Joint Program Exam in Real Analysis
September 2013

Instructions:

1. Print your student ID and the problem number on each page. Write on one side of each paper sheet only. Start each problem on a new sheet. Write legibly using a dark pencil or pen.

2. You may use up to three and a half hours to complete this exam.

3. The exam consists of 7 problems. All the problems are weighted equally. You need to do ALL of them for full credit.

4. For each problem which you attempt try to give a complete solution. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used.

5. Throughout the exam all the integrals mean the Lebesgue integrals. $L^p(E)$ denotes the L^p space with respect to Lebesgue measure on the Lebesgue measurable set E. $[a, b]$ denotes a bounded and closed interval in $\mathbb{R}$.

1. Let $f, f_1, f_2, \ldots$ be measurable functions on $\mathbb{R}$ and finite a.e in $\mathbb{R}$. Suppose $f_n \to f$ in measure and $f_n \to g$ almost everywhere. Is it always true that $f(x) = g(x)$ a.e. on $\mathbb{R}$? (Prove or give a counterexample.)

2. (i) Suppose that f is a non-decreasing function on $\mathbb{R}$, and let E be the set of all discontinuous points of f. Show that E is a countable set.

   (ii) Suppose that f is a Riemann integrable function with $f(x) > 0$ for every $x \in [a, b]$. Prove that $\int_a^b f(x)\, dx > 0$.

3. Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f_n \colon X \to \mathbb{C}$ a sequence of measurable functions. Assume that $\int_X |f_n|\, d\mu \to 0$ as $n \to \infty$ and $|f_n|^2 \leq g \in L^1_\mu(X)$. Prove that $\int_X |f_n|^2\, d\mu \to 0$ as $n \to \infty$.

4. Let $E = [0, 1] \times [0, 1]$ and

$$f(x, y) = \frac{x^2 y^2}{(x^3 + y^3)^2}$$

   for $0 < x, y \leq 1$ and 0 otherwise. Show that $f \notin L^1(E)$.

5. Suppose that $f \in L^1[a, b]$. Prove that

$$\int_a^b f(x) \sin(nx)\, dx \to 0 \qquad (n \to \infty).$$

6. Prove that the function $f(x) = x \sin(1/x)$ for $0 < x \leq 1$ and $f(0) = 0$ is uniformly continuous on $[0, 1]$ but not of bounded variation.

7. A function f is said to satisfy the Lipschitz condition on $[a, b]$ if for some constant $L > 0$

$$|f(x_1) - f(x_2)| \leq L|x_1 - x_2| \qquad \forall\, x_1, x_2 \in [a, b].$$

   Prove that f satisfies the Lipschitz condition on $[a, b]$ if and only if f' exists a.e. in $[a, b]$, f' is essentially bounded on $[a, b]$, and

$$f(x) = f(a) + \int_a^x f'(t)\, dt \qquad (a \leq x \leq b).$$