Joint Program Exam in Real Analysis

May 2013

Instructions

1. You may take up to three and a half hours to complete the exam.
2. The exam consists of 7 problems. All the problems are weighted equally. You need to do ALL of them for full credit.
3. For each problem which you attempt try to give a complete solution. Justify the steps in your solutions by referring to theorems by name, when appropriate, and by verifying the hypotheses of these theorems. You do not need to reprove the theorems you used. Completeness is important: a correct and complete solution to one problem will gain more credit than two "half solutions" to two problems.
4. Throughout the exam:
   - m stands for the Lebesgue measure (in $\mathbb{R}^k$ for any $k \geq 1$).
   - Notation such as $\int_{[0,1]} f \, dm$ and $\int_{[0,1]} f(t) \, dt$ is used for the Lebesgue integral.
   - $V_a^b(f)$ stands for the total variation of a function f on an interval $[a, b]$.
   - For any measurable set $E \subset \mathbb{R}^k$ the notation $L^p(E)$ means the L^p space of functions $f \colon E \to \mathbb{C}$ with respect to the Lebesgue measure on E.

1. Find the limit
$$\lim_{k\to\infty} \int_{[1/k,\,k]} \left(1 - \frac{x}{k}\right)^k dx$$
and justify your assertion using appropriate limit theorems (and calculus).

2. Given $p = 1, 2$ and $f \in L^p([0,\infty))$, prove that
$$\lim_{n\to\infty} \int_{[0,\infty)} f(x)e^{-nx}\,dx = 0.$$
Note: you need to prove this for $p = 1$ and $p = 2$ separately.

3. If f is Lebesgue integrable on $[0,1]$ and $g(x) = \int_{[x,1]} t^{-1} f(t)\,dt$, then prove that g is Lebesgue integrable on $[0,1]$ and $\int_{[0,1]} g\,dm = \int_{[0,1]} f\,dm$.

4. Let $E \subset \mathbb{R}^k$ with $m(E) > 0$. Show that for any $p \in [1,\infty)$ there is a function $f \in L^p(E)$ such that $f \notin L^\infty(E)$.

5. Let E be a Lebesgue measurable subset of $\mathbb{R}^n$. Let us define the notion of a 'locally measurable function' on E as follows. Let $f\colon E \to \mathbb{R}$. Call f *locally measurable* if for every $x \in E$ there exists a non-empty open ball B_x centered at x such that the restriction of f to $E \cap B_x$ is a measurable function. Prove that this definition is misguided in the sense that f is locally measurable if and only if f is measurable.

6. Let $E \subset [a,b]$ be a Lebesgue measurable subset of a finite interval $[a,b] \subset \mathbb{R}$. Suppose there exists a *continuous* function $f\colon [a,b] \to \mathbb{R}$ such that for any subinterval $I \subset [a,b]$ we have $m(E \cap I) = \int_I f\,dm$. Prove that either $m(E) = 0$ or $m(E) = b - a$.

7. Let $f\colon [a,b] \to \mathbb{R}$ be a function of bounded variation. Let $V_f(x) = V_a^x(f)$ denote the total variation of f on $[a,x]$ for each $x \in [a,b]$.

   (i) Prove that if $f \in C^1[a,b]$ (continuously differentiable), then $V_f \in C^1[a,b]$.

   (ii) Find a function $f \in C^2[a,b]$ (having a continuous second derivative) such that $V_f \notin C^2[a,b]$.