omega}^\mu$ around $\tilde{\Gamma}^0(\varepsilon)$ for $\mu$ sufficiently small. Choose a section $\Sigma$ transversal to $\tilde{\Gamma}^0(\varepsilon)$ through an arbitrary
point \( O \in \tilde{\Gamma}^0(\varepsilon) \), and choose points \( P_1^0, Q_1^0 \in \Sigma \) sufficiently near \( O \) as shown in figure 9. The solutions of (26) through \( P_1^0 \) and \( Q_1^0 \) will cross \( \Sigma \) again at some other points, say at \( P_2^0 \) and \( Q_2^0 \) respectively. Since \( \tilde{\Gamma}^0(\varepsilon) \) is asymptotically stable for system (26) under time reversal, it follows that \( P_2^0 \) and \( Q_2^0 \) lie in the interiors of the segments \( \overline{OP}_1^0 \) and \( \overline{OQ}_1^0 \), respectively. Now for sufficiently small parameter \( \mu > 0 \), the continuous dependence of solutions with respect to parameters imply that the orbits of (27) through \( P_1^0 \) and \( Q_1^0 \) will intersect \( \Sigma \) at some other points, say \( P_2^\mu \) and \( Q_2^\mu \) respectively. Since \( \tilde{\Gamma}^\mu(\varepsilon) \) is asymptotically stable for system (27), it follows that \( P_2^\mu \) and \( Q_2^\mu \) lie in the interiors of \( \overline{OP}_1^\mu \) and \( \overline{OQ}_1^\mu \), respectively. Noticing \( \tilde{\Omega}^\mu \) does not contain any equilibria, we conclude by the Poincaré Bendixson theorem that (27) admits a limit cycle \( \tilde{\Gamma}^\mu(\varepsilon) \) that lies in \( \tilde{\Omega}^\mu \), as desired.

Summarizing the above discussions, we have

**Theorem 3.** Assume that

\[
0 < \alpha < 1, \quad 0 < \beta < \frac{1 - \alpha}{1 + \alpha}, \quad 0 < \gamma < \frac{4}{(1 + \alpha)^2} \left( \frac{1 - \alpha}{1 + \alpha} - \beta \right).
\]

Then the following hold:

(i) For every sufficiently small \( \varepsilon > 0 \), system (26) has a unique relaxation oscillation limit cycle, which is orbitally stable.

(ii) Fix \( \varepsilon > 0 \) sufficiently small. Then for every sufficiently small \( \mu > 0 \), system (27) has a periodic traveling wave solution, whose profile exhibits relaxation oscillations.

## 5 Multiple relaxation oscillations

In this section we consider system (3) under the additional assumption that the \( u \)-nullcline \( f = 0 \) has a minimum located on the left side of the maximum \( P(u_1^*, v_1^*) \), namely, the curve \( f(u, v) = 0 \) looks like an “S-shaped” curve. Precisely, in addition to the assumptions (D0)-(D3) we assume that

\( (D6) \) \( v = v_f(u) \) has exactly one minimum point \( P_2(u_2^*, v_2^*) \) where \( u_2^* \in (0, u_1^*) \) and \( v_2^* \in (b_1, v_Q) \).

\( (D7) \) The nullcline \( g(u, v) = 0 \) intersects the \( u \)-nullcline \( f(u, v) = 0 \) at finitely many points \( E_i(u_i, v_i) \) all lying on the left stable branch of \( f = 0 \), namely, with \( u_i \in (0, u_2^*) \).

\( (D8) \) \( v_0^* < v_2^* \).
Let \( v_3^* > 0 \) such that

\[
\int_{v_2^*}^{v_3^*} \frac{f(0, v)}{g(0, v)} \, dv = 0.
\]

From the assumptions (D1)-(D3) and (D8) we have \( v_3^* < v_1^* \). For convenience, we let \( u = u_m(v) \) be the middle branch of the graph \( v = v_f(u) \) with \( u \in [u_2^*, u_1^*] \). We define the second singular orbit \( \gamma_0 \) by

\[
\gamma_0 := l_1 \cup l_2 \cup l_3 \cup l_4,
\]

where \( l_1 \) is the segment joining the points \( (u_m(v_3^*), v_3^*) \) and \( P_2(u_2^*, v_2^*) \) along the middle arc \( PP_2 \), \( l_2 \) is the horizontal line segment connecting the vertex \( P_2 \) to the point \( (0, v_2^*) \), \( l_3 \) is the vertical line segment along the \( v \)-axis joining the points \( (0, v_2^*) \) and \( (0, v_3^*) \) and \( l_4 \) is the horizontal line segment connecting \( (0, v_3^*) \) to \( (u_m(v_3^*), v_3^*) \) as shown in figure 10.

Figure 10: Representation of the singular orbits \( \Gamma_0 \) and \( \gamma_0 \).

**Theorem 4.** Assume (D0)-(D3) and (D6)-(D8). Then for sufficiently small \( \varepsilon > 0 \), system (3) has exactly two limit cycles \( \Gamma_\varepsilon \) and \( \gamma_\varepsilon \) with \( \Gamma_\varepsilon \to \Gamma_0 \) and \( \gamma_\varepsilon \to \gamma_0 \) as \( \varepsilon \to 0 \); \( \Gamma_\varepsilon \) is asymptotically orbitally stable and \( \gamma_\varepsilon \) is orbitally unstable.

**Proof.** The existence of \( \Gamma_\varepsilon \) follows from Theorem 1 (I). By switching \(-t\) to \( t\) and applying Theorem 1 (I) yields the existence of \( \gamma_\varepsilon \). It follows from the vector field
directions of (3) along the boundaries of the annular regions lying in the vicinities of $\Gamma_0$ and $\gamma_0$ that $\Gamma_\varepsilon \to \Gamma_0$ and $\gamma_\varepsilon \to \gamma_0$ as $\varepsilon \to 0$. The corresponding asymptotic formula in Lemma 2 can also be established for $\gamma_\varepsilon$, giving the orbital unstability of $\gamma_\varepsilon$. The nonexistence of any other closed orbit can be proved in a similar manner to that in the proof of Lemma 3.

As an example for the application of Theorem 4, we consider the Holling-Tanner prey-predator model with Holling type IV functional response:

$$\begin{align*}
\frac{du}{dt} &= u \left(1 - u - \frac{mv}{1 + \alpha u^2}\right), \\
\frac{dv}{dt} &= \varepsilon v \left(1 - \frac{\beta v}{u}\right),
\end{align*}$$

(34)

which, by rescaling time, can be written as

$$\begin{align*}
\frac{du}{dt} &= u^2 \left[(1 - u)(1 + \alpha u^2) - mv\right], \\
\frac{dv}{dt} &= \varepsilon v(1 + \alpha u^2)(u - \beta v).
\end{align*}$$

This is a particular system (3) with $b_1 > 0$ that can be taken arbitrarily small, $b_2 = \infty$, $a = 1$, $k = 2$, and $f(u, v) = (1 - u)(1 + \alpha u^2) - mv$, $g(u, v) = v(1 + \alpha u^2)(u - \beta v)$, and so $f(u, v) = 0$ and $g(u, v) = 0$ for $(u, v) \in [0, 1] \times (0, \infty)$ give, respectively,

$$v = v_f(u) = \frac{1}{m}(1 - u)(1 + \alpha u^2), \quad u = u_g(v) = \beta v.$$

Assume that $\alpha > 3$. Since $v'_f = \frac{1}{m}(-3\alpha u^2 + 2\alpha u - 1) = 0$, it follows that

$$u^*_1 = \frac{1}{3} \left(1 + \sqrt{1 - \frac{3}{\alpha}}\right), \quad u^*_2 = \frac{1}{3} \left(1 - \sqrt{1 - \frac{3}{\alpha}}\right).$$

thus, $v_f(u)$ reaches its maximum and minimum at $u^*_1$ and $u^*_2$ respectively, with

$$v^*_1 = v_f(u^*_1) = \frac{2\alpha u^*_1}{m}(1 - u^*_1)^2, \quad v^*_2 = v_f(u^*_2) = \frac{2\alpha u^*_2}{m}(1 - u^*_2)^2,$$

where we used the fact that $1 + \alpha(u^*_i)^2 = 2\alpha u^*_i(1 - u^*_i)$ for $i = 1, 2$.

We now compute $v^*_0$ by evaluating the integral on the loss of stability delay

$$\int_{v^*_0}^{v^*_1} \frac{f(0, v)}{g(0, v)} dv = \int_{v^*_0}^{v^*_1} \frac{1 - mv}{-\beta v^2} dv = \frac{1}{\beta} \left[m \ln \frac{v^*_1}{v^*_0} + \frac{1}{v^*_1} - \frac{1}{v^*_0}\right] = 0,$$
yielding that \( v_0^* \) is determined implicitly by the equation
\[
\ln \frac{v_1^*}{v_0^*} = m \left( \frac{1}{v_0^*} - \frac{1}{v_1^*} \right). \tag{35}
\]
The condition (D8) requires \( v_0^* < v_2^* \). Since this inequality cannot be written out explicitly in terms of the coefficients in (34), we now show that it holds at least for sufficiently large \( \alpha \). To see this, we use the above formulas for \( u_1^*, u_2^*, v_1^* \) and \( v_2^* \) to get
\[
u_1^* \to \frac{2}{3}, \quad u_2^* \to 0, \quad \alpha u_1^* \to \infty, \quad \alpha u_2^* \to 1 \quad (\alpha \to \infty)
\]
and so
\[
v_1^* \to \infty, \quad v_2^* \to \frac{1}{m} \quad (\alpha \to \infty).
\]
This together with \( v_0^* < v_f(0) = \frac{1}{m} \) and (35) yields that \( v_0^* \to 0 \) as \( \alpha \to 0 \), hence \( v_0^* < v_2^* \) for sufficiently large \( \alpha \). Finally, noting that the value of \( u_2^* \) does not depend on \( \beta \), by taking \( \beta > 0 \) sufficiently small, the unique positive equilibrium \( E^*(u^*, v^*) \) of (34) lies to the left of the minimum point of \( P_2(u_2^*, v_2^*) \). We now can apply Theorem 4 to obtain the following:

**Theorem 5.** Let \( m > 0 \), let \( \alpha > 0 \) be sufficiently large such that \( v_0^* < v_2^* \), and let \( \beta > 0 \) be sufficiently small such that \( E^*(u^*, v^*) \) lies to the left of the minimum point of \( P_2(u_2^*, v_2^*) \). Then the conclusions in Theorem 4 hold for the system (34).

The results on the existence of three and more relaxation oscillations of (3) can be obtained similarly. Due to length of the paper we will not give these results here. We also refer to [12] on this subject for a different class of models.

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### References


